Project 1

Linear Advection

1.1 Introduction

The hydrodynamic equations are a set of partial differential equations with spatial and temporal derivatives. Due to the non-linearities in the Euler equations wave fronts begin to steepen and it may come to discontinuities (shock waves). The numerical solution requires a special treatment of these non-linear terms.

As a test problem on how such discontinuities behave numerically, in this project the linear advection equation should be investigated.

1.2 Linear Advection

As a simple test system for numerical hydrodynamics often the linear advection equation is used. In a one dimensional setup it reads

$$\frac{\partial \psi}{\partial t} + a \frac{\partial \psi}{\partial x} = 0\,. \tag{1.1}$$

Here, ψ is a quantity to be transported (e.g. the density) and the constant a is the velocity.

Verify that with the initial condition

$$\psi(x, 0) = \psi_0(x)$$

the linear advection equation (1.1) has the following analytical solution

$$\psi(x, t) = \psi_0(x - at)\,.$$

This implies that the initial conditions will be transported with constant velocity a to the right if $a > 0$, and to the left if $a < 0$, i.e. the quantity a is the *transport velocity* of the 1D wave.

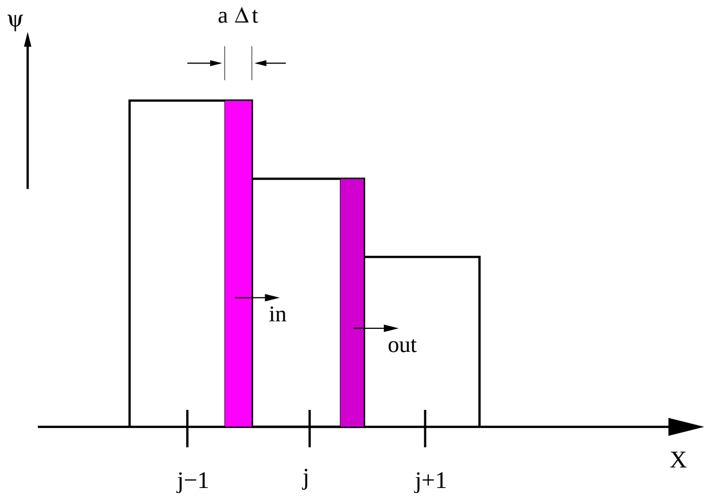

Figure 1.1: Illustration of the fluxes across the boundaries of a grid-cell using the upwind method. Shown is the j-th gridcell where the flux $F_{in}(j)$ (denoted F_j) is located at point $x_{j-1/2}$ and the flux $F_{out}(j)$ (denoted F_{j+1}) at the right cell boundary $x_{j+1/2}$. The shaded area symbolizes the amount of 'material' transported accross the boundaries.

1.3 Numerical Solution

To integrate the linear advection equation a finite difference scheme should be used. Here, the partial differential equation (pde) is translated in a finite difference equation (fde) that can be solved numerically.

The numerical solution should be calculated on a grid. Let the length of each gridcell be Δx, and the time step Δt be constant. The discretisation in time and in space will be denoted by an upper and lower index, respectively, such that

$$\psi_j^n = \psi(j\Delta x, n\Delta t)$$

will denote the numerical value of the quantity ψ at time $t = n\Delta t$ (after n timesteps) and at spatial location $x = j\Delta x$ (the j-th gridpoint).

1.3.1 Two numerical methods

In this project two different methods should be implemented and compared to each other. The first one is the upwind method which is the most simple stable evolution method for the advection problem. This comes in two versions (1st and 2nd order) which both should be implemented. As an alternative the Lax-Wendroff method should be implemented as well.

1) **Upwind (1st Order)**
For the time derivative a *forward* and for the spatial derivative a *backward* derivative is used, a methodology sometimes named *Forward Time Backward Space* (FTBS)

$$\frac{\psi_j^{n+1} - \psi_j^n}{\Delta t} + a\frac{\psi_j^n - \psi_{j-1}^n}{\Delta x} = 0\,. \tag{1.2}$$

Rearranging

$$\boxed{\psi_j^{n+1} = \psi_j^n - \sigma(\psi_j^n - \psi_{j-1}^n)} \tag{1.3}$$

where

$$\sigma \equiv a\frac{\Delta t}{\Delta x} \; > \; 0 \tag{1.4}$$

is the so called **CFL-number**, after Courant-Friedrich-Lewy. σ plays an important role for the stability of the method. For physical and numerical purposes it is convenient to use a different approach that makes the conservative properties of the problems more clear. Rewriting eq. 1.1 as

$$\frac{\partial \psi}{\partial t} + \frac{\partial a\psi}{\partial x} = 0 \tag{1.5}$$

we can integrate this to

$$\psi_j^{n+1}\Delta x = \psi_j^n \Delta x + \Delta t(F_{in} - F_{out}) \tag{1.6}$$

where F_{in} and F_{out} denote the fluxes of a conserved quantity (e.g. mass with $F =$ velocity $\cdot$ density) accross the boundaries of the cell. Denoting on the left side $F_{in} = F_j$ and at the right boundary $F_{out} = F_{j+1}$ (see Fig. 1.1) then eq. (1.6) can be written as

$$\psi_j^{n+1} = \psi_j^n - \frac{\Delta t}{\Delta x}(F_{j+1} - F_j)\,. \tag{1.7}$$

For the fluxes the upstream values need to be taken. Assuming that the velocity a is positive, i.e. the transport is to the right we find for the fluxes

$$F_j = a\,\psi_{j-1}\,. \tag{1.8}$$

Using this in eq. (1.7) corresponds to the first order upwind scheme as it assumes constant states in each gridcell.

2) **Upwind (2nd Order)**
In case of non-constant states the above method can be extended considering the variation of the quantity within one gridcell. Taking a linear behavior we obtain

$$\underbrace{\underbrace{F_j = a\left[\psi_{j-1}^n\right.}_{1st\ Order} + \left.\frac{1}{2}(1-\sigma)\Delta\psi_{j-1}\right]}_{2nd\ Order} \tag{1.9}$$

where $\Delta\psi_j$ denotes the slope across the gridcell j, sometimes called the undivided difference. It is an approximation of

$$\Delta\psi \approx \frac{\partial \psi}{\partial x}\,\Delta x$$

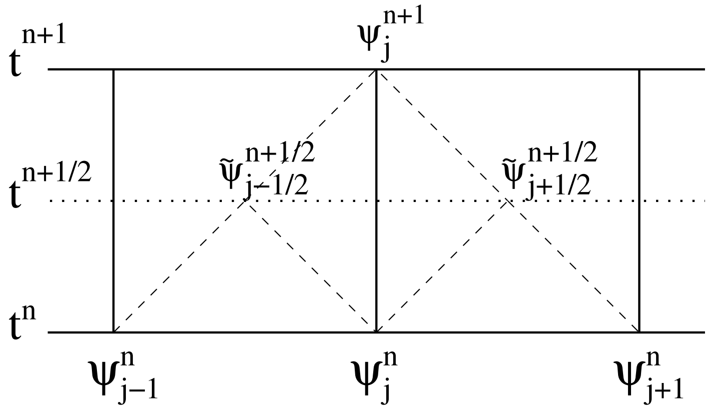

Figure 1.2: The procedure of the Lax-Wendroff method to integrate the hydrodynamic equations. The method proceeds in two steps. First intermediate quantities are calculated at time $t^{n+1/2}$, these are then used to update to the new level at time $t = n + 1$.

Here, we use a 2nd Order Upwind Scheme (van Leer) where $\Delta\psi$ is approximated by a geometric mean of the neighboring slopes. This choice preserves *Monotonicity.*

$$\Delta\psi_j = \begin{cases} 2\frac{(\psi_{j+1}-\psi_j)(\psi_j-\psi_{j-1})}{(\psi_{j+1}-\psi_{j-1})} & \text{if} \quad (\psi_{j+1}-\psi_j)(\psi_j-\psi_{j-1}) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.10}$$

3) **Lax-Wendroff Scheme**
The derivatives are now implemented by a two-step *forward Differencing* in time and using central spatial derivatives, as shown in Fig. 1.2. The first step is the *predictor-step* (advancing to time $t^{n+1/2}$). For the gridpoint at $x_{j+1/2}$ this reads

$$\tilde{\psi}^{n+1/2}_{j+1/2} = \frac{1}{2}\left(\psi^n_j + \psi^n_{j+1}\right) - \frac{\sigma}{2}\left(\psi^n_{j+1} - \psi^n_j\right) \tag{1.11}$$

and the value for $x_{j-1/2}$ is obained by shifting one cell to the left (Fig. 1.2). Then the second *corrector-step* advances to time t^{n+1})

$$\psi^{n+1}_j = \psi^n_j - \sigma\left(\tilde{\psi}^{n+1/2}_{j+1/2} - \tilde{\psi}^{n+1/2}_{j-1/2}\right) \tag{1.12}$$

Combining these two steps, one obtains the **Lax-Wendroff** scheme

$$\boxed{\psi^{n+1}_j = \psi^n_j - \frac{\sigma}{2}(\psi^n_{j+1} - \psi^n_{j-1}) + \frac{\sigma^2}{2}(\psi^n_{j+1} - 2\psi^n_j + \psi^n_{j-1})} \tag{1.13}$$

Instead of this last formulation it is advantageous to use the methodology that we introduced for the upwind method above and use a 2nd order method of eq. (1.9). Indeed, the Lax-Wendroff method of eq. (1.13) can be written as a 2nd order upwind method (eq. 1.9) with the following form of the undivided difference

$$\Delta\psi_j = \psi_{j+1} - \psi_j \,. \tag{1.14}$$

Verify that this definition leads to the Lax-Wendroff method of eq. (1.13).

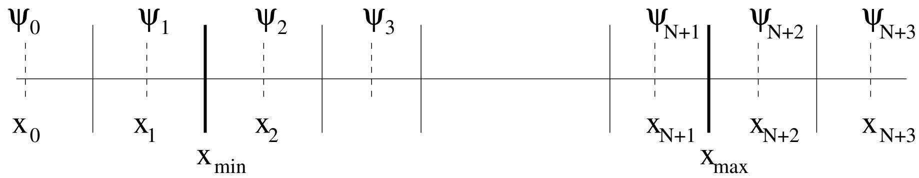

Figure 1.3: The grid structure for the advection problem. The actual domain where the computations are done are the gridcells between x_{min} and x_{max}. This region is covered by N gridcells with variables ψ_2 to ψ_{N+1}.

1.4 Excercise: Linear Advection

Write a program to solve the linear advection equation (1.1). Use the Upwind method in the form of (1.7) using the first and second order slopes of (1.9). Then use the Lax-Wendroff method (1.13) using the 2nd order upwind method with the slope (1.14) and compare all results on the linear advection problem.

Test your programme on the following example (square wave), see Fig. 1.4:
Let $a = 1$ the positive transport velocity. The computational domain is $[-1, 1]$. The **boundary conditions are periodic**, i.e. $\psi(-1, t) = \psi(1, t)$ for all times t.

The **Initial Conditions** at time $t = 0$, $\psi(x, t = 0)$ are given by

$$\psi(x, t=0) = \begin{cases} 1.0 & \text{for} \quad |x| < \frac{1}{3} \\ 0.0 & \text{for} \quad \frac{1}{3} < |x| \leq 1 \end{cases}$$

Define

$$\sigma = a\frac{\Delta t}{\Delta x} = 0.8$$

and calculate, using $\sigma = 0.8$, the solution for $\psi(x, 4)$ using 400 gridpoints.

The positions of the variables ψ_i in the grid is displayed in Fig. 1.3. Within x_{min} and x_{max} are exactly N gridpoints. To implement the periodic boundary conditions we have to identify the following points

$$\begin{aligned} \psi_0 &= \psi_N \\ \psi_1 &= \psi_{N+1} \\ \psi_{N+2} &= \psi_2 \\ \psi_{N+3} &= \psi_3 \end{aligned} \tag{1.15}$$

This has to be done at each time step t^n after the interior gridpoints ψ_j with $j = 2$ to $N + 1$ have been updated.

Compare the obtained numerical solution with the analytical one and prepare plots where the analytical solution is overplotted with the numerical one.

Describe the differences between the 1st and 2nd order upwind method and the Lax-Wendroff method.

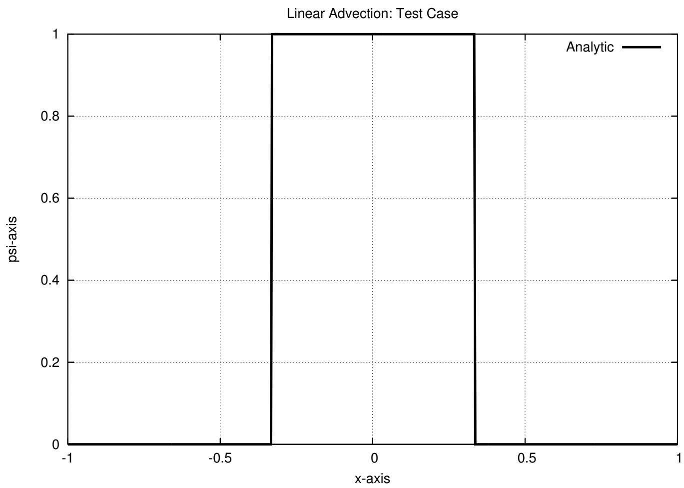

Figure 1.4: The initial condition at $t = 0$ for the linear advection problem. For the transport velocity $a = 1$ the final state (at time $t = 4$) is identical. The square wave has made two 'rounds'.