Numerical Hydrodynamics: A Primer

Wilhelm Kley
Institut für Astronomie & Astrophysik
& Kepler Center for Astro and Particle Physics Tübingen

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

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The Euler-Equations in conservative form read

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \tag{1}$$

$$\frac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) = -\nabla p + \rho \vec{k} \tag{2}$$

$$\frac{\partial (\rho \epsilon)}{\partial t} + \nabla \cdot (\rho \epsilon \vec{u}) = -p \nabla \cdot \vec{u} \tag{3}$$

$\vec{u}$: Velocity, $\vec{k}$: external forces, ϵ specific internal energy
The equations describe the conservation of mass, momentum and energy.

For completion we need an equation of state (eos):

$$p = (\gamma - 1)\, \rho \epsilon \tag{4}$$

Using this and eq. (3), we can rewrite the energy equation as an equation for the pressure

$$\frac{\partial p}{\partial t} + \nabla \cdot (p \vec{u}) = -(\gamma - 1) p \, \nabla \cdot \vec{u} \tag{5}$$

Reformulating

Expanding the divergences on the left side and use for the momentum and energy equation the continuity equation

$$\frac{\partial \rho}{\partial t} + (\vec{u} \cdot \nabla)\rho = -\rho \nabla \cdot \vec{u} \quad (6)$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho} \nabla p + \vec{k} \quad (7)$$

$$\frac{\partial p}{\partial t} + (\vec{u} \cdot \nabla)p = -\gamma p \nabla \cdot \vec{u} \quad (8)$$

Since all quantities depend on space ($\vec{r}$) and time (t), for example $\rho(\vec{r}, t)$, we can use for the left side the total time derivative (Lagrange-Formulation). For example, for the density one obtains

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\vec{u} \cdot \nabla)\rho = -\rho \nabla \cdot \vec{u} \,. \quad (9)$$

The Operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \quad (10)$$

is called material derivative (equivalent to the total time derivative, d/dt).

Lagrange-Formulation

Use now the material derivative

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\vec{u} \tag{11}$$

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{k} \tag{12}$$

$$\frac{Dp}{Dt} = -\gamma p\nabla\cdot\vec{u} \tag{13}$$

These equations describe the change of the quantities in the comoving frame = Lagrange-Formulation.

For the Euler-Formulation, one analysed the changes at a specific, fixed point in space !

The Lagrange-Formulation can be used conveniently for 1D-problems, for example the radial stellar oscillations, using comoving mass-shells.

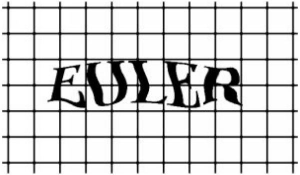
For the Euler-Formulation a fixed grid is used.

The problem

Consider the evolution of the full time-dependent hydrodynamic equations.
The non-linear partial differential equations of hydrodynamics will be solved numerically Continuum $\Rightarrow$ Discretisation

Numerical Hydrodynamics: Method of solution

Grid-based methods (Euler)



fixed Grid
- matter flows through grid

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla p$$

Methods:

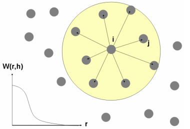
- finite differences
  non-conservative
- Control Volume
  conservative
- Riemann-solver
  wave properties
- Problem: Discontinuities

Particle methods (Lagrange)

moving Grid/Particle
- flow moved grid

$$\rho \frac{d\vec{u}}{dt} = -\nabla p$$

Well known method:
Smoothed Particle Hydrodynamics, SPH



'smeared out particles'
good for free boundaries, self-gravity

consider: 1D Euler equations

describe conservation of mass, momentum and energy

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \tag{14}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u u}{\partial x} = -\frac{\partial p}{\partial x} \tag{15}$$

$$\frac{\partial \rho \epsilon}{\partial t} + \frac{\partial \rho \epsilon u}{\partial x} = -p\frac{\partial u}{\partial x} \tag{16}$$

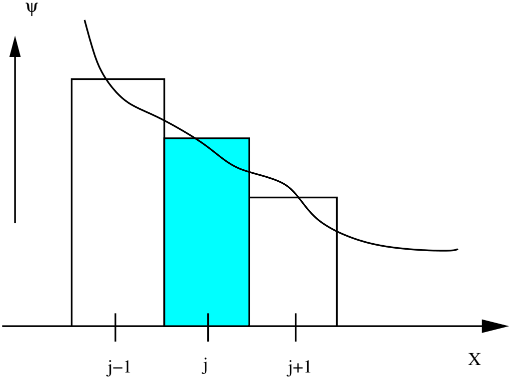
ρ: density
u: velocity
p: pressure
ϵ: internal specific energy (Energy/Mass)

with the equation of state

$$p = (\gamma - 1)\rho\epsilon \tag{17}$$

γ: adiabatic exponent

partial differential equation in space and time
$\rightarrow$ need discretisation in space and time.



Consider funktion: $\psi(x,t)$

discretisation in space

cover with a grid

$$\Delta x = \frac{x_{max} - x_{min}}{N}$$

ψ_j^n cell average of $\psi(x,t)$ at the grdipoint x_j at time t^n

$$\psi_j^n = \psi(x_j, t^n) \approx \frac{1}{\Delta x} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \psi(x, n\Delta t) dx$$

ψ_j^n is piecewise constant. j spatial index, n time step.

Consider general equation

$$\frac{\partial \psi}{\partial t} = \mathcal{L}(\psi(x,t)) \tag{18}$$

with a (spatial) differential operator $\mathcal{L}$.
typical Discretisation (1. order in time), at time: $t = t^n = n\Delta t$

$$\frac{\partial \psi}{\partial t} \approx \frac{\psi(t+\Delta t) - \psi(t)}{\Delta t} = \frac{\psi^{n+1} - \psi^n}{\Delta t} = L(\psi^n) \tag{19}$$

now at a special location, the grid point x_j (with moving terms)

$$\psi_j^{n+1} = \psi_j^n + \Delta t L(\psi_k^n) \tag{20}$$

$L(\psi_k^n)$: discretised differential operator $\mathcal{L}$ (here explicit)
- k in $L(\psi_k)$: set of spatial indices:
- typical for 2. order: $k \in \{j-2, j-1, j, j+1, j+2\}$
  (need information from left and right, 5 point 'Stencil')

$$\frac{\partial \vec{A}}{\partial t} = \mathcal{L}_1(\vec{A}) + \mathcal{L}_2(\vec{A}) \tag{21}$$

$\mathcal{L}_i(\vec{A}), i = 1, 2$ individul (Differential-)operators applied to the quantities $\vec{A} = (\rho, u, \epsilon)$.
Here, for 1D ideal hydrodynamics

$$\mathcal{L}_1 : \text{Advection}$$

$$\mathcal{L}_2 : \text{pressure, or external forces}$$

To solve the full equation the solution is split in several substeps

$$\begin{aligned} \vec{A}^1 &= \vec{A}^n + \Delta t L_1(\vec{A}^n) \\ \vec{A}^{n+1} = \vec{A}^2 &= \vec{A}^1 + \Delta t L_2(\vec{A}^1) \end{aligned} \tag{22}$$

L_i is the differential operator to $\mathcal{L}_i$.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u}{\partial x}$$
$$\frac{\partial (\rho u)}{\partial t} = -\frac{\partial (\rho u u)}{\partial x}$$
$$\frac{\partial (\rho \epsilon)}{\partial t} = -\frac{\partial (\rho \epsilon u)}{\partial x}$$

In explicit conservation form

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{f}(\vec{u})}{\partial x} = 0 \qquad (23)$$

for $\vec{u} = (u_1, u_2, u_3)$ and $\vec{f} = (f_1, f_2, f_3)$ we have:

$\vec{u} = (\rho, \rho u, \rho \epsilon)$ and $\vec{f} = (\rho\, u, \rho u u, \rho \epsilon u)$.

This step yields: $\rho^n \to \rho^1 = \rho^{n+1}, \quad u^n \to u^1, \quad \epsilon^n \to \epsilon^1$

Momentum equation

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{24}$$

$$u_j^{n+1} = u_j - \Delta t \, \frac{1}{\bar{\rho}_j^{n+1}} \, \frac{\left(p_j - p_{j-1}\right)}{\Delta x} \quad \text{for} \quad j = 2, N \tag{25}$$

energy equation

$$\frac{\partial \epsilon}{\partial t} = -\frac{p}{\rho}\frac{\partial u}{\partial x} \tag{26}$$

$$\epsilon_j^{n+1} = \epsilon_j - \Delta t \, \frac{p_j}{\rho_j^{n+1}} \, \frac{\left(u_{j+1} - u_j\right)}{\Delta x} \quad \text{for} \quad j = 1, N \tag{27}$$

on the right hand side we use the actual values for u, ϵ and p, i.e. here u^1, p^1, ϵ^1.

This step yields: $u^1 \rightarrow u^{n+1}, \quad \epsilon^1 \rightarrow \epsilon^{n+1}$

Model equation for advection

The continuity equation was

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \tag{28}$$

Here, $F^m = \rho u$ is the mass flow

Using the notation $\rho \to \psi$ and $u \to a = const$. we obtain the Linear Advection Equation

$$\frac{\partial \psi}{\partial t} + a\frac{\partial \psi}{\partial x} = 0 \,. \tag{29}$$

with a constant velocity a, the solution is a wave traveling to the right

$$\text{using} \quad \psi(x, t = 0) = f(x) \quad \text{we get} \quad \psi(x, t) = f(x - at)$$

Here $f(x)$ is the initial condition at time $t = 0$, that is shifted by the advection with a constant velocity a to the right.

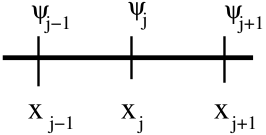
The numerics should maintain this property as accurately as possible.

Numerical Hydrodynamics: Linear Advection

FTCS: *Forward Time Centered Space* algorithm

$$\frac{\partial\psi}{\partial t} + a\frac{\partial\psi}{\partial x} = 0 \tag{30}$$

Specify the grid :



and write

$$\left.\frac{\partial\psi}{\partial t}\right|_j^n = \frac{\psi_j^{n+1} - \psi_j^n}{\Delta t} \tag{31}$$

$$\left.\frac{\partial\psi}{\partial x}\right|_j^n = \frac{\psi_{j+1}^n - \psi_{j-1}^n}{2\,\Delta x} \tag{32}$$

it follows

$$\psi_j^{n+1} = \psi_j^n - \frac{a\Delta t}{2\Delta x}\left(\psi_{j+1}^n - \psi_{j-1}^n\right) \tag{33}$$

The method looks well motivated: but it is unstable for all Δt !

$$\frac{\partial \psi}{\partial t} + \frac{\partial a\psi}{\partial x} = 0 \qquad (34)$$

or

$$\frac{\partial \psi}{\partial t} + a\frac{\partial \psi}{\partial x} = 0 \qquad (35)$$

a: constant (velocity) > 0
$\psi(x,t)$ arbitrary transport quantity

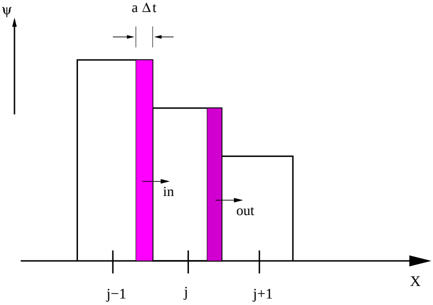
change of ψ in grid cell j

$$\psi_j^{n+1}\Delta x = \psi_j^n \Delta x + \Delta t (F_{in} - F_{out}) \qquad (36)$$

The flux F_{in} is for constant ψ_j

$$F_{in} = a\,\psi_{j-1}^n \qquad (37)$$
$$F_{out} = a\,\psi_j^n \qquad (38)$$

Upwind-Method
Information comes from upstream



purple regions will be transported into the next neighbour cell

Upwind-Method II

Extension for non-constant states

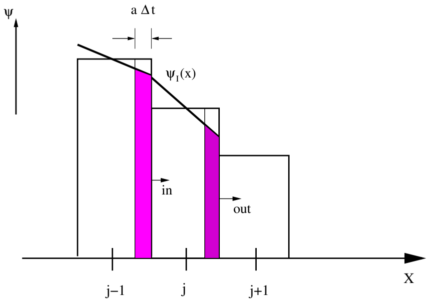
$$F_{in} = a\,\psi_I\left(x_{j-1/2} - \frac{a\Delta t}{2}\right) \qquad (39)$$

$\psi_I(x)$ interpolation polynomial
Here linear interpolation (straight line)
This yields

$$\underbrace{\underbrace{F_{in} = a\,\left[\psi^n_{j-1}\right.}_{1st\ Order} + \left.\frac{1}{2}(1-\sigma)\Delta\psi_{j-1}\right]}_{2nd\ Order} \qquad (40)$$

with $\sigma = a\Delta t/\Delta x$

$$\Delta\psi_j \approx \left.\frac{\partial\psi}{\partial x}\right|_{x_j} \Delta x$$



$$\psi_I(x) = \psi^n_j + \frac{x - x_j}{\Delta x}\Delta\psi_j \quad (41)$$

$\Delta\psi_j$ undivided differences

2nd order upwind

$\psi_I(x)$ is evaluated in the middle of the purple area.

a) $\Delta\psi_j = 0$ *Upwind, 1st Order*, piece-wise constant

b) $\Delta\psi_j = \frac{1}{2}\left(\psi_{j+1} - \psi_{j-1}\right)$ *Fromm*, centered derivative

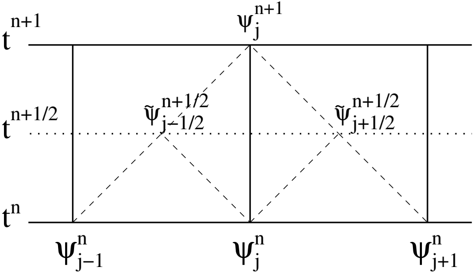
c) $\Delta\psi_j = \psi_j - \psi_{j-1}$ *Beam-Warming*, upwind slope

d) $\Delta\psi_j = \psi_{j+1} - \psi_j$ *Lax-Wendroff*, downwind slope

Often used is the 2nd Order Upwind (van Leer) Geometric Mean
(maintains the Monotonicity)

$$\Delta\psi_j = \begin{cases} 2\frac{(\psi_{j+1}-\psi_j)(\psi_j-\psi_{j-1})}{(\psi_{j+1}-\psi_{j-1})} & \text{if} \quad (\psi_{j+1} - \psi_j)(\psi_j - \psi_{j-1}) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

The derivatives are evaluated at the corresponding time step level or the intermediate time step



Schematic overview of the method

uses centered spatial and temporal differences

that makes it 2nd order in space and time

Using two steps:

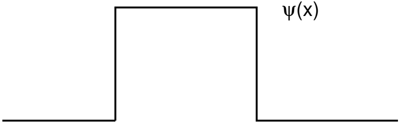
predictor-step (at intermediate time $t^{n+1/2}$)

$$\tilde{\psi}_{j+1/2}^{n+1/2} = \frac{1}{2}\left(\psi_j^n + \psi_{j+1}^n\right) - \frac{\sigma}{2}\left(\psi_{j+1}^n - \psi_j^n\right) \qquad (43)$$

The *corrector-step* (to new time t^{n+1})

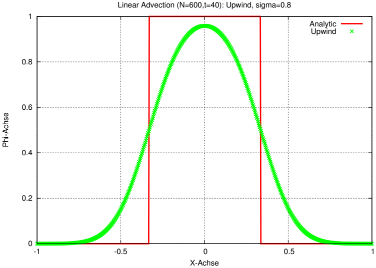
$$\psi_j^{n+1} = \psi_j^n - \sigma\left(\tilde{\psi}_{j+1/2}^{n+1/2} - \tilde{\psi}_{j-1/2}^{n+1/2}\right) \qquad (44)$$

Example: Linar Advection

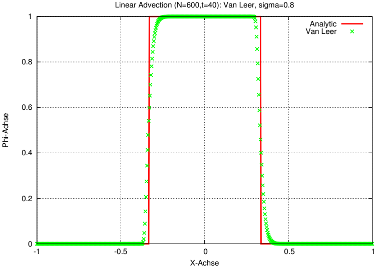


Square function:
Width 0.6 in the intervall $[-1, 1]$
velocity $a = 1$, until $t = 40$
periodic boundaries
$\sigma = a\Delta t/\Delta x = 0.8$ - (Courant no.)

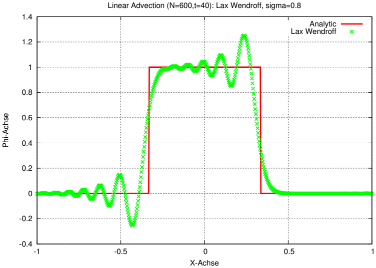
Upwind - (Diffusive)



Van Leer



Lax-Wendroff - (Dispersive)



Substitute for the solution a Fourier series (von Neumann 1940/50s) consider simplifying one component, and analyse its growth properties

$$\psi_j^n = V^n \, e^{i\theta j} \tag{45}$$

here, θ is defined through grid size Δx and the total length L

$$\theta = \frac{2\pi\Delta x}{L} \tag{46}$$

Consider simple Upwind method with $\sigma = a\Delta t/\Delta x$

$$\psi_j^{n+1} - \psi_j^n + \sigma(\psi_j^n - \psi_{j-1}^n) = 0 \tag{47}$$

Substituting eq. (45)

$$V^{n+1} e^{i\theta j} = V^n e^{i\theta j} + \sigma V^n \left[e^{i\theta(j-1)} - e^{i\theta j} \right]$$

dividing by V^n and $e^{i\theta j}$ yields

$$\frac{V^{n+1}}{V^n} = 1 + \sigma \left(e^{-i\theta} - 1 \right) \tag{48}$$

For the square of the absolute value one obtains

$$\begin{aligned}\lambda(\theta) \equiv \left|\frac{V^{n+1}}{V^n}\right|^2 &= \left[1+\sigma\left(e^{-i\theta}-1\right)\right]\left[1+\sigma\left(e^{i\theta}-1\right)\right] \\ &= 1+\sigma\left(e^{-i\theta}+e^{i\theta}-2\right)-\sigma^2\left(e^{-i\theta}+e^{i\theta}-2\right) \\ &= 1+\sigma(1-\sigma)(2\cos\theta-2) \\ &= 1-4\sigma(1-\sigma)\sin^2\left(\frac{\theta}{2}\right) \qquad (49)\end{aligned}$$

The method is now stable, if the magnitude of the *amplification factor* $\lambda(\theta)$ is smaller than unity. The upwind-method is stable for $0 < \sigma < 1$, the $|\lambda(\theta)| < 1$. Rewritten

$$\Delta t < f_{\mathrm{CFL}} \frac{\Delta x}{a} \qquad (50)$$

with the Courant-faktor $f_{\mathrm{CFL}} < 1$. Typically $f_{\mathrm{CFL}} = 0.5$.

Theorem: *Courant-Friedrich-Levy*

There is no *explizit*, consistent and stable finite difference method which is unconditionally stable (i.e. for all Δt).

Modified Equation I

Consider again Upwind method mit $\sigma = a\Delta t/\Delta x$

$$\psi_j^{n+1} - \psi_j^n + \sigma(\psi_j^n - \psi_{j-1}^n) = 0 \tag{51}$$

substitute differences by derivatives, i.e. Taylor-series (up to 2. order)

$$\frac{\partial \psi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial \psi}{\partial \tilde{t}}\Delta t^2 + \mathcal{O}(\Delta t^3) + \sigma\left(\frac{\partial \psi}{\partial x}\Delta x - \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2}\Delta x^2\right) + \mathcal{O}(\Delta t \Delta x^2) = 0 \tag{52}$$

divied by Δt, and substitute for σ

$$\frac{\partial \psi}{\partial t} + a\frac{\partial \psi}{\partial x} + \frac{1}{2}\left(\frac{\partial \psi}{\partial \tilde{t}}\Delta t - a\frac{\partial^2 \psi}{\partial x^2}\Delta x\right) + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2) = 0 \tag{53}$$

Use wave equation $\psi_{tt} = a^2\psi_{xx} \Rightarrow$ modified equation (index M)

$$\frac{\partial \psi_M}{\partial t} + a\frac{\partial \psi_M}{\partial x} = \frac{1}{2}a\Delta x\,(1-\sigma)\,\frac{\partial^2 \psi_M}{\partial x^2} \tag{54}$$

The FDE adds a new diffusive term to the original PDE

Modified Equation II

with the diffusion coefficient

$$D = \frac{1}{2} a \Delta x \,(1 - \sigma) \tag{55}$$

Note: only for $D > 0$ this is a diffusion equation, and it follows $\sigma < 1$ for stability. (**Hirt**-method). For Upwind-Method $D > 0 \;\Rightarrow$ Diffusion.

Lax-Wendroff yields

$$\frac{\partial \psi_M}{\partial t} + a \frac{\partial \psi_M}{\partial x} = \frac{\Delta t^2 a}{\sigma} \left(\sigma^2 - 1\right) \frac{\partial^3 \psi_M}{\partial x^3} \tag{56}$$

The equation has the form

$$\psi_t + a\psi_x = \mu \psi_{xxx} \tag{57}$$

$$\mu = \frac{\Delta t^2 a}{\sigma} \left(\sigma^2 - 1\right) \tag{58}$$

This implies Dispersion. Here: waves are too slow $(\mu < 0)$
$\Rightarrow$ Oscillations behind the diskontinuity (cp. square function)

From the above analysis: the time step Δt has to be limited for a stable numerical evolution.
For the linear Advection (with the velocity a) we find

$$\Delta t < \frac{\Delta x}{a} \tag{59}$$

In the more general case the sound speed has to be included and it follows the Courant-Friedrich-Lewy-condition

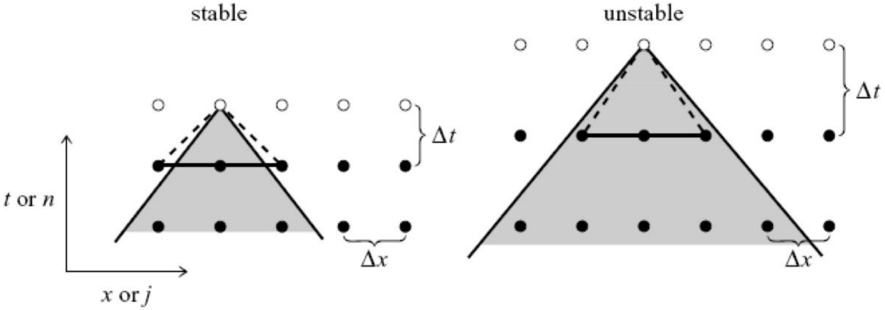
$$\Delta t < \frac{\Delta x}{c_s + |\vec{u}|} \tag{60}$$

physically this means that information cannot travel in one timestep more than one gridcell. Typically one writes

$$\Delta t = f_C \frac{\Delta x}{c_s + |\vec{u}|} \tag{61}$$

with the Courant-Factor f_C. For 2D situations: $f_C \sim 0.5$.
Only for impliciten methods there are (theoretically) no limitations of Δt.

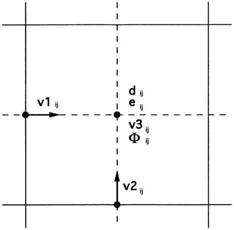
Time step size - graphically



The numerical region of dependence (dashed line) should be larger than the physical one (gray shaded area) since $\Delta x/\Delta t > a$.

The complete information from inside the physical 'sound cone' should be considered.

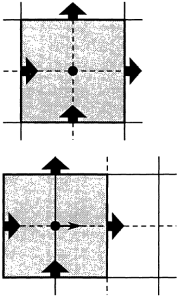
Grid definition (in 2D, staggered):
skalers in cell centers
(hier: $\rho, \epsilon, p, v_3, \psi$)
Vectors ar interfaces
(here: v_1, v_2)



Fluxes across cell boundaries

Top: mass flux

bottom: X-momentum (griud shifted)



from: *ZEUS-2D: A radiation magnetohydrodynamics code for astrophysical flows in two space dimensions. I* in *The Astrophysical Journal Suppl.*, by Jim Stone and Mike Norman, 1992.

Use Operator-Splitting and Directional Splitting: The x and y direction are dealt with subseuqently. First x-scans, then y-scans.

Numerical methods should resemble the conservation properties.
write equations in conservative form
Numerical Methods should resemble the wave properties.
Shock-Capturing methods, Riemann-solver
Numerical Methods should control discontinuites.
need diffusion ($\Rightarrow$ stability)
either explicitly (artificial viscosity) or intrinsically (through method)
Numerical methods should be accurate
min. 2nd order in space and time

Freely available codes:

`ZEUS:` http://www.astro.princeton.edu/~jstone/zeus.html
classical Upwind-Code, 2nd order, staggered grid, RMHD

`ATHENA:` https://trac.princeton.edu/Athena/
successo of ZEUS: Riemann solver, centered grid, MHD

`NIRVANA:` http://www.aip.de/Members/uziegler/nirvana-code
3D, AMR, finite volume code, MHD

`PLUTO:` http://plutocode.ph.unito.it/
3D, relativistic, Riemann-solver/finite volume, MHD

Consider one-dimensional equtions (motion in x-direction):

From Euler equations: With $p = (\gamma - 1)\rho\epsilon$ and separation of derivatives

$$\left.\begin{array}{rcr} \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} & = & 0 \\ \frac{\partial \rho u}{\partial t} + \frac{\partial \rho u u}{\partial x} & = & -\frac{\partial p}{\partial x} \\ \frac{\partial \rho\epsilon}{\partial t} + \frac{\partial \rho\epsilon u}{\partial x} & = & -p\frac{\partial u}{\partial x} \end{array}\right\} \Longrightarrow \begin{array}{rcl} \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} & = & 0 \\ \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + \frac{1}{\rho}\,\frac{\partial p}{\partial x} & = & 0 \\ \frac{\partial p}{\partial t} + u\,\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} & = & 0 \end{array}$$

As Vector equation

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A}\,\frac{\partial \mathbf{W}}{\partial x} = 0 \tag{62}$$

mit

$$\mathbf{W} = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} \qquad \text{und} \qquad \mathbf{A} = \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{pmatrix} \tag{63}$$

Equations are non-linear and coupled.
Try decoupling: $\Rightarrow$ Diagonalisation of **A**

Hydrodynamics: Diagonalisation

Eigenvalues (EV)

$$\det(\mathbf{A}) = \begin{vmatrix} u-\lambda & \rho & 0 \\ 0 & u-\lambda & 1/\rho \\ 0 & \gamma p & u-\lambda \end{vmatrix} = (u-\lambda) \begin{vmatrix} u-\lambda & 1/\rho \\ \gamma p & u-\lambda \end{vmatrix}$$

$$= (u-\lambda)\left[(u-\lambda)^2 - \gamma p/\rho\right] = 0 \qquad (64)$$

It follows

$$\lambda_0 = u$$
$$\lambda_\pm = u \pm c_s \qquad (65)$$

with the sound speed

$$c_s^2 = \frac{\gamma p}{\rho} \qquad (66)$$

The Eigenvalues are the charakteristic velocities,
with which the information is spreading.
It is a combination of fluid velocity (u) and sound speed (c_s)
3 real Eigenvalues $\Rightarrow$ **A** diagonalisable

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \Lambda \qquad (67)$$

Q is built up from the Eigenvalues to the EV (λ_i, $i = 0, +, -$), Λ is a diagonal matrix.

For **Q** it follows

$$\mathbf{Q} = \begin{pmatrix} 1 & \frac{1}{2}\frac{\rho}{c_s} & -\frac{1}{2}\frac{\rho}{c_s} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2}\rho c_s & -\frac{1}{2}\rho c_s \end{pmatrix} \qquad \text{und} \qquad \mathbf{Q}^{-1} = \begin{pmatrix} 1 & 0 & -\frac{1}{c_s^2} \\ 0 & 1 & \frac{1}{\rho c_s} \\ 0 & 1 & -\frac{1}{\rho c_s} \end{pmatrix}$$

We had

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{W}}{\partial x} = 0 \qquad (68)$$

and

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \Lambda$$

Define:

$$d\mathbf{v} \equiv \mathbf{Q}^{-1} d\mathbf{W} \qquad \text{also} \qquad d\mathbf{W} = \mathbf{Q}\, d\mathbf{v} \qquad (69)$$

Multiply eq. (68) with $\mathbf{Q}^{-1}$

$$\frac{\partial \mathbf{v}}{\partial t} + \Lambda \frac{\partial \mathbf{v}}{\partial x} = 0 \qquad (70)$$

$v = (v_0, v_+, v_-)$ are the charakteristic Variables: $v_i = const.$ in curves

$$\frac{dx}{dt} = \lambda_i$$

from the definitions

$$dv_0 = d\rho - \frac{1}{c_s^2}\, dp \tag{71}$$

$$\frac{\partial v_0}{\partial t} + \lambda_0 \frac{\partial v_0}{\partial x} = 0 \qquad \text{mit} \qquad \lambda_0 = u \tag{72}$$

What is dv_0 ?
From thermodynamics (1. Law) for specific quantities)

$$Tds = d\epsilon + p\, d\left(\frac{1}{\rho}\right) = d\epsilon - \frac{p}{\rho^2}\, d\left(\frac{1}{\rho}\right) \tag{73}$$

with $p = (\gamma - 1)\rho\epsilon$, $\epsilon = c_v T$, $\gamma = c_p/c_v$ it follows

$$ds = -\frac{c_p}{\rho}\left[d\rho - \frac{dp}{c_s^2}\right] = -\frac{c_p}{\rho}\, dv_0 \tag{74}$$

$$\Longrightarrow \quad \frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} = 0 \tag{75}$$

i.e. *s is const. along stream lines*, hence

$$\frac{ds}{dt} = 0 \tag{76}$$

For the additional variables

$$\frac{\partial v_\pm}{\partial t} + (u \pm c_s)\,\frac{\partial v_\pm}{\partial x} = 0 \tag{77}$$

with

$$dv_\pm = du \pm \frac{1}{\rho c_s} dp \tag{78}$$

it follows

$$v_\pm = u \pm \int \frac{dp}{\rho c_s}\,. \tag{79}$$

Let the entropy constant everywhere (i.e. $p = K\rho^\gamma$)

$$\Longrightarrow \quad v_\pm = u \pm \frac{2c_s}{\gamma - 1} \tag{80}$$

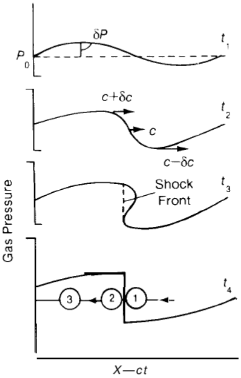
Riemann-Invariants: $v_\pm = const.$ on curves

$$\frac{dx}{dt} = u \pm c_s$$

Linearisation of the Euler-equation resukts in the wave equation for the perturbations:

$$\frac{\partial \rho_1}{\partial \tilde{t}} = c_s^2 \frac{\partial^2 \rho_1}{\partial x^2} \qquad (81)$$

but: c_s is not constant $\Rightarrow$ steepening

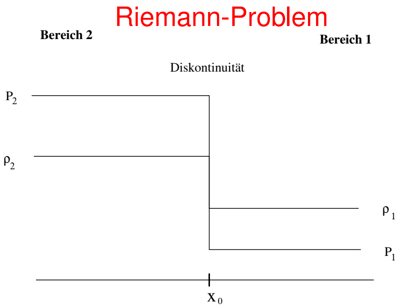


Example for (receeding) shock wave

$\Rightarrow$ Diskontinuities

Examples: Shocktube

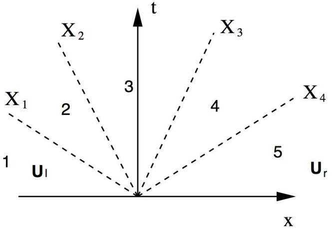
Initial discontinuity in a tube at position x_0 (one-dimensional)

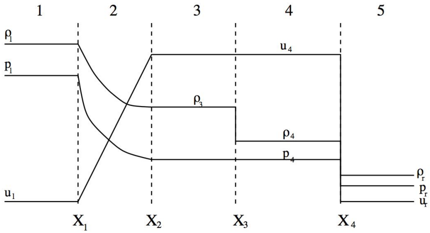


Jump in pressure (p) and density (ρ)

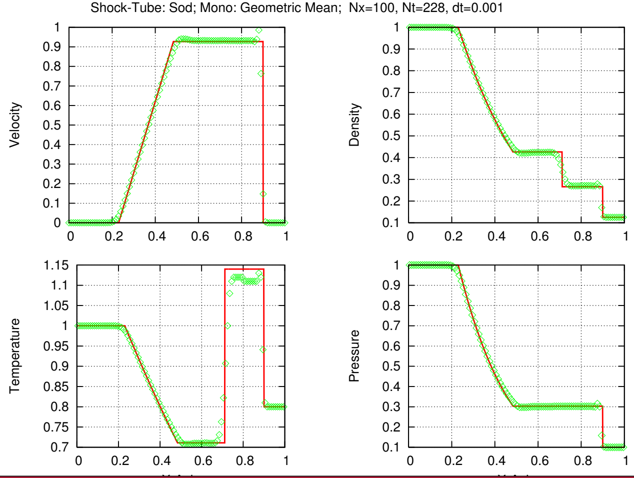
Evolution:

- a shock wave to the right (X_4)
  (supersonically $u_{sh} > c_s$)
- a contact discontinuity
  density jump (along X_3)
- a rarefaction wave
  (between X_1 and X_2)





A standard test problem for numerical hydrodynamics, $x \in [0, 1]$ with
$X_0 = 0.5$, $\gamma = 1.4$
$\rho_1 = 1.0, p_1 = 1.0, \epsilon_1 = 2.5,\ T_1 = 1$ and $\rho_2 = 0.1, p_2 = 0.125, \epsilon_2 = 2.0,\ T_2 = 0.8$
Hier solution with van Leer method (at time $t = 0.228$ after 228 time steps:)



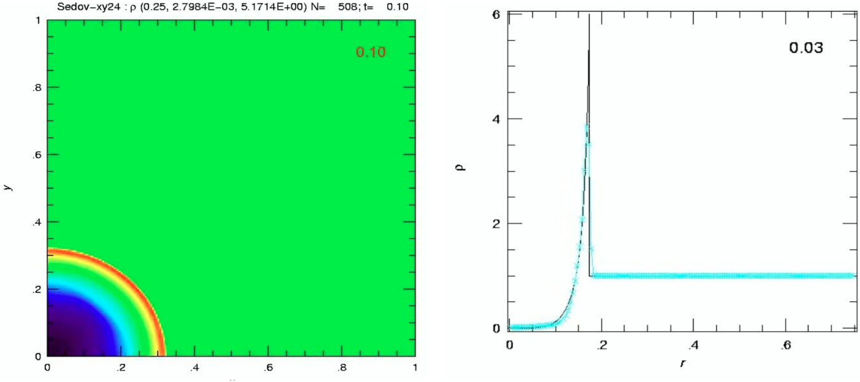
Red: Exact
Green: Numerics

The solution is: self similar obtained through stretching

Sedov-Explosion

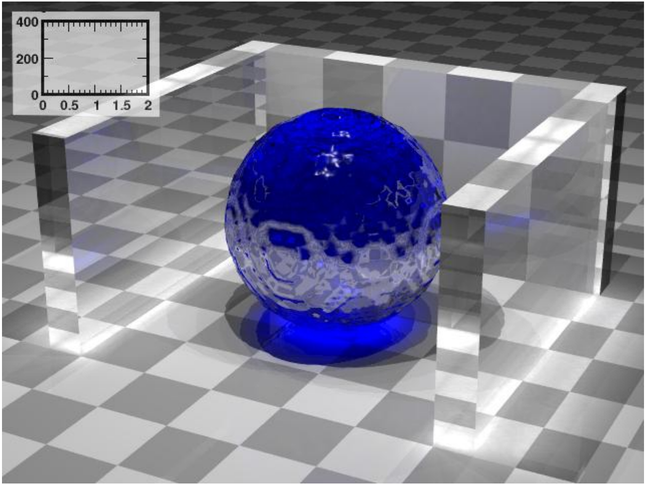
An example for bomb explosions (Sedov & Taylor, 1950s), analytical solution (Sedov)
Basic setup for Supernovae-outbusts, e.g. estimate of the remnant size

Standard test problem of multi-dimensional hydrodynamics, e.g. for
$x, y \in [0, 1] \times [0, 1]$
Energy-Input at origin, $E = 1$, in $\rho = 1$, $\gamma = 1.4$, 200×200 grid points
Here: solution with van Leer method (solve for total energy variable). Plotted: density



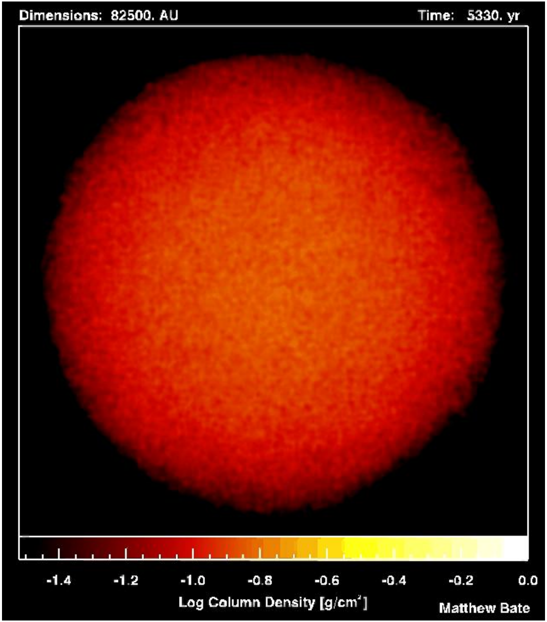
Water droplet: SPH

Water sphere (R=30cm), Basin (1x1 m, 60cm high) Incl. surface tension, time in seconds (TU-München, 2002)



(Website)

Ster formation: SPH



Molecular Cloud

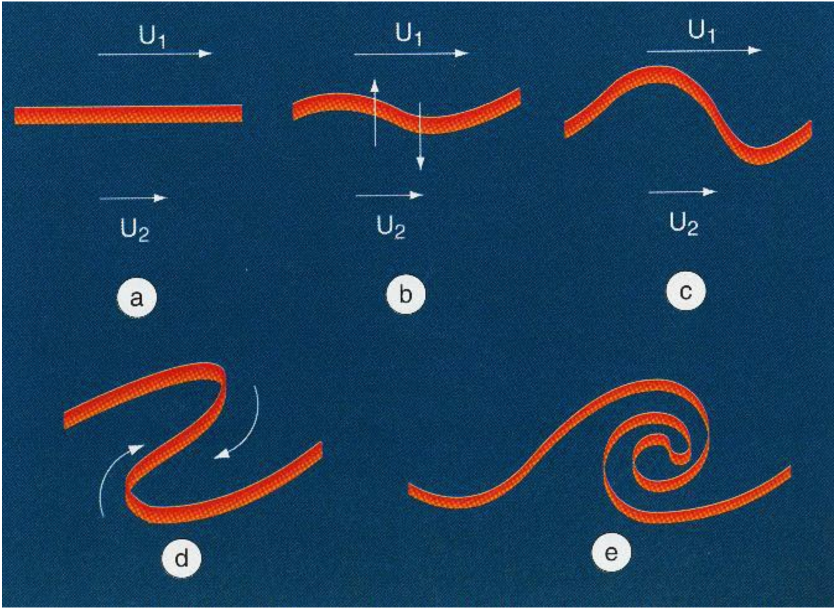
Mass:
50 $M_\odot$

Diameter:
1.2 LJ = 76,000 AU

Temperature:
10 K

(M. Bate, 2002)

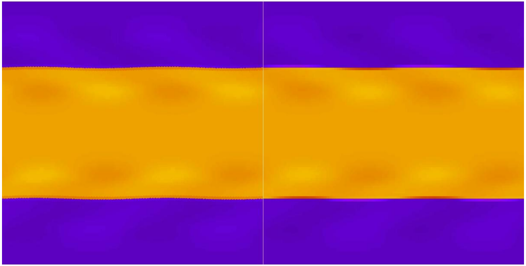
Examples: Kelvin-Helmholtz Instability I

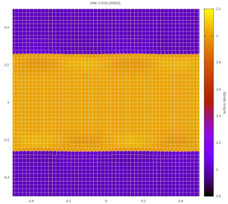


Direct comparision: moving $< - >$ fixed grid

Left: Moving grid (Voronoi-Tesselation)
Right: fixed square grid (Euler)

with grid motion displayed





(Kevin Schaal, Tübingen)

Youtube channel

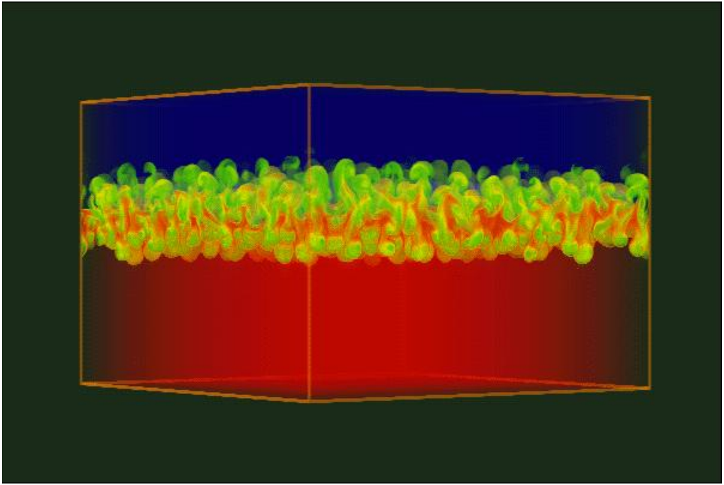
Kelvin-Helmholtz Instability III

(Boulder (NCAR), USA)

Rayleigh-Taylor Instability

PPM Code, 128 Nodes, ASCI Blue-Pacific ID System at LLNL

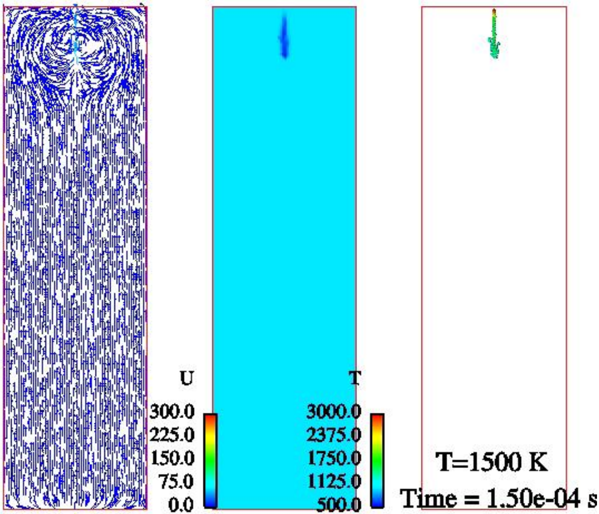
512^3 Grid Cells (LLNL, 1999)



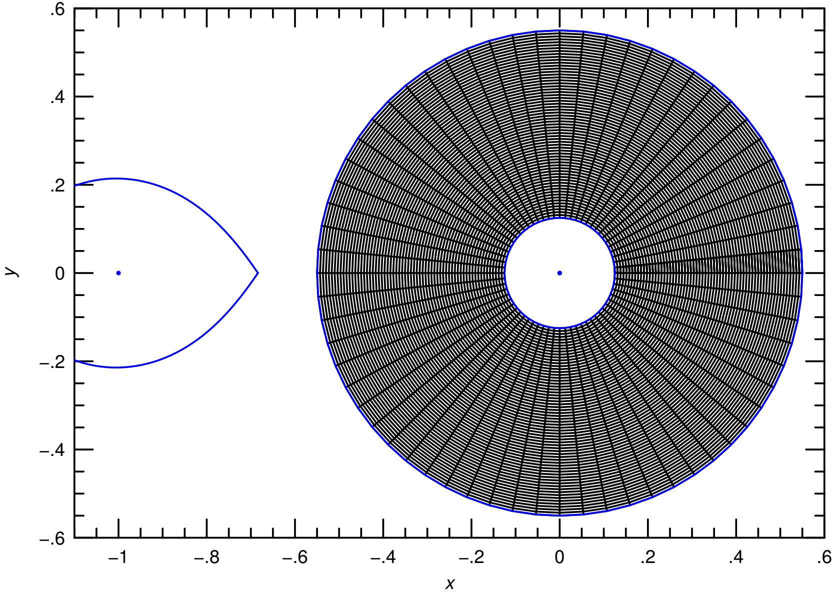
(Web-Link)

Finite Volumen Method (FOAM)

Velocity, Temperature, Particles (+Isosurfaces) (Nabla Ltd, 2004)



Examples: Cataclysmic Variable: Grid



Kataclysmic Variable: Disk formation

RH2D Code, Van Leer slope

512^2 Gridpoints, mass ratio: $q = m_2/m_1 = 0.15$