Post-Newtonian oscillations of a rotating disk of dust

Wilhelm Kley and Gerhard Schäfer
Max-Planck-Gesellschaft, AG-Gravitationstheorie, Friedrich-Schiller-Universität, D-07743 Jena, Germany
(Received 19 July 1994)

At the first post-Newtonian approximation to general relativity, analytic solutions are presented for the motions of generalized MacLaurin disks of dust. The main result consists in the solution for a rotating and oscillating disk. This disk has the remarkable property that, in contrast with its Newtonian analogue, it is rotating nonuniformly. Even in the stationary limit of a vanishing oscillation amplitude, the rotation law does not become rigid. On the other hand, by imposing initially uniform rotation and nonoscillatory motion, the motion is found in agreement with earlier results by Bardeen and Wagoner. The solution of the collapsing generalized MacLaurin disk without rotation is presented also.

PACS number(s): 04.25.Nx, 04.40.−b

I. INTRODUCTION

Thin disks consisting of an ensemble of noninteracting dust particles are, apart from spherical symmetric cases, among the simplest axisymmetric systems that incorporate the essential new features of general relativity beyond Newtonian physics. Evolving disks emit gravitational radiation and the formation of black holes is possible under certain conditions. Consequently, general relativistic disks of dust have recently enjoyed a renewed interest and are presently an active topic of research. Bardeen and Wagoner [1] were the first to begin with the study of relativistic stationary rotating disks. They obtained a solution for rigid rotation in the form of a series expansion. However, terms beyond the second post-Newtonian approximation are only given numerically. Recently, Neugebauer and Meinel [2], using an inverse method, were able to obtain the exact solution for the rigidly rotating stationary disk of dust in full general relativity.

The aforementioned solutions only treat the rigidly rotating stationary case. However, evolving disks might be important for a better understanding of black-hole formation and accretion processes. As they are sources of gravitational radiation, nonstationary disks are also of interest for future gravitational-wave astronomy. The general-relativistic problem of the internal motion of extended, nonspherically symmetric bodies is intrinsically very difficult and analytic treatments are known only for stationary situations. Before referring to extensive numerical calculations in general relativity it is very instructive to investigate the solution space in the vicinity of the Newtonian solutions. These so-called post-Newtonian approximations have been widely employed in the study of binary stars. After the discovery of the Hulse-Taylor binary pulsar PSR1913+16 in 1974, much effort has been devoted to understanding the relativistic motion of compact binary systems. Presently, the motion of binary star systems with pointlike components is known analytically up to the second post-Newtonian approximation to general relativity [3].

Here we treat the oscillations of an infinitely thin disk in the first post-Newtonian approximation. We consider so-called MacLaurin disks, which are obtained as a limiting case of the well-known MacLaurin spheroids. The solution extends the known self-similar oscillating Newtonian case (see Hunter [4], and more recently [5]) into the weakly relativistic regime. At this level of approximation the problem can be solved fully analytically, and the solution resembles closely the one obtained for a binary system as given by Damour and Deruelle [6]. A word of caution is appropriate here. It is well known that Newtonian pressureless, cold dusty disks are dynamically unstable to ring formation (e.g., [7]), a result which most likely carries over to general relativistic disks [8]. As in the Newtonian case we assume a solution exists, and by using an appropriate ansatz, we transform the problem into a set of ordinary differential equations. The introduction of a surface pressure, i.e., heating the disk, will stabilize it. The analytical zero pressure solution presented may be considered as modeling such a situation, and can serve, for example, as a test case for fully relativistic calculations of thin disks. For the case of zero net angular momentum, Abrahams *et al.* [8] have followed the evolution of (counter)rotating thin disks numerically in full general relativity using a particle method.

In Sec. II we review briefly the Newtonian motion, oscillation, and collapse of the disk. In Sec. III we present the post-Newtonian solution, and analyze in Sec. IV the oscillating rotating disk. In Sec. V, the special cases of the nonrotating collapsing disk and the rigidly rotating stationary case are discussed. Finally, in Sec. VI we present our conclusions.

II. NEWTONIAN MOTION

The Newtonian motion of MacLaurin disks has been treated for example by Hunter [4] and recently by Shapiro and Teukolsky [5]. In order to elucidate the post-Newtonian extension, we review first the nonrelativistic Newtonian treatment. In Eulerian form the axisymmetric pressureless hydrodynamic equations, in cylindrical coordinates (r, φ, z), for a motion confined to the plane $z = 0$ are

$$\frac{\partial \Sigma}{\partial t} + v_r \frac{\partial \Sigma}{\partial r} + \Sigma \frac{1}{r} \frac{\partial (r v_r)}{\partial r} = 0, \tag{1}$$

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} - \frac{\partial U}{\partial r} - \frac{v_\varphi^2}{r} = 0, \tag{2}$$

and

$$\frac{\partial v_\varphi}{\partial t} + v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_r v_\varphi}{r} = 0, \tag{3}$$

where Σ is the surface density, v_r, v_φ are the radial and azimuthal velocity, and U is the gravitational potential. To solve these equations for a disk, one seeks a self-similar solution and makes the following ansatz for Σ, v_r, and v_φ:

$$\Sigma(r,t) = \sigma(t) \sqrt{1 - \left(\frac{r}{r_d(t)}\right)^2}, \tag{4}$$

$$v_r(r,t) = -r\, f(t), \tag{5}$$

$$v_\varphi(r,t) = r\, \Omega(t). \tag{6}$$

The surface density has the profile of a MacLaurin disk with $\sigma(t)$ being the central surface density and $r_d(t)$ being the maximum disk radius; both are functions of time. The disk is rigidly rotating at all times with the angular velocity $\Omega(t)$. For this density distribution the potential in all space is calculated from Poisson's equation

$$\Delta U = -4\pi G \Sigma(r) \delta(z),$$

and within the disk $(z = 0, r < r_d)$ it has the form

$$U(r,t) = G\sigma(t) r_d(t)\, \frac{\pi^2}{4} \left[2 - \left(\frac{r}{r_d(t)}\right)^2\right]. \tag{7}$$

Substituting the ansatz (4)–(6) and (7) into the evolution equations (1)–(3), and collecting terms of equal powers r/r_d yields four ordinary differential equations for the unknowns σ, r_d, f, and Ω:

$$\frac{\dot{\sigma}}{\sigma} - 2f = 0, \tag{8}$$

$$\frac{\dot{r}_d}{r_d} + f = 0, \tag{9}$$

$$-\dot{f} + f^2 + G\frac{\pi^2}{2}\frac{\sigma}{r_d} - \Omega^2 = 0, \tag{10}$$

$$\frac{\dot{\Omega}}{\Omega} - 2f = 0, \tag{11}$$

where the overdot represents differentiation with respect to time. These evolution equations are integrated applying the initial conditions $f(0) = 0$ and $r_d(0) = R_d$. The first two equations can be combined and solved to yield

$$M_N = \frac{2\pi}{3}\, \sigma(t) r_d^2(t), \tag{12}$$

where the constant of integration, M_N, is the total conserved (Newtonian) mass of the disk. With Ω_0 and R_d denoting the initial values of $\Omega(t)$ and $r_d(t)$, respectively, Eq. (11) yields

$$\Omega(t) r_d^2(t) = \Omega_0 R_d^2, \tag{13}$$

stating the conservation of angular momentum. By introducing the dimensionless radius X through

$$r_d(t) = R_d\, X(t), \tag{14}$$

we obtain from Eq. (10) a second-order differential equation for X:

$$\ddot{X} + \frac{2C}{R_d^3}\left(\frac{1}{X^2} - \frac{\xi_N^2}{X^3}\right) = 0, \tag{15}$$

where ξ_N is defined through

$$\Omega_0^2 = \frac{2C\xi_N^2}{R_d^3}, \tag{16}$$

and the constant C is given by

$$C = \frac{3\pi G M_N}{8} = \frac{\pi^2 G \sigma_0 R_d^2}{4}, \tag{17}$$

where σ_0 denotes the initial central surface density $\sigma(0)$. Equation (15) resembles closely the equation for the radial motion of a binary system with total angular momentum $l = \sqrt{2CR_d}\xi_N$. Hence ξ_N is the fraction of the specific equilibrium angular momentum of the disk (see, e.g., [5]). Since the disk motion is self-similar, we can write for all radii $r < r_d$ an identical equation of motion (14). The solution of (15) with the initial conditions $X = 1$ and $\dot{X} = 0$ at $t = 0$ can be written in parametrized form as

$$\frac{2\pi}{P} t = u - e \sin u - \pi,$$
$$X = \alpha\,(1 - e\cos u), \tag{18}$$

where, in the equivalent two-body problem, α is the relative semimajor axis, e the eccentricity, and P the period. Here the constants are given by (see, e.g., [4,5])

$$\alpha = \frac{1}{2 - \xi_N^2},$$
$$e = 1 - \xi_N^2, \tag{19}$$
$$P = 2\pi\left(\frac{\alpha^3 R_d^3}{2C}\right)^{1/2}.$$

For the time dependences of radial and rotational velocity one obtains

$$f^2(t) = \frac{4C}{R_d^3}\frac{1 - X}{X^3}\left[1 - \frac{\xi_N^2}{2X}(1 + X)\right], \tag{20}$$

$$\Omega(t) = \frac{\xi_N\sqrt{2CR_d}}{R_d^2\, X^2(t)}. \tag{21}$$

The limiting case $\xi_N = 1$ leads to a stationary nonoscillating equilibrium disk, where centrifugal forces balance gravity exactly. On the other hand, $\xi_N = 0$ describes a case with no rotation, i.e., zero angular momentum. There the whole disk collapses homologeously, and all parts reach the center simultaneously after the finite time

$$t_c = \frac{1}{2}\left(\frac{R_d}{G\sigma_0}\right)^{1/2}. \tag{22}$$

The scaled radius $X(t)$ is then obtained from

$$X(t) = \cos^2\beta(t), \quad \text{where} \quad \beta(t) + \frac{1}{2}\sin[2\beta(t)] = \frac{\pi}{2}\frac{t}{t_c}. \tag{23}$$

In this parametrization the solution of the disk collapse resembles very closely that of a collapsing homogeneous sphere considered by Hunter [4]. The parameter β is related to u through

$$\beta = \frac{1}{2}(u - \pi).$$

For intermediate cases $0 < \xi_N < 1$ the disk oscillates according to (18), rotating always rigidly.

III. POST-NEWTONIAN MOTION

To study the evolution of the disk in the post-Newtonian approximation we start out from a formulation of the hydrodynamic equations by Chandrasekhar [9] which read for the pressureless case

$$\frac{dv_\alpha}{dt} - \frac{\partial U}{\partial x_\alpha} = \frac{1}{c^2}\left[-(v^2+4U)\frac{dv_\alpha}{dt} + 2v^2\frac{\partial U}{\partial x_\alpha} - 4v_\beta\frac{\partial U_\beta}{\partial x_\alpha} - \frac{1}{2}\frac{\partial}{\partial x_\alpha}\left(\frac{\partial^2\chi}{\partial t^2}\right)\right.$$
$$\left. + 2\frac{\partial\Phi}{\partial x_\alpha} + 4\frac{dU_\alpha}{dt} - v_\alpha\frac{d}{dt}\left(\frac{1}{2}v^2 + 3U\right)\right], \tag{24}$$

$$\frac{d\rho}{dt} + \rho\frac{\partial v_\alpha}{\partial x_\alpha} = -\frac{\rho}{c^2}\frac{d}{dt}\left(\frac{1}{2}v^2 + 3U\right). \tag{25}$$

To study the above one-dimensional disk case we define the two-dimensional surface density Σ through

$$\rho = \frac{\Sigma\delta(z)}{\sqrt{-g_{zz}}}. \tag{26}$$

Note that the space-time signature in Ref. [9] is -2. Then we obtain, for the continuity equation and the equations of motion,

$$\frac{\partial\Sigma}{\partial t} + v_r\frac{\partial\Sigma}{\partial r} + \frac{\Sigma}{r}\frac{\partial r v_r}{\partial r} = -\frac{\Sigma}{c^2}\frac{d}{dt}\left(\frac{1}{2}v^2 + 2U\right), \tag{27}$$

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} - \frac{\partial U}{\partial r} - \frac{v_\varphi^2}{r} = \frac{1}{c^2}\left\{4\frac{\partial U_r}{\partial t} + 2\frac{\partial\Phi}{\partial r} - \frac{1}{2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial\chi}{\partial r}\right) - 3v_r\frac{dU}{dt} + \frac{\partial U}{\partial r}(v_\varphi^2 - 4U) - 4\frac{v_\varphi}{r}\frac{\partial(rU_\varphi)}{\partial r}\right\}, \tag{28}$$

and

$$\frac{\partial v_\varphi}{\partial t} + v_r\frac{\partial v_\varphi}{\partial r} + \frac{v_r v_\varphi}{r} = \frac{1}{c^2}\left[\frac{4}{r}\frac{d(rU_\varphi)}{dt} - v_\varphi\frac{d}{dt}\left(\frac{1}{2}v^2 + 3U\right)\right]. \tag{29}$$

The left hand sides of these equations have the standard Newtonian form and the right hand sides are proportional to $1/c^2$ and represent the first post-Newtonian extension. The additional "potentials" χ, U_r, and Φ are defined as solutions of the equations

$$\Delta\chi = -2U, \quad \Delta U_i = -4\pi G\Sigma\delta(z)v_i, \quad (i = r,\varphi), \quad \Delta\Phi = -4\pi G\Sigma\delta(z)\phi, \tag{30}$$

where

$$\phi = v^2 + U. \tag{31}$$

To solve Eqs. (27)–(29) we can substitute the known Newtonian solution of the previous section into the correction terms of the right hand side and look for solutions of the now post-Newtonian density Σ and velocities v_r and v_φ. For later convenience let us define the distance in units of the maximum disk radius

$$a(r,t) = \frac{r}{r_d(t)}. \tag{32}$$

In the Newtonian case the total (Lagrangian) time derivative of a vanishes ($\dot{a} = 0$), a consequence of the self-similarity of the collapse. In the post-Newtonian case this is not necessarily true. Substituting the Newtonian solution (to be noted with a superscript N where necessary) into the right hand side of the continuity equation (27) we find

$$\frac{d}{dt}\left(\frac{1}{2}v^2 + 2U\right) = v_r\frac{\partial U}{\partial r} + 2\dot{U} = 4C\frac{f}{r_d} \tag{33}$$

and the continuity equation becomes

$$\frac{\partial \Sigma}{\partial t} + v_r \frac{\partial \Sigma}{\partial r} + \frac{\Sigma}{r}\frac{\partial (r v_r)}{\partial r} = -4\,\frac{C}{c^2}\,\frac{\Sigma f^N}{r_d^N}. \tag{34}$$

To evaluate the equations of motion we shall first calculate the additional potentials by using Poisson integrals over the area $\mathcal{D}$ of the disk. Let us start with

$$\Phi = G \int_{\mathcal{D}} \frac{\phi(r',\varphi')\,\Sigma(r',\varphi')\,r'dr'd\varphi'}{|\mathbf{r}-\mathbf{r}'|}. \tag{35}$$

Using the definition for ϕ (31) and a (32) we obtain

$$\Phi = 2G\sigma C\, I_1 + G\sigma\left[(f^2+\Omega^2)r_d^3 - C\right] I_2, \tag{36}$$

where I_1 and I_2 are the integrals

$$I_1(a) = \int_0^{2\pi}\int_0^1 \frac{\sqrt{1-x'^2}\,x'dx'd\varphi'}{(a^2-2ax'\cos\varphi'+x'^2)^{1/2}}, \tag{37}$$

$$I_2(a) = \int_0^{2\pi}\int_0^1 \frac{\sqrt{1-x'^2}\,x'^2\,x'dx'd\varphi'}{(a^2-2ax'\cos\varphi'+x'^2)^{1/2}}. \tag{38}$$

These integrals are easily solved by first performing a coordinate transformation $(x',\varphi') \mapsto (x,\varphi)$ that is equivalent to shifting the origin of the coordinate system to the point a where the potential is to be evaluated. Specifically the transformation reads

$$x'^2 = x^2 - 2ax\cos\varphi + a^2,$$
$$x^2 = x'^2 - 2ax'\cos\varphi' + a^2.$$

This coordinate transformation leaves the area element formally unchanged and we obtain

$$I_1(a) = \int_0^{2\pi}\int_0^{x_{\max}} \sqrt{Y}\,dxd\varphi,$$
$$I_2(a) = \int_0^{2\pi}\int_0^{x_{\max}} \sqrt{Y}\,(x^2-2ax\cos\varphi+a^2)\,dxd\varphi,$$

where we used

$$Y = 1 - a^2 + 2ax\cos\varphi - x^2, \tag{39}$$

and where the integration limit $x_{\max}$ is

$$x_{\max}(\varphi) = a\cos\varphi + \sqrt{1-a^2\sin^2\varphi}\;. \tag{40}$$

The occurring integrals of the type I_1 and I_2 can be integrated analytically consisting of the parts

$$\int_0^{2\pi}\int_0^{x_{\max}} \sqrt{Y}\,dxd\varphi = \frac{\pi^2}{4}\,(2-a^2)\,, \tag{41}$$

$$\int_0^{2\pi}\int_0^{x_{\max}} \sqrt{Y}\,x\cos\varphi\,dxd\varphi = \frac{\pi^2}{4}\,a\left(1-\frac{1}{4}a^2\right)\,, \tag{42}$$

$$\int_0^{2\pi}\int_0^{x_{\max}} \sqrt{Y}\,x^2\,dxd\varphi = \frac{\pi^2}{8}\left(1+a^2-\frac{1}{8}a^4\right)\,, \tag{43}$$

$$\int_0^{2\pi}\int_0^{x_{\max}} \sqrt{Y}\,x\sin\varphi\,dxd\varphi = 0\,. \tag{44}$$

For the potential Φ we obtain finally

$$\Phi = \frac{C^2}{r_d^2}\left(\frac{7}{2}-\frac{5}{2}a^2+\frac{9}{16}a^4\right) + Cr_d(f^2+\Omega^2)\left(\frac{1}{2}+\frac{1}{2}a^2-\frac{9}{16}a^4\right). \tag{45}$$

The potential χ turns out to be

$$\chi = -G\int_{\mathcal{D}} \Sigma(r',\varphi')\,|\mathbf{r}-\mathbf{r}'|\,r'dr'd\varphi'$$
$$= -G\sigma r_d^3 \int_0^{2\pi}\int_0^{x_{\max}} \sqrt{Y}\,x^2\,dxd\varphi = -\frac{1}{2}Cr_d\left(1+a^2-\frac{1}{8}a^4\right). \tag{46}$$

With

$$v_r = -v_r'\cos\varphi' + v_\varphi'\sin\varphi' = (x\cos\varphi - a)f(t) + \Omega(t)x\sin\varphi\,, \tag{47}$$

$$v_\varphi = -v_r'\sin\varphi' - v_\varphi'\cos\varphi' = f(t)x\sin\varphi + (a - x\cos\varphi)\Omega(t)\,, \tag{48}$$

we find, for the velocity potentials,

$$U_r = G\int_{\mathcal{D}} \frac{\Sigma(r',\varphi')\,v_r(r',\varphi')\,r'dr'd\varphi'}{|\mathbf{r}-\mathbf{r}'|}$$
$$= G\sigma r_d^2 a \int_0^{2\pi}\int_0^{x_{\max}} \sqrt{Y}\;\left[(x\cos\varphi - a)f(t) + \Omega(t)x\sin\varphi\right]\,dxd\varphi$$
$$= -Cf(t)\,a\left(1-\frac{3}{4}a^2\right) \tag{49}$$

and

$$
\begin{aligned}
U_\varphi &= G\int_{\mathcal{D}} \frac{\Sigma(r',\varphi')\,v_\varphi(r',\varphi')\,r'dr'd\varphi'}{|\mathbf{r}-\mathbf{r}'|} \\
&= G\sigma r_d^2 a \int_0^{2\pi}\int_0^{x_{\max}} \sqrt{Y}\,\left[f(t)x\sin\varphi + (a - x\cos\varphi)\Omega(t)\right]\,dx d\varphi \\
&= C\Omega(t)\,a\left(1-\frac{3}{4}a^2\right).
\end{aligned} \tag{50}
$$

Using now

$$
\frac{\partial U_r}{\partial t} = -Ca\left[\left(1-\frac{3}{4}a^2\right)\dot{f} + \left(1-\frac{9}{4}a^2\right)f^2\right] = \frac{\partial}{\partial r}\left(\frac{\partial^2\chi}{\partial t^2}\right), \tag{51}
$$

$$
\frac{\partial\Phi}{\partial r} = \frac{C^2 a}{r_d^3}\left[-5+\frac{9}{4}a^2 + \frac{2\xi_N^2 R_d}{r_d}\left(1-\frac{9}{4}a^2\right)\right] + Cf^2 a\left(1-\frac{9}{4}a^2\right), \tag{52}
$$

$$
\frac{v_\varphi}{r}\frac{\partial r U_\varphi}{\partial r} = \frac{4C^2\xi_N^2 R_d a}{r_d^4}\left(1-\frac{3}{2}a^2\right), \tag{53}
$$

and Eq. (10) we find, for the momentum equations,

$$
\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} - \frac{\partial U}{\partial r} - \frac{v_\varphi^2}{r} = \frac{Ca}{c^2}\left[\left(-1+\frac{7}{4}a^2\right)\frac{C}{r_d^3} + (1+3a^2)f^2 + \left(-\frac{5}{2}+\frac{23}{8}a^2\right)\Omega^2\right], \tag{54}
$$

$$
\frac{\partial v_\varphi}{\partial t} + v_r\frac{\partial v_\varphi}{\partial r} + \frac{v_r v_\varphi}{r} = -\frac{2Ca}{c^2}f\Omega(1+a^2). \tag{55}
$$

The post-Newtonian corrections to the evolution equation show a polynomial behavior in terms of a. Thus, to solve the system (34), (54), and (55) we make the ansatz

$$
\Sigma(r,t) = \sigma(t)\sqrt{1-a^2}\left(1+\frac{1}{2}\kappa(t)\,a^2\right), \tag{56}
$$

$$
v_r(r,t) = -r\left[f(t) + \lambda(t)\,a^2\right], \tag{57}
$$

$$
v_\varphi(r,t) = r\left[\Omega(t) + \omega(t)\,a^2\right]. \tag{58}
$$

These are the Newtonian expressions plus correction terms $\kappa(t)$, $\lambda(t)$, and $\omega(t)$ which are $O(c^{-2})$ small. Note that a is time as well as radius dependent. Using the density ansatz (56), we find, for the (post-Newtonian) gravitational potential,

$$
\begin{aligned}
U &= G\sigma r_d \int_{\mathcal{D}} \frac{\sqrt{1-x'^2}\left(1+\frac{1}{2}\kappa x'^2\right)\left(1-\frac{U_N}{c^2}\right)x'dx'd\varphi'}{(a^2 - 2ax'\cos\varphi' + x'^2)^{1/2}} \\
&= G\sigma r_d \int_0^{2\pi}\int_0^{x_{\max}} \sqrt{Y}\left[1-\frac{2C}{c^2 r_d} + \frac{1}{2}\left(\kappa + \frac{2C}{c^2 r_d}\right)(x^2 - 2ax\cos\varphi + a^2)\right]dx d\varphi \\
&= G\sigma r_d \frac{\pi^2}{4}\left[\left(1-\frac{2C}{c^2 r_d}\right)(2-a^2) + \frac{1}{4}\left(\kappa + \frac{2C}{c^2 r_d}\right)\left(1+a^2-\frac{9}{8}a^4\right)\right].
\end{aligned} \tag{59}
$$

Substituting all expansions into the equations of motion (34), (54), and (55) and comparing equal powers of a, we obtain finally

$$
\frac{\dot\sigma}{\sigma} - 2f = -4\frac{C}{c^2 R_d}\frac{d}{dt}\left(\frac{1}{X}\right), \tag{60}
$$

$$
\frac{\dot X}{X} + f + \lambda = 0, \tag{61}
$$

$$
\dot\kappa - 10\lambda = 0, \tag{62}
$$

$$
-\dot f + f^2 + \frac{\pi^2}{4}\frac{G\sigma}{R_d X}\left(2-\frac{1}{2}\kappa\right) - \Omega^2 = \frac{2C^2}{c^2 R_d^4 X^4}\left[4 - 2X - \frac{\xi_N^2}{X}\left(\frac{7}{2} - X^2\right)\right], \tag{63}
$$

$$
-\dot\lambda + 2\lambda f - 2\omega\Omega + \frac{9C}{8R_d^3 X^3}\kappa = \frac{2C^2}{c^2 R_d^4 X^4}\left[\frac{23}{4} - 6X - \frac{\xi_N^2}{X}\left(\frac{1}{8} - 3X^2\right)\right], \tag{64}
$$

$$\dot{\Omega} - 2f\Omega = -\frac{2C\Omega f}{c^2 R_d X}, \tag{65}$$

$$\dot{\omega} - 2f\omega - 2\lambda\Omega = -\frac{2C\Omega f}{c^2 R_d X}, \tag{66}$$

where the first three equations result from the continuity equation (34), and the latter four from the momentum equations (54) and (55). The first three equations can easily be combined and integrated. Noting that $X(0) = 1$, and expanding the occurring exponential we obtain, for the central surface density,

$$\sigma(t) = \frac{\sigma_0}{X^2(t)}\left\{1 - \frac{4C}{c^2 R_d}\left[\frac{1}{X(t)} - 1\right] - \frac{1}{5}\left[\kappa(t) - \kappa_0\right]\right\}, \tag{67}$$

where σ_0 and κ_0 denote the initial values of $\sigma(t)$ and $\kappa(t)$ at $t = 0$, respectively. The Newtonian expression (12) is corrected by the small terms in the curly brackets on the right hand side. Defining

$$\tilde{\Omega} = \Omega + \omega \quad \text{and} \quad \tilde{f} = f + \lambda = -\frac{\dot{X}}{X}, \tag{68}$$

we find upon combining Eqs. (65) and (66),

$$\dot{\tilde{\Omega}} - 2\tilde{f}\tilde{\Omega} = -\frac{4C\Omega f}{c^2 R_d X}. \tag{69}$$

Integration then yields

$$\tilde{\Omega}(t) = \frac{\tilde{\Omega}_0}{X^2}\left[1 - \frac{4C}{c^2 R_d}\left(\frac{1}{X} - 1\right)\right], \tag{70}$$

with $\tilde{\Omega}_0 = \tilde{\Omega}(0)$. Let us define the two dimensionless functions $K(t)$ and $W(t)$ through

$$\kappa(t) = \frac{10C}{c^2}\frac{K(t)}{r_d(t)} \quad \text{and} \quad \omega(t) = \frac{C}{c^2}\frac{\Omega_N(t)W(t)}{r_d(t)}. \tag{71}$$

From Eq. (62) we then find

$$\lambda = \frac{C}{c^2}\frac{d}{dt}\left(\frac{K}{r_d}\right). \tag{72}$$

Using this in combination with Eq. (66), we get, after integration,

$$\frac{W}{X} - W_0 = 2\left[\left(\frac{K}{X} - K_0\right) - \left(\frac{1}{X} - 1\right)\right], \tag{73}$$

with $W_0 = W(0)$ and $K_0 = K(0)$. Combination with Eq. (64) yields

$$-\frac{R_d^3 X^3}{2C}\ddot{K} + \frac{9}{8} + K\left(\frac{37}{8} + \frac{\xi_N^2}{X}\right) - 2\xi_N^2\left[W_0 + 2\left(\frac{K}{X} - K_0\right) - 2\left(\frac{1}{X} - 1\right)\right] = \frac{37}{8} - 6X - \frac{\xi_N^2}{X}\left(\frac{1}{8} - 3X^2\right). \tag{74}$$

For the corrections to the density and angular velocity K and W we make the ansatz

$$K(t) = k_1 + k_2 X(t) \quad \text{and} \quad W(t) = w_1 + w_2 X(t), \tag{75}$$

where k_i and w_i are constants and $K_0 = k_1 + k_2$ and $W_0 = w_1 + w_2$. Equating now equal powers of X we find, for the coefficients,

$$k_1 = \frac{11}{8}, \quad k_2 = -\frac{16}{15}\left(1 - \frac{\xi_N^2}{2}\right), \tag{76}$$

$$w_1 = \frac{3}{4}, \quad w_2 = \frac{39}{128\,\xi_N^2}. \tag{77}$$

Substituting (75) and the solution for $\tilde{\Omega}$ and combining Eqs. (63) and (64) we get a second-order ordinary differential equation for X which has the form

$$\ddot{X} + \frac{a_1}{X^2} + \frac{a_2}{X^3} + \frac{a_3}{X^4} = 0. \tag{78}$$

Multiplying with $\dot{X}$ this can be integrated to yield [remember $X(0) = 1$ and $\dot{X}(0) = 0$]

$$\dot{X}^2 = A + \frac{2B}{X} + \frac{C'}{X^2} + \frac{D}{X^3}, \tag{79}$$

where the coefficients are given by

$$\begin{aligned} B &= \frac{\pi^2 G\sigma_0}{2R_d} + p_c\left(12 + 2k_1 + \frac{25}{8}k_2 - 4\xi_N^2\right), \\ C' &= -\tilde{\Omega}_0^2 + p_c\left(-\frac{55}{4} + \frac{9}{8}k_1 - 8\xi_N^2\right), \\ D &= +p_c\frac{31}{4}\xi_N^2, \\ A &= -(2B + C' + D). \end{aligned} \tag{80}$$

Here p_c is a small quantity denoting the $O(c^{-2})$ corrections to the otherwise Newtonian parts:

$$p_c = \frac{2C^2}{c^2 R_d^4} = 2\left(\frac{3\pi G M_N}{8cR_d^2}\right)^2. \tag{81}$$

Equation (79) has the same structure as the post-Newtonian equation for the orbiting binary star problem.

IV. GENERAL OSCILLATING SOLUTION

Before we state the general solution of (79) it is instructive to restore the dimensional version and express the coefficients in terms of the globally conserved quantities, baryonic mass M_0, angular momentum L, and the binding energy E, as stated in the Appendix. We then find

$$\dot{r}_d^2 = \tilde{A} + \frac{2\tilde{B}}{r_d} + \frac{\tilde{C}}{r_d^2} + \frac{\tilde{D}}{r_d^3}, \tag{82}$$

where the coefficients are given by

$$\tilde{A} = \frac{5E}{M_0}\left\{1 - \frac{5E/c^2}{56M_0(\xi_N^2-2)^2}\left[-101 - \frac{113}{2}k_1 - 25k_2 - 79\xi_N^2 + 40\xi_N^4 + 80K_0\xi_N^2 - 24W_0\xi_N^2\right] + \frac{E/c^2}{M_0(\xi_N^2-2)}\left[13 + \xi_N^2 + 5K_0\right]\right\}, \tag{83}$$

$$\tilde{B} = \frac{3\pi G M_0}{4}\left\{1 + \frac{5E/c^2}{2M_0(\xi_N^2-2)}\left[\frac{22}{5}(2-\xi_N^2) + \frac{9}{8}k_2\right]\right\}, \tag{84}$$

$$\tilde{C} = -\frac{25L^2}{4M_0^2}\left\{1 + \frac{5E/c^2}{7M_0(\xi_N^2-2)}\left[-\frac{58}{5} - \frac{26}{5}\xi_N^2 - 6K_0 + 3W_0\right]\right\} + \frac{9\pi^2G^2M_0^2}{32c^2}\left[-\frac{55}{4} + \frac{9}{8}k_1 - 8\xi_N^2\right], \tag{85}$$

and

$$\tilde{D} = \frac{2325\pi G L^2}{128c^2M_0}. \tag{86}$$

The general solution of (82) is now given by, e.g., Damour and Deruelle [6], and can be written in a parametrized form very similar to the Newtonian solution (18) of an orbiting binary star system:

$$\frac{2\pi}{P}t = u - e_t \sin u - \pi, \tag{87}$$

$$r_d = a(1 - e_r \cos u), \tag{88}$$

with

$$\frac{2\pi}{P} = \frac{(-\tilde{A})^{3/2}}{\tilde{B}}, \tag{89}$$

$$e_t = \left[1 - \frac{\tilde{A}}{\tilde{B}^2}\left(\tilde{C} - \frac{\tilde{B}\tilde{D}}{\tilde{C}_0}\right)\right]^{1/2}, \tag{90}$$

$$a = -\frac{\tilde{B}}{\tilde{A}} + \frac{\tilde{D}}{2\tilde{C}_0}, \tag{91}$$

$$e_r = \left(1 + \frac{\tilde{A}\tilde{D}}{2\tilde{B}\tilde{C}_0}\right)e_t, \tag{92}$$

where $\tilde{C}_0$ is the first term of $\tilde{C}$,

$$\tilde{C}_0 = -\tilde{\Omega}_0^2R_d^4 = -\frac{25L^2}{4M_0^2}. \tag{93}$$

To stress the similarity with the Newtonian solution of Sec. II we define the post-Newtonian ξ through

$$e_r = 1 - \xi^2, \tag{94}$$

and obtain

$$a = \frac{R_d}{2-\xi^2}. \tag{95}$$

In the equivalent two-body problem P is the time of return to the periastron, i.e., the orbital period. The parameter u is an "eccentric" anomaly and e_t and e_r are a "time" and "relative" eccentricity, respectively [6]. The post-Newtonian approximation breaks down if the correction terms are of order 1. In the present situation, the conditions

$$\frac{10|E|}{M_0c^2} \ll 1 \quad \text{and} \quad \left(\frac{3\pi G M_0^2}{40Lc}\right)^2 \ll 1 \tag{96}$$

must hold, which follow from Eqs. (92) and (A3). Having calculated the solution for $X(t)$, the solutions for $\kappa(t)$ and $\omega(t)$ follow directly from the definitions (71), and $\sigma(t)$ and $\tilde{\Omega}$ are obtained from (67) and (70), respectively. Finally, $\lambda(t)$ and $f(t)$ can be calculated from (62) and (61). This completes the general post-Newtonian oscillating solution of a pressureless dusty disk.

The question arises as to how to observationally determine the degree of "post-Newtoness" of a rotating disk. To answer this, let us assume that an observer is able to measure the gravitational mass M_G, the total angular momentum L, and the period P of the oscillations of the system. These three quantities determine the motion completely and are given by Eqs. (A7), (A3), and (89). Now Newtonian and post-Newtonian oscillations of different systems with identical L, P, M_G can be compared by monitoring, for example, the time variations of the orbital period $\tilde{\Omega}$ at the outer disk radius, as seen by an observer at infinity. The ratio of the maximum and minimum values then displays most clearly the post-Newtonian deviations from the nonrelativistic case. To illustrate the post-Newtonian influence on the motion we generated a sequence of test cases in which the parameter

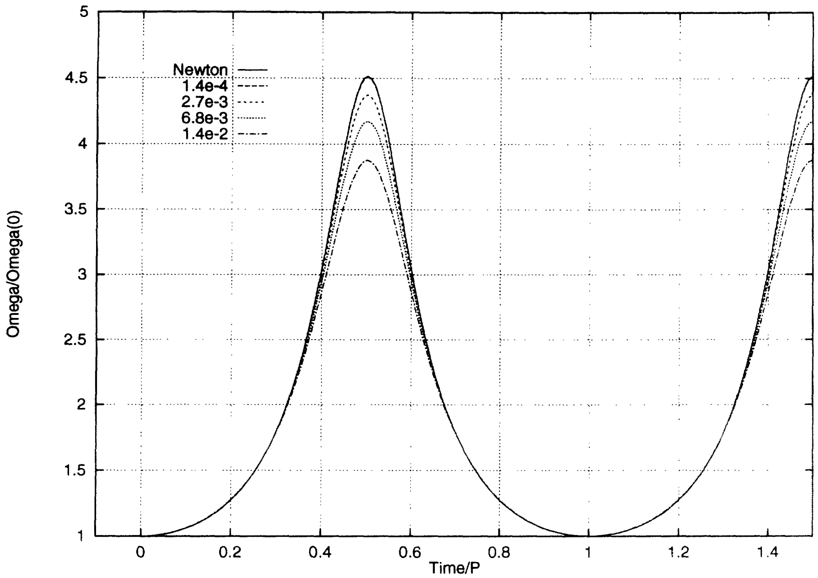

FIG. 1. The angular velocity at the outer edge of the disk, as seen by an observer at infinity, normalized to the initial value versus time normalized to the orbital period. Plotted are curves for models with identical ξ_N and R_d and varying mass M_G. The individual curves are labeled by the dimensionless parameter $|E|/(M_0c^2)$, and the Newtonian limit is given by the solid line.

ξ_N and R_d are held fixed and the gravitational mass M_G is varied. In Fig. 1 we plot the rotational period at the outer rim of the disk ($\tilde{\Omega}$) versus time for such a sequence with $\xi_N = 0.8$. The individual curves are labeled with the dimensionless parameter $\epsilon_c = |E|/(M_0c^2)$. The minimum occurs always at the maximum extension of the disk and the maximum at the disk's minimum radius. Since we normalized the ordinate to the initial rotation $\tilde{\Omega}(0)$ and the abscissa to the rotational period P, the Newtonian curve is identical in all instances and given by the solid line labeled "Newton." Small $\epsilon_c < 10^{-3}$ cases are very nonrelativistic and show only little differences between post-Newtonian and Newtonian evolution. The trend is obviously a reduced ratio $\tilde{\Omega}_{\rm max}/\tilde{\Omega}_{\rm min}$ for more relativistic situations, and in the limit of the approximation, for $\epsilon_c \approx 0.025$, the deviation has reached 20%.

V. SPECIAL CASES

In addition to the oscillating rotating disk solution, it is illuminating to discuss the solution of the post-Newtonian motion for the dusty disk in the limiting cases of the nonrotating collapsing disk and the rigidly rotating stationary disk. We shall discuss these limits now in turn.

A. Rigidly rotating stationary disk

The oscillating solution as presented above displays the surprising and interesting property that by setting $\xi = 1$, one obtains a stationary post-Newtonian solution with $X = X_0 = 1$ which is not rigidly rotating, although the Newtonian limit is. This can most easily be seen by noticing that the constant W_0, or $\omega(0)$, is nonzero even when the limit of vanishing amplitude of the oscillation is taken. It is possible, however, starting directly from the stationary equations, to construct a solution that rotates rigidly, i.e., has $\omega = 0$. Equations (60)–(66) reduce in the stationary limit, where $f = \lambda = 0$, to

$$\frac{\pi^2}{4}\frac{G\sigma}{R_d}\left(2 - \frac{1}{2}\kappa\right) - \Omega^2 = -\frac{C^2}{c^2 R_d^4}, \tag{97}$$

$$-2\omega\Omega + \frac{9C}{8R_d^3}\kappa = \frac{21}{4}\frac{C^2}{c^2 R_d^4}. \tag{98}$$

Since we are interested in rigid rotation, we have to set $\omega = 0$, and substitute the ansatz for κ (71) and (75). From (98) we obtain

$$K = \frac{7}{15}, \tag{99}$$

and from (97) can calculate Ω:

$$\Omega^2 = \frac{\pi^2 G\sigma}{2R_d} + \frac{C^2}{c^2 R_d^4}(1 - 5K).$$

Let us expand the conserved quantities M_0, L, and E of this stationary case in terms of ν_c, the absolute value of the central potential of the disk at $r = 0$. In the post-Newtonian approximation [9], ν_c is given by

$$\nu_c = \frac{U_c}{c^2} + \frac{2\Phi_c}{c^4},$$

where the Φ_c and U_c are the values of U and Φ at $a = 0$. We then find for the rotational velocity at the disk's outer radius

$$\frac{\Omega^2 R_d^2}{c^2} = \nu_c\left(1 - 2\nu_c\right). \tag{100}$$

The total conserved mass, angular momentum, and energy can be written as

$$M_0 = \frac{4c^2}{3\pi G}R_d\nu_c\left(1 + \frac{3}{5}\nu_c\right), \tag{101}$$

$$L = \frac{8c^3}{15\pi G} R_d^2 (\nu_c)^{3/2} (1+\nu_c)\,, \qquad (102)$$

and

$$E = -\frac{4c^4 R_d}{15\pi G} (\nu_c)^2 \left(1 + \frac{1}{2}\nu_c\right). \qquad (103)$$

We note in passing that this solution is identical to the series expansion given by Bardeen and Wagoner [1] if only the first (post-Newtonian) correction is included. This can be seen most easily be expressing their expansion parameter γ, which is related to the central redshift as seen by an observer at infinity, through the expansion parameter ν_c as defined above. The parameter γ is given by

$$\gamma = 1 - e^{-\nu_c}.$$

Note that Bardeen and Wagoner use ν with the opposite sign. By noting further that their radial coordinate ρ is identical to r used here, the identity of the two solutions is established.

B. Nonrotating collapsing disk

At the other extreme, it is interesting to study the evolution of an initially nonrotating disk $\xi = 0$. In the Newtonian case the solution is given by the parametric solution in Sec. II. Surprisingly, again the post-Newtonian case cannot be obtained directly from the oscillatory solution be setting $\xi = 0$ and $W = 0$, but instead one has to start from Eq. (74) and then set $\xi = 0$ and $W = 0$. Upon equating once again equal powers of X in the differential equation for $\ddot{K}$ one finds, for the coefficients of K,

$$k_1 = \frac{46}{37}, \qquad k_2 = -\frac{16}{15}. \qquad (104)$$

Thus, the constants k_1 and k_2 which determine the mass distribution are not identical with those obtained in the oscillatory case. The constants in the equation of motion then read

$$\begin{aligned} B &= \frac{\pi^2 G\sigma_0}{2R_d} + p_c\left(12 + 2k_1 + \frac{25}{8}k_2\right), \\ C' &= \qquad\quad p_c\left(-\frac{55}{4} + \frac{9}{8}k_1\right), \\ A &= -(2B + C'). \end{aligned} \qquad (105)$$

Using now the solution of the full case (89)–(92), and noticing that $e_t = e_r = e$, we can cast the solution for the collapsing nonrotating disk in parametrized form as

$$\frac{\pi}{2}\frac{t}{\tilde{t}_c} = \beta + \frac{1}{2}\left(1 - \frac{457}{37}\frac{C}{R_d c^2}\right)\sin(2\beta). \qquad (106)$$

The collapse time is given by

$$\tilde{t}_c = t_c\left(1 + \frac{1637}{444}\frac{C}{R_d c^2}\right), \qquad (107)$$

where t_c is given by Eq. (22). The radius can then be calculated from

$$r_d(t) = R_d\cos^2\beta(t)\left[1 + \frac{457}{74}\frac{C}{R_d c^2}\tan^2\beta(t)\right]. \qquad (108)$$

These relations directly extend the Newtonian collapse solution of Sec. II. The solutions for $\kappa(t)$ and $\sigma(t)$ follow from (71) and (67). As above, $\lambda(t)$ and $f(t)$ are obtained from (62) and (61), respectively. The collapse time $\tilde{t}_c$ is larger compared to the Newtonian value t_c. The conditions for the validity of the post-Newtonian approximation are now given by

$$\frac{457}{37}\frac{C}{R_d c^2} \ll 1 \quad\text{and}\quad \frac{457}{74}\frac{C}{R_d c^2}\tan^2\beta(t) \ll 1.$$

The approximation obviously breaks down before the collapse time $\tilde{t}_c$ is reached. This is in contrast to the above oscillatory case, where the conditions for validity are not time dependent.

VI. CONCLUSIONS

In this paper we derived an analytic solution for a weakly relativistic, rotating, and oscillating disk of dust in the first post-Newtonian approximation. Although in the nonoscillating case uniform rotation exists, in the limit of zero oscillation amplitudes the angular velocity Ω of the oscillating rotating disk is not uniform. This contrasts sharply with the Newtonian dynamics where in the oscillating case the rotation is uniform even for finite amplitudes. Thus, to obtain a rigidly rotating disk the dust has to be redistributed radially. This fact demonstrates the generality of the class of post-Newtonian solutions, in which additional constraints such as rigid rotation have to be imposed to pick a specific solution. The result, however, appears not completely unexpected if one remembers the phenomenon of the periastron shift that occurs in the relativistic two-body problem, where the equal-mass case is nearest to our situation. It is worthwhile to speculate about an oscillating and rotating disk of dust where the oscillation comes to rest by some damping mechanism and where the final rotation is a nonuniform one. We note that the total energy for the two nonoscillatory cases, for the same angular momentum, radius, and baryonic mass, is identical, as can be seen directly from Eq. (A5). Thus the two solutions appear to be degenerate in this case. However, most dissipative processes are likely to destroy the limiting nonuniform pattern of rotation. In a forthcoming paper we shall investigate how gravitational-radiation damping influences the oscillatory and rotational motion of a relativistic disk of dust.

ACKNOWLEDGMENTS

We gratefuly acknowledge stimulating discussions within the Jena relativity group, in particular, with Reinhard Meinel.

APPENDIX: CONSERVED QUANTITIES

During the collapse certain physical quantities are conserved. The first is the total baryonic mass of the disk,

$$M_0 = \int \rho^* d^3x,$$

where the density ρ^* is given by [9]

$$\rho^* = \rho\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + 3U\right)\right].$$

Substituting for the potential and the velocity we find, after integrating over the volume (surface) of the disk,

$$M_0 = \frac{2\pi\sigma_0 R_d^2}{3}\left[1 + \frac{2}{5}\frac{C}{c^2 R_d}(8 + \xi_N^2 + 5K_0)\right]. \quad \text{(A1)}$$

The total baryonic mass M_0 is identical with the Newtonian mass M_N. From the linear momentum [9]

$$\boldsymbol{\pi} = \rho\mathbf{v} + \frac{1}{c^2}\rho\left[\mathbf{v}(v^2 + 6U) - \mathbf{P}\right],$$

we find, for the conserved total angular momentum,

$$\mathbf{L} = \int \mathbf{x} \times \boldsymbol{\pi} d^3x = \mathbf{e_z}\int r\pi_\varphi d^3x.$$

The last equal sign follows for the considered disk case where the angular momentum vector is parallel to the z axis. The vector $\mathbf{P}$ is defined through

$$P_\mu = 4U_\mu - \frac{1}{2}\frac{\partial^2}{\partial t \partial x_\mu}\chi,$$

and in our situation it reduces to

$$P_r = \frac{7}{2}U_r, \qquad P_\varphi = 4U_\varphi. \quad \text{(A2)}$$

For the total angular momentum we then get

$$L = \frac{4\pi\sigma_0\tilde{\Omega}_0 R_d^4}{15}\left[1 + \frac{4}{7}\frac{C}{c^2 R_d}\left(\frac{17}{2} + 2\xi_N^2 + 5K_0 - \frac{3}{4}W_0\right)\right]. \quad \text{(A3)}$$

The binding energy of the disk is given by

$$E = \int \mathcal{E} d^3x,$$

where the energy density $\mathcal{E}$ has the form [9]

$$\mathcal{E} = \rho\left[\frac{1}{2}(v^2 - U) + \frac{1}{c^2}\left(\frac{5}{8}v^4 + \frac{5}{2}v^2U - \frac{5}{2}U^2 - \frac{1}{2}v_\mu P_\mu\right)\right]. \quad \text{(A4)}$$

Substituting now again the potentials U, P_μ, velocity v, and density Σ, we find, after integrating over the volume (surface) of the disk,

$$E = -\frac{\pi\sigma_0 R_d^2}{15}\left\{2\left[\pi^2 G\sigma_0 R_d - R_d^2\tilde{\Omega}_0^2\right] - \frac{4}{7}\frac{C^2}{c^2 R_d^2}\left[50K_0\left(\frac{2}{5}\xi_N^2 - 1\right) - 6W_0\xi_N^2 + 10\xi_N^4 + 24\xi_N^2 - 69\right]\right\}. \quad \text{(A5)}$$

In the Newtonian limit this reduces to

$$E_N = -\frac{3\pi}{20}\frac{GM_N^2}{R_d}(2 - \xi_N^2). \quad \text{(A6)}$$

Relation (A6) can be cast into the form

$$\frac{C}{R_d} = \frac{5E_N}{2M_N(\xi_N^2 - 2)},$$

which proved to be useful in obtaining the expressions for the constants $\tilde{A}$ to $\tilde{D}$ in Sec. III above. The total gravitational mass of the disk, to the order c^{-2}, is then

$$M_G = M_0 + \frac{E}{c^2}. \quad \text{(A7)}$$

[1] J. M. Bardeen and R. V. Wagoner, Astrophys. J. **167**, 359 (1971).
[2] G. Neugebauer and R. Meinel, Astrophys. J. **414**, L97 (1993).
[3] G. Schäfer and N. Wex, Phys. Lett. A **174**, 196 (1993); **177**, 461(E) (1993); T. Damour and G. Schäfer, Nuovo Cimento B **101**, 127 (1988).
[4] C. Hunter, Mon. Not. R. Astron. Soc. **126**, 299 (1963); **129**, 321 (1965).
[5] S. L. Shapiro and S. A. Teukolsky, Phys. Rev. D **49**, 1886 (1994).
[6] T. Damour and N. Deruelle, Ann. Inst. Henri Poincaré **43**, 107 (1985).
[7] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, 1987).
[8] A. M. Abrahams, S. L. Shapiro, and S. A. Teukolsky, report, 1994 (unpublished).
[9] S. Chandrasekhar, Astrophys. J. **142**, 1488 (1965); **148**, 621 (1967).