Hydrodynamics: Hydrodynamic Equations

The Euler-Equations in conservative form read

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \tag{1}
\]

\[
\frac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) = -\nabla p + \rho k \tag{2}
\]

\[
\frac{\partial (\rho \epsilon)}{\partial t} + \nabla \cdot (\rho \epsilon \vec{u}) = -p \nabla \cdot \vec{u} \tag{3}
\]

\(\vec{u}\): Velocity, \(k\): external forces, \(\epsilon\) specific internal energy

The equations describe the conservation of mass, momentum and energy.

For completion we need an equation of state (eos):

\[
p = (\gamma - 1) \rho \epsilon \tag{4}
\]

Using this and eq. (3), we can rewrite the energy equation as an equation for the pressure

\[
\frac{\partial p}{\partial t} + \nabla \cdot (p \vec{u}) = -(\gamma - 1)p \nabla \cdot \vec{u} \tag{5}
\]
Hydrodynamics: Reformulating

Expanding the divergences on the left side and use for the momentum and energy equation the continuity equation

\[
\frac{\partial \rho}{\partial t} + (\bar{u} \cdot \nabla) \rho = -\rho \nabla \cdot \bar{u} 
\]  
(6)

\[
\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} = -\frac{1}{\rho} \nabla p + \vec{k} 
\]  
(7)

\[
\frac{\partial p}{\partial t} + (\bar{u} \cdot \nabla) p = -\gamma p \nabla \cdot \bar{u} 
\]  
(8)

Since all quantities depend on space ($\vec{r}$) and time ($t$), for example $\rho(\vec{r}, t)$, we can use for the left side the total time derivative (Lagrange-Formulation). For example, for the density one obtains

\[
\frac{D \rho}{Dt} = \frac{\partial \rho}{\partial t} + (\bar{u} \cdot \nabla) \rho = -\rho \nabla \cdot \bar{u} . 
\]  
(9)

The Operator

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u} \cdot \nabla 
\]  
(10)

is called material derivative (equivalent to the total time derivative, $d/dt$).
Hydrodynamics: Lagrange-Formulation

Use now the material derivative

\[
\frac{D\rho}{Dt} = -\rho \nabla \cdot \vec{u}
\]

(11)

\[
\frac{D\vec{u}}{Dt} = -\frac{1}{\rho} \nabla p + \vec{k}
\]

(12)

\[
\frac{Dp}{Dt} = -\gamma p \nabla \cdot \vec{u}
\]

(13)

These equations describe the change of the quantities in the comoving frame = Lagrange-Formulation.

For the Euler-Formulation, one analysed the changes at a specific, fixed point in space!

The Lagrange-Formulation can be used conveniently for 1D-problems, for example the radial stellar oscillations, using comoving mass-shells.

For the Euler-Formulation a fixed grid is used.
Numerical Hydrodynamics: The problem

Consider the evolution of the full time-dependent hydrodynamic equations. The non-linear partial differential equations of hydrodynamics will be solved numerically Continuum $\Rightarrow$ Discretisation
Numerical Hydrodynamics: Method of solution

Grid-based methods (Euler)

fixed Grid
- matter flows through grid

\[ \rho \left( \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla p \]

Methods:
- finite differences
  non-conservative
- Control Volume
  conservative
- Riemann-solver
  wave properties
- Problem: Discontinuities

Particle methods (Lagrange)

moving Grid/Particle
- flow moved grid

\[ \rho \frac{d\vec{u}}{dt} = -\nabla p \]

Well known method:
Smoothed Particle Hydrodynamics, SPH

’smeared out particles’
good for free boundaries, self-gravity
Numerical Hydrodynamics: consider: 1D Euler equations

describe conservation of mass, momentum and energy

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \tag{14}
\]

\[
\frac{\partial \rho u}{\partial t} + \frac{\partial \rho uu}{\partial x} = -\frac{\partial p}{\partial x} \tag{15}
\]

\[
\frac{\partial \rho \epsilon}{\partial t} + \frac{\partial \rho \epsilon u}{\partial x} = -p \frac{\partial u}{\partial x} \tag{16}
\]

\(\rho\): density

\(u\): velocity

\(p\): pressure

\(\epsilon\): internal specific energy (Energy/Mass)

with the equation of state

\[
p = (\gamma - 1) \rho \epsilon \tag{17}
\]

\(\gamma\): adiabatic exponent

partial differential equation in space and time

→ need discretisation in space and time.
Consider function: \( \psi(x, t) \)

discretisation in space

cover with a grid

\[
\Delta x = \frac{x_{\text{max}} - x_{\text{min}}}{N}
\]

\( \psi_j^n \) cell average of \( \psi(x, t) \) at the gridpoint \( x_j \) at time \( t^n \)

\[
\psi_j^n = \psi(x_j, t^n) \approx \frac{1}{\Delta x} \int_{(j-1)/2}^{(j+1)/2} \psi(x, n\Delta t) dx
\]

\( \psi_j^n \) is piecewise constant. \( j \) spatial index, \( n \) time step.
Consider general equation

\[ \frac{\partial \psi}{\partial t} = \mathcal{L}(\psi(x, t)) \]  

with a (spatial) differential operator \( \mathcal{L} \).

Typical Discretisation (1. order in time), at time: \( t = t^n = n\Delta t \)

\[ \frac{\partial \psi}{\partial t} \approx \frac{\psi(t + \Delta t) - \psi(t)}{\Delta t} = \frac{\psi^{n+1} - \psi^n}{\Delta t} = \mathcal{L}(\psi^n) \]  

Now at a special location, the grid point \( x_j \) (with moving terms)

\[ \psi_j^{n+1} = \psi_j^n + \Delta t \mathcal{L}(\psi_k^n) \]  

\( \mathcal{L}(\psi_k^n) \): discretised differential operator \( \mathcal{L} \) (here explicit)

- \( k \) in \( \mathcal{L}(\psi_k) \): set of spatial indices:

- Typical for 2. order: \( k \in \{j - 2, j - 1, j, j + 1, j + 2\} \)

(need information from left and right, 5 point ’Stencil’)
\begin{equation}
\frac{\partial \vec{A}}{\partial t} = \mathcal{L}_1(\vec{A}) + \mathcal{L}_2(\vec{A})
\end{equation}

\(\mathcal{L}_i(\vec{A})\), \(i = 1, 2\) individual (Differential-)operators
applied to the quantities \(\vec{A} = (\rho, u, \epsilon)\).
Here, for 1D ideal hydrodynamics

\[\mathcal{L}_1 : \text{Advection}\]

\[\mathcal{L}_2 : \text{pressure, or external forces}\]

To solve the full equation the solution is split in several substeps

\[\vec{A}^1 = \vec{A}^n + \Delta t L_1(\vec{A}^n)\]
\[\vec{A}^{n+1} = \vec{A}^2 = \vec{A}^1 + \Delta t L_2(\vec{A}^1)\]  \(\text{(22)}\)

\(L_i\) is the differential operator to \(\mathcal{L}_i\).
Numerical Hydrodynamics: advection-step

\[
\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u}{\partial x}, \\
\frac{\partial (\rho u)}{\partial t} = -\frac{\partial (\rho uu)}{\partial x}, \\
\frac{\partial (\rho \varepsilon)}{\partial t} = -\frac{\partial (\rho \varepsilon u)}{\partial x}
\]

In explicit conservation form

\[
\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{f}(\vec{u})}{\partial x} = 0 \quad (23)
\]

for \(\vec{u} = (u_1, u_2, u_3)\) and \(\vec{f} = (f_1, f_2, f_3)\) we have:

\(\vec{u} = (\rho, \rho u, \rho \varepsilon)\) and \(\vec{f} = (\rho u, \rho uu, \rho \varepsilon u)\).

This step yields: \(\rho^n \rightarrow \rho^1 = \rho^{n+1}\), \(u^n \rightarrow u^1\), \(\varepsilon^n \rightarrow \varepsilon^1\)
Momentum equation

\[ \frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \]  \hspace{1cm} (24)

\[ u^{n+1}_j = u_j - \Delta t \frac{1}{\rho^{n+1}_j} \left( \frac{p_j - p_{j-1}}{\Delta x} \right) \]  \hspace{1cm} for \hspace{0.5cm} j = 2, N \hspace{1cm} (25)

Energy equation

\[ \frac{\partial \epsilon}{\partial t} = -\frac{p}{\rho} \frac{\partial u}{\partial x} \]  \hspace{1cm} (26)

\[ \epsilon^{n+1}_j = \epsilon_j - \Delta t \frac{p_j}{\rho^{n+1}_j} \left( \frac{u_{j+1} - u_j}{\Delta x} \right) \]  \hspace{1cm} for \hspace{0.5cm} j = 1, N \hspace{1cm} (27)

On the right hand side we use the actual values for \( u, \epsilon \) and \( p \), i.e. here \( u^1, p^1, \epsilon^1 \).

This step yields: \( u^1 \rightarrow u^{n+1}, \epsilon^1 \rightarrow \epsilon^{n+1} \)
The continuity equation was

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \] (28)

Here, \( F^m = \rho u \) is the mass flow.

Using the notation \( \rho \rightarrow \psi \) and \( u \rightarrow a = \text{const.} \) we obtain the Linear Advection Equation

\[ \frac{\partial \psi}{\partial t} + a \frac{\partial \psi}{\partial x} = 0. \] (29)

with a constant velocity \( a \), the solution is a wave traveling to the right.

Using \( \psi(x, t = 0) = f(x) \) we get \( \psi(x, t) = f(x - at) \).

Here \( f(x) \) is the initial condition at time \( t = 0 \), that is shifted by the advection with a constant velocity \( a \) to the right.

The numerics should maintain this property as accurately as possible.
Numerical Hydrodynamics: Linear Advection

**FTCS: Forward Time Centered Space algorithm**

\[ \frac{\partial \psi}{\partial t} + a \frac{\partial \psi}{\partial x} = 0 \quad (30) \]

Specify the grid:

\[
\begin{array}{c|c|c}
\psi_{j-1} & \psi_j & \psi_{j+1} \\
\hline
x_{j-1} & x_j & x_{j+1}
\end{array}
\]

and write

\[ \left. \frac{\partial \psi}{\partial t} \right|_j^n = \frac{\psi_{j+1}^n - \psi_j^n}{\Delta t} \quad (31) \]

\[ \left. \frac{\partial \psi}{\partial x} \right|_j^n = \frac{\psi_{j+1}^n - \psi_{j-1}^n}{2 \Delta x} \quad (32) \]

it follows

\[ \psi_{j+1}^{n+1} = \psi_j^n - \frac{a \Delta t}{2 \Delta x} (\psi_{j+1}^n - \psi_{j-1}^n) \quad (33) \]

The method looks well motivated: but it is **unstable** for all \( \Delta t \)!
\[
\frac{\partial \psi}{\partial t} + \frac{\partial a \psi}{\partial x} = 0 \quad \text{(34)}
\]
or
\[
\frac{\partial \psi}{\partial t} + a \frac{\partial \psi}{\partial x} = 0 \quad \text{(35)}
\]

\(a\): constant (velocity) \(> 0\)

\(\psi(x, t)\) arbitrary transport quantity

change of \(\psi\) in grid cell \(j\)

\[
\psi^{n+1}_j \Delta x = \psi^n_j \Delta x + \Delta t (F_{\text{in}} - F_{\text{out}}) \quad \text{(36)}
\]

The flux \(F_{\text{in}}\) is for constant \(\psi_j\)

\[
F_{\text{in}} = a \psi^{n-1}_{j-1} \quad \text{(37)}
\]

\[
F_{\text{out}} = a \psi^n_j \quad \text{(38)}
\]

**Upwind-Method**

Information comes from upstream

purple regions will be transported into the next neighbour cell
Extension for non-constant states

\[ F_{in} = a \psi_I \left( x_{j-1/2} - \frac{a \Delta t}{2} \right) \]  

(39)

\( \psi_I(x) \) interpolation polynomial

Here linear interpolation (straight line)

This yields

\[ F_{in} = a \left[ \psi_{j-1}^n + \frac{1}{2} (1 - \sigma) \Delta \psi_{j-1} \right] \]  

(40)

with \( \sigma = a \Delta t / \Delta x \)

\[ \Delta \psi_j \approx \left. \frac{\partial \psi}{\partial x} \right|_{x_j} \Delta x \]

\( \Delta \psi_j \) undivided differences

2nd order upwind

\( \psi_I(x) \) is evaluated in the middle of the purple area.
a) \( \Delta \psi_j = 0 \) \textit{Upwind, 1st Order}, piece-wise constant

b) \( \Delta \psi_j = \frac{1}{2} (\psi_{j+1} - \psi_{j-1}) \) \textit{Fromm}, centered derivative

c) \( \Delta \psi_j = \psi_j - \psi_{j-1} \) \textit{Beam-Warming}, upwind slope

d) \( \Delta \psi_j = \psi_{j+1} - \psi_j \) \textit{Lax-Wendroff}, downwind slope

Often used is the 2nd Order Upwind (van Leer) Geometric Mean (maintains the Monotonicity)

\[
\Delta \psi_j = \begin{cases} 
2 \frac{(\psi_{j+1} - \psi_j)(\psi_j - \psi_{j-1})}{(\psi_{j+1} - \psi_{j-1})} & \text{if } (\psi_{j+1} - \psi_j)(\psi_j - \psi_{j-1}) > 0 \\
0 & \text{otherwise}
\end{cases}
\]  

(42)

The derivatives are evaluated at the corresponding time step level or the intermediate time step
Numerical Hydrodynamics: Lax-Wendroff Method

Schematic overview of the method uses centered spatial and temporal differences that makes it 2nd order in space and time.

Using two steps:

**predictor-step** (at intermediate time \( t^{n+1/2} \))

\[
\tilde{\psi}_{j+1/2}^{n+1/2} = \frac{1}{2} \left( \psi_j^n + \psi_{j+1}^n \right) - \frac{\sigma}{2} \left( \psi_{j+1}^n - \psi_j^n \right) \tag{43}
\]

The **corrector-step** (to new time \( t^{n+1} \))

\[
\psi_{j}^{n+1} = \psi_j^n - \sigma \left( \tilde{\psi}_{j+1/2}^{n+1/2} - \tilde{\psi}_{j-1/2}^{n+1/2} \right) \tag{44}
\]
Numerical Hydrodynamics: Example: Linear Advection

Square function:
Width 0.6 in the interval $[-1, 1]$
velocity $a = 1$, until $t = 40$
periodic boundaries
$\sigma = a \Delta t / \Delta x = 0.8$ - (Courant no.)

Van Leer

Lax-Wendroff - (Dispersive)
Substitute for the solution a Fourier series (von Neumann 1940/50s)
consider simplifying one component, and analyse its growth properties

$$\psi_j^n = V^n e^{i\theta_j}$$  \hspace{1cm} (45)

here, $\theta$ is defined through grid size $\Delta x$ and the total length $L$

$$\theta = \frac{2\pi \Delta x}{L}$$  \hspace{1cm} (46)

Consider simple Upwind method with $\sigma = a\Delta t/\Delta x$

$$\psi_j^{n+1} - \psi_j^n + \sigma(\psi_j^n - \psi_{j-1}^n) = 0$$  \hspace{1cm} (47)

Substituting eq. (45)

$$V^{n+1} e^{i\theta_j} = V^n e^{i\theta_j} + \sigma V^n \left[ e^{i\theta(j-1)} - e^{i\theta_j} \right]$$

dividing by $V^n$ and $e^{i\theta_j}$ yields

$$\frac{V^{n+1}}{V^n} = 1 + \sigma \left( e^{-i\theta} - 1 \right)$$  \hspace{1cm} (48)
For the square of the absolute value one obtains

\[
\lambda(\theta) \equiv \left| \frac{V^{n+1}}{V^n} \right|^2 = \left[ 1 + \sigma \left( e^{-i\theta} - 1 \right) \right] \left[ 1 + \sigma \left( e^{i\theta} - 1 \right) \right] = 1 + \sigma \left( e^{-i\theta} + e^{i\theta} - 2 \right) - \sigma^2 \left( e^{-i\theta} + e^{i\theta} - 2 \right) = 1 + \sigma(1 - \sigma)(2 \cos \theta - 2) = 1 - 4\sigma(1 - \sigma) \sin^2 \left( \frac{\theta}{2} \right) \quad (49)
\]

The method is now stable, if the magnitude of the amplification factor \( \lambda(\theta) \) is smaller than unity. The upwind-method is stable for \( 0 < \sigma < 1 \), the \( |\lambda(\theta)| < 1 \). Rewritten

\[
\Delta t < f_{CFL} \frac{\Delta x}{a} \quad (50)
\]

with the Courant-factor \( f_{CFL} < 1 \). Typically \( f_{CFL} = 0.5 \).

**Theorem: Courant-Friedrich-Levy**

There is no explizit, consistent and stable finite difference method which is unconditionally stable (i.e. for all \( \Delta t \)).
Consider again Upwind method mit $\sigma = a\Delta t/\Delta x$

$$\psi_j^{n+1} - \psi_j^n + \sigma (\psi_j^n - \psi_{j-1}^n) = 0 \quad (51)$$

substitute differences by derivatives, i.e. Taylor-series (up to 2. order)

$$\frac{\partial \psi}{\partial t} \Delta t + \frac{1}{2} \frac{\partial \psi}{\partial \tilde{t}} \Delta t^2 + O(\Delta t^3) + \sigma \left( \frac{\partial \psi}{\partial x} \Delta x - \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} \Delta x^2 \right) + O(\Delta t \Delta x^2) = 0 \quad (52)$$

divided by $\Delta t$, and substitute for $\sigma$

$$\frac{\partial \psi}{\partial t} + a \frac{\partial \psi}{\partial x} + \frac{1}{2} \left( \frac{\partial \psi}{\partial \tilde{t}} \Delta t - a \frac{\partial^2 \psi}{\partial x^2} \Delta x \right) + O(\Delta t^2) + O(\Delta x^2) = 0 \quad (53)$$

Use wave equation $\psi_{tt} = a^2 \psi_{xx}$ ⇒ modified equation (index $M$)

$$\frac{\partial \psi_M}{\partial t} + a \frac{\partial \psi_M}{\partial x} = \frac{1}{2} a \Delta x (1 - \sigma) \frac{\partial^2 \psi_M}{\partial x^2} \quad (54)$$

The FDE adds a new diffusive term to the original PDE
Numerical Hydrodynamics: Modified Equation II

with the diffusion coefficient

\[ D = \frac{1}{2} a \Delta x (1 - \sigma) \]  

(55)

Note: only for \( D > 0 \) this is a diffusion equation, and it follows \( \sigma < 1 \) for stability. (Hirt-method). For Upwind-Method \( D > 0 \) \( \Rightarrow \) Diffusion.

Lax-Wendroff yields

\[ \frac{\partial \psi^M}{\partial t} + a \frac{\partial \psi^M}{\partial x} = \Delta t^2 a \frac{\sigma}{\sigma^2 - 1} \frac{\partial^3 \psi^M}{\partial x^3} \]  

(56)

The equation has the form

\[ \psi_t + a \psi_x = \mu \psi_{xxx} \]  

(57)

\[ \mu = \frac{\Delta t^2 a}{\sigma} (\sigma^2 - 1) \]  

(58)

This implies Dispersion. Here: waves are too slow \( (\mu < 0) \) \( \Rightarrow \) Oscillations behind the diskontinuity (cp. square function)
Numerical Hydrodynamics: The time step

From the above analysis: the time step $\Delta t$ has to be limited for a stable numerical evolution. For the linear Advection (with the velocity $a$) we find

$$\Delta t < \frac{\Delta x}{a}$$

(59)

In the more general case the sound speed has to be included and it follows the Courant-Friedrich-Lewy-condition

$$\Delta t < \frac{\Delta x}{c_s + |\vec{u}|}$$

(60)

physically this means that information cannot travel in one timestep more than one gridcell. Typically one writes

$$\Delta t = f_C \frac{\Delta x}{c_s + |\vec{u}|}$$

(61)

with the Courant-Factor $f_C$. For 2D situations: $f_C \sim 0.5$. Only for impliciten methods there are (theoretically) no limitations of $\Delta t$. 

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The numerical region of dependence (dashed line) should be larger than the physical one (gray shaded area) since $\Delta x / \Delta t > a$. The complete information from inside the physical 'sound cone' should be considered.
Grid definition (in 2D, staggered): scalers in cell centers (hier: $\rho, \epsilon, p, v_3, \psi$)
Vectors at interfaces (here: $v_1, v_2$)

Fluxes across cell boundaries
Top: mass flux
bottom: X-momentum (grid shifted)


Use Operator-Splitting and Directional Splitting: The $x$ and $y$ direction are dealt with subseuqently. First $x$-scans, then $y$-scans.
Numerical methods should resemble the conservation properties.
- write equations in conservative form

Numerical Methods should resemble the wave properties.
- *Shock-Capturing methods*, Riemann-solver

Numerical Methods should control discontinuities.
- need diffusion ($\Rightarrow$ stability)
  - either explicitly (artificial viscosity) or intrinsically (through method)

Numerical methods should be accurate
- min. 2nd order in space and time

Freely available codes:

- classical Upwind-Code, 2nd order, staggered grid, RMHD

**ATHENA**: [https://trac.princeton.edu/Athena/](https://trac.princeton.edu/Athena/)
- successo of ZEUS: Riemann solver, centered grid, MHD

**NIRVANA**: [http://www.aip.de/Members/uziegler/nirvana-code](http://www.aip.de/Members/uziegler/nirvana-code)
- 3D, AMR, finite volume code, MHD

**PLUTO**: [http://plutocode.ph.unito.it/](http://plutocode.ph.unito.it/)
- 3D, relativistic, Riemann-solver/finite volume, MHD
Consider one-dimensional equations (motion in $x$-direction):

From Euler equations: With $p = (\gamma - 1)\rho \epsilon$ and separation of derivatives

$$\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} &= 0 \\
\frac{\partial \rho u}{\partial t} + \frac{\partial \rho uu}{\partial x} &= -\frac{\partial p}{\partial x}
\end{align*}$$

$$\begin{align*}
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} &= 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} &= 0 \\
\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \gamma p \frac{\partial u}{\partial x} &= 0
\end{align*}$$

As Vector equation

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{W}}{\partial x} = 0 \quad (62)$$

mit

$$\mathbf{W} = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} \quad \text{und} \quad \mathbf{A} = \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{pmatrix} \quad (63)$$

Equations are non-linear and coupled.

Try decoupling: $\Rightarrow$ Diagonalisation of $\mathbf{A}$
Hydrodynamics: Diagonalisation

Eigenvalues (EV)

\[
\det(A) = \begin{vmatrix}
    u - \lambda & \rho & 0 \\
    0 & u - \lambda & 1/\rho \\
    0 & \gamma p & u - \lambda \\
\end{vmatrix}
= (u - \lambda) \begin{vmatrix}
    u - \lambda & 1/\rho \\
    \gamma p & u - \lambda \\
\end{vmatrix}
= (u - \lambda) \left[ (u - \lambda)^2 - \gamma p / \rho \right] = 0
\]

It follows

\[
\lambda_0 = u \\
\lambda_{\pm} = u \pm c_s
\]

with the sound speed

\[
c_s^2 = \frac{\gamma p}{\rho}
\]

The Eigenvalues are the characteristic velocities, with which the information is spreading.

It is a combination of fluid velocity \( u \) and sound speed \( c_s \)

3 real Eigenvalues \( \Rightarrow \) \( A \) diagonalisable

\[
Q^{-1}AQ = \Lambda
\]

\( Q \) is built up from the Eigenvalues to the EV \( \lambda_i, i = 0, +, - \), \( \Lambda \) is a diagonal matrix.
Hydrodynamics: Characteristic Variables

For \( \mathbf{Q} \) it follows

\[
\mathbf{Q} = \begin{pmatrix}
1 & \frac{1}{2} \rho c_s & -\frac{1}{2} \rho c_s \\
0 & \frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{2} \rho c_s & -\frac{1}{2} \rho c_s
\end{pmatrix}
\]

and

\[
\mathbf{Q}^{-1} = \begin{pmatrix}
1 & 0 & -\frac{1}{c_s^2} \\
0 & 1 & -\frac{1}{\rho c_s} \\
0 & 1 & -\frac{1}{\rho c_s}
\end{pmatrix}
\]

We had

\[
\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{W}}{\partial x} = 0 \tag{68}
\]

and

\[
\mathbf{Q}^{-1} \mathbf{A} \mathbf{Q} = \Lambda
\]

Define:

\[
d\mathbf{v} \equiv \mathbf{Q}^{-1} d\mathbf{W} \quad \text{also} \quad d\mathbf{W} = \mathbf{Q} d\mathbf{v} \tag{69}
\]

Multiply eq. (68) with \( \mathbf{Q}^{-1} \)

\[
\frac{\partial \mathbf{v}}{\partial t} + \Lambda \frac{\partial \mathbf{v}}{\partial x} = 0 \tag{70}
\]

\( \mathbf{v} = (v_0, v_+, v_-) \) are the \textbf{characteristic Variables}: \( v_i = \text{const.} \) in curves

\[
\frac{dx}{dt} = \lambda_i
\]
Hydrodynamics: The variable $v_0$

from the definitions

$$dv_0 = d\rho - \frac{1}{c_s^2} dp$$ (71)

$$\frac{\partial v_0}{\partial t} + \lambda_0 \frac{\partial v_0}{\partial x} = 0 \quad \text{mit} \quad \lambda_0 = u$$ (72)

What is $dv_0$?

From thermodynamics (1. Law) for specific quantities)

$$Tds = d\varepsilon + p \, d \left( \frac{1}{\rho} \right) = d\varepsilon - \frac{p}{\rho^2} \, d \left( \frac{1}{\rho} \right)$$ (73)

with $p = (\gamma - 1) \rho \varepsilon$, $\varepsilon = c_v \, T$, $\gamma = c_p / c_v$ it follows

$$ds = -\frac{c_p}{\rho} \left[ d\rho - \frac{dp}{c_s^2} \right] = -\frac{c_p}{\rho} \, dv_0$$ (74)

$$\implies \frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} = 0$$ (75)

i.e. $s$ is const. along stream lines, hence

$$\frac{ds}{dt} = 0$$ (76)
Hydrodynamics: Riemann-Invariants

For the additional variables

\[ \frac{\partial v_{\pm}}{\partial t} + (u \pm c_s) \frac{\partial v_{\pm}}{\partial x} = 0 \]  

(77)

with

\[ dv_{\pm} = du \pm \frac{1}{\rho c_s} dp \]  

(78)

it follows

\[ v_{\pm} = u \pm \int \frac{dp}{\rho c_s} . \]  

(79)

Let the entropy constant everywhere  
(i.e. \( p = K \rho^\gamma \))

\[ \Rightarrow v_{\pm} = u \pm \frac{2c_s}{\gamma - 1} \]  

(80)

Riemann-Invariants: \( v_{\pm} = \text{const.} \) on curves

\[ \frac{dx}{dt} = u \pm c_s \]
Hydrodynamics: Steepening of sound waves

Linearisation of the Euler-equation results in the wave equation for the perturbations:

\[
\frac{\partial \rho_1}{\partial \tilde{t}} = c_s^2 \frac{\partial^2 \rho_1}{\partial x^2}
\]

but: \(c_s\) is not constant \(\Rightarrow\) steepening

\(\Rightarrow\) Diskontinuities

Example for (receding) shock wave
Examples: Shocktube

Initial discontinuity in a tube at position $x_0$ (one-dimensional)

Riemann-Problem

Jump in pressure ($\rho$) and density ($\rho$)

Evolution:
- a shock wave to the right ($X_4$) (supersonically $u_{sh} > c_s$)
- a contact discontinuity density jump (along $X_3$)
- a rarefaction wave (between $X_1$ and $X_2$)
Examples: Sod-Shocktube

A standard test problem for numerical hydrodynamics, $x \in [0, 1]$ with $X_0 = 0.5$, $\gamma = 1.4$

$\rho_1 = 1.0$, $\rho_1 = 1.0$, $\epsilon_1 = 2.5$, $T_1 = 1$ and $\rho_2 = 0.1$, $p_2 = 0.125$, $\epsilon_2 = 2.0$, $T_2 = 0.8$

Hier solution with van Leer method (at time $t = 0.228$ after 228 time steps:)

Red: Exact
Green: Numerics

The solution is: self similar obtained through stretching
Examples: Sedov-Explosion

An example for bomb explosions (Sedov & Taylor, 1950s), analytical solution (Sedov)
Basic setup for Supernovae-outbursts, e.g. estimate of the remnant size
Standard test problem of multi-dimensional hydrodynamics, e.g. for
$x, y \in [0, 1] \times [0, 1]$
Energy-Input at origin, $E = 1$, in $\rho = 1$, $\gamma = 1.4$, 200 $\times$ 200 grid points
Here: solution with van Leer method (solve for total energy variable). Plotted: density

![Density plot](image_url)
Examples: Water droplet: SPH

Water sphere (R=30cm), Basin (1x1 m, 60cm high) Incl. surface tension, time in seconds (TU-München, 2002)
Examples: Ster formation: SPH

Molecular Cloud

Mass: 50 \, M_\odot

Diameter: 1.2 \, LJ = 76,000 \, AU

Temperature: 10 \, K

(M. Bate, 2002)
Examples: Kelvin-Helmholtz Instability I
Examples: Kelvin-Helmholtz Instability II

Direct comparison: moving $< -$ > fixed grid

Left: Moving grid (Voronoi-Tessellation)
Right: fixed square grid (Euler)

(Kevin Schaal, Tübingen)

Youtube channel
Examples: Kelvin-Helmholtz Instability III

(Boulder (NCAR), USA)
Examples: Rayleigh-Taylor Instability

PPM Code, 128 Nodes, ASCI Blue-Pacific ID System at LLNL
512³ Grid Cells (LLNL, 1999)
Examples: Diesel Injection

Finite Volumen Method (FOAM)

Velocity, Temperature, Particles (+Isosurfaces) (Nabla Ltd, 2004)
Examples: Cataclysmic Variable: Grid
Examples: Kataclysmic Variable: Disk formation

RH2D Code, Van Leer slope

$512^2$ Gridpoints, mass ratio: $q = m_2/m_1 = 0.15$