4H FLUID DYNAMICS

Vector Formulae and Theorems of Vector Calculus

Vector Formulae

$$
\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\
\boldsymbol{\nabla} \times \boldsymbol{\nabla} f &= 0 \\
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) &= 0 \\
\boldsymbol{\nabla} \cdot (f\mathbf{F}) &= \mathbf{F} \cdot \boldsymbol{\nabla} f + f \boldsymbol{\nabla} \cdot \mathbf{F} \\
\boldsymbol{\nabla} \times (f\mathbf{F}) &= \boldsymbol{\nabla} f \times \mathbf{F} + f \boldsymbol{\nabla} \times \mathbf{F} \\
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{F}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \\
\boldsymbol{\nabla}(\mathbf{F} \cdot \mathbf{G}) &= (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} + \mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G}) + \mathbf{G} \times (\boldsymbol{\nabla} \times \mathbf{F}) \\
\boldsymbol{\nabla} \cdot (\mathbf{F} \times \mathbf{G}) &= \mathbf{G} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) - \mathbf{F} \cdot (\boldsymbol{\nabla} \times \mathbf{G}) \\
\boldsymbol{\nabla} \times (\mathbf{F} \times \mathbf{G}) &= \mathbf{F}(\boldsymbol{\nabla} \cdot \mathbf{G}) - \mathbf{G}(\boldsymbol{\nabla} \cdot \mathbf{F}) + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} - (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G}
\end{aligned}
$$

If $\mathbf{x} = r\mathbf{e}_r$ is the position vector with respect to some origin, then

$$\boldsymbol{\nabla} \cdot \mathbf{x} = 3\,, \quad \boldsymbol{\nabla} \times \mathbf{x} = 0\,, \quad \boldsymbol{\nabla} \cdot \mathbf{e}_r = \frac{2}{r}\,, \quad \boldsymbol{\nabla} \times \mathbf{e}_r = 0$$

The Divergence Theorem and Corollaries

In the following V is a volume bounded by a closed surface S,

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F}\, dV = \int_S \mathbf{F} \cdot d\mathbf{S} \qquad \text{(Divergence theorem)}$$

$$\int_V \boldsymbol{\nabla} f\, dV = \int_S f\, d\mathbf{S}$$

$$\int_V \boldsymbol{\nabla} \times \mathbf{F}\, dV = \int_S d\mathbf{S} \times \mathbf{F}$$

$$\int_V \left(f\nabla^2 g + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g\right) dV = \int_S f \boldsymbol{\nabla} g \cdot d\mathbf{S} \qquad \text{(Green's first identity)}$$

$$\int_V \left(f\nabla^2 g - g\nabla^2 f\right) dV = \int_S (f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f) \cdot d\mathbf{S} \qquad \text{(Green's theorem)}$$

Stoke's Theorem and a Corollary

In the following S is an open surface bounded by the contour C,

$$\int_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{x} \qquad \text{(Stoke's theorem)}$$

$$\int_S d\mathbf{S} \times \boldsymbol{\nabla} f = \oint_C f\, d\mathbf{x}$$

Common Vector Differential Quantities in Polar Coordinates

Cylindrical Polars (s, ϕ, z)

$$
\begin{aligned}
\boldsymbol{\nabla} f &= \frac{\partial f}{\partial s}\mathbf{e}_s + \frac{1}{s}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z \\
\boldsymbol{\nabla}\cdot\mathbf{F} &= \frac{1}{s}\frac{\partial}{\partial s}(sF_s) + \frac{1}{s}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \\
\boldsymbol{\nabla}\times\mathbf{F} &= \left(\frac{1}{s}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\mathbf{e}_s + \left(\frac{\partial F_s}{\partial z} - \frac{\partial F_z}{\partial s}\right)\mathbf{e}_\phi \\
&\quad + \left(\frac{1}{s}\frac{\partial}{\partial s}(sF_\phi) - \frac{1}{s}\frac{\partial F_s}{\partial \phi}\right)\mathbf{e}_z \\
\nabla^2 f &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \\
\nabla^2\mathbf{F} &= \left(\nabla^2 F_s - \frac{F_s}{s^2} - \frac{2}{s^2}\frac{\partial F_\phi}{\partial \phi}\right)\mathbf{e}_s \\
&\quad + \left(\nabla^2 F_\phi + \frac{2}{s^2}\frac{\partial F_s}{\partial \phi} - \frac{F_\phi}{s^2}\right)\mathbf{e}_\phi + \left(\nabla^2 F_z\right)\mathbf{e}_z
\end{aligned}
$$

Spherical Polars (r, θ, ϕ)

$$
\begin{aligned}
\boldsymbol{\nabla} f &= \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi \\
\boldsymbol{\nabla}\cdot\mathbf{F} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} \\
\boldsymbol{\nabla}\times\mathbf{F} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(\sin\theta F_\phi) - \frac{\partial F_\theta}{\partial \phi}\right)\mathbf{e}_r \\
&\quad + \left(\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rF_\phi)\right)\mathbf{e}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta) - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right)\mathbf{e}_\phi \\
\nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \\
\nabla^2\mathbf{F} &= \left(\nabla^2 F_r - \frac{2F_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) - \frac{2}{r^2\sin\theta}\frac{\partial F_\phi}{\partial \phi}\right)\mathbf{e}_r \\
&\quad + \left(\nabla^2 F_\theta + \frac{2}{r^2}\frac{\partial F_r}{\partial \theta} - \frac{F_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\phi}{\partial \phi}\right)\mathbf{e}_\theta \\
&\quad + \left(\nabla^2 F_\phi + \frac{2}{r^2\sin\theta}\frac{\partial F_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\theta}{\partial \phi} - \frac{F_\phi}{r^2\sin^2\theta}\right)\mathbf{e}_\phi
\end{aligned}
$$

The Navier-Stokes Equations in Cylindrical Polar Coordinates

The Navier-Stokes equations in cylindrical polars (s, ϕ, z) are:

$$\frac{\partial u_s}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})u_s - \frac{u_\phi^2}{s} \;=\; -\frac{1}{\rho}\frac{\partial p}{\partial s} + \nu\left(\nabla^2 u_s - \frac{u_s}{s^2} - \frac{2}{s^2}\frac{\partial u_\phi}{\partial \phi}\right) \,,$$

$$\frac{\partial u_\phi}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})u_\phi + \frac{u_s u_\phi}{s} \;=\; -\frac{1}{\rho s}\frac{\partial p}{\partial \phi} + \nu\left(\nabla^2 u_\phi - \frac{u_\phi}{s^2} + \frac{2}{s^2}\frac{\partial u_s}{\partial \phi}\right) \,,$$

$$\frac{\partial u_z}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})u_z \;=\; -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z \,,$$

$$\frac{1}{s}\frac{\partial}{\partial s}(s u_s) + \frac{1}{s}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} \;=\; 0 \,.$$

where

$$\mathbf{u}\cdot\boldsymbol{\nabla} f \;=\; u_s\frac{\partial f}{\partial s} + \frac{u_\phi}{s}\frac{\partial f}{\partial \phi} + u_z\frac{\partial f}{\partial z} \,.$$