4H FLUID DYNAMICS

4. Waves and Instability

4.1 Introduction

In this section, we turn our attention to the time-dependence of flows. In particular, we shall be interested in investigating what happens to a steady (not necessarily stationary) *basic state* when it is given an infinitessimal perturbation. That the perturbation is infinitessimal, permits a linearisation of the governing equations about the basic state; simplifying the mathematical analysis. In an ideal fluid, the system can respond to the perturbation in two ways:

1. The forces generated by the perturbation can act against the perturbation, i.e. to reduce it and hence try to restore the system to its original basic state. When no damping is present, the system will generally overshoot the basic state, leading to an oscillation or a wave motion.
2. The forces generated by the perturbation can act in the direction of the perturbation, i.e to amplify it, leading to instability.

In case (1), the system remains close to the basic state. In case (2), the perturbation grows exponentially with time (until the linearisation approximation breaks down).

Examples

Here, we give some examples of the *restoring forces* that give rise to oscillations or wave motion:

Gravity

Pressure

Surface Tension

In the case of gravity, it can also act to amplify a perturbation, giving instability. This happens when the heavy fluid overlies the lighter fluid.

4.2 Internal Gravity Waves

Consider a fluid with density distribution $\rho_0(y)$, with gravity $\mathbf{g} = -g\hat{\mathbf{y}}$. Neglecting viscous effects, the motion of the fluid is described by the Euler equations

$$\rho\frac{D\mathbf{u}}{Dt} \;=\; -\nabla p + \rho\mathbf{g}\,, \qquad \nabla\cdot\mathbf{u} \;=\; 0\,, \tag{4.1}$$

together with a condition on the density ρ. We shall assume that the density of an individual fluid parcel does not change with time (i.e. there is no diffusion of density), giving

$$\frac{D\rho}{Dt} \;=\; 0\,. \tag{4.2}$$

The *equilibrium* or *basic state* has $\mathbf{u} = \mathbf{0}$ and is described by

$$\mathbf{0} \;=\; -\nabla p_0 + \rho_0\mathbf{g} \qquad \Rightarrow \qquad 0 \;=\; -\frac{dp_0}{dy} - \rho_0(y)\,g\,. \tag{4.3}$$

Given $\rho_0(y)$, this determines the "hydrostatic" pressure $p_0(y)$.

Let us consider a 2-dimensional perturbation $\mathbf{u}_1 = \Big(u_1(x,y,t), v_1(x,y,t), 0\Big)$, $p_1(x,y,t)$, $\rho_1(x,y,t)$ to the basic state $\mathbf{u} = \mathbf{u}_0 = \mathbf{0}$, $p = p_0(y)$, $\rho = \rho_0(y)$, such that

$$\mathbf{u} \;=\; \mathbf{u}_1\,, \quad p \;=\; p_0 + p_1\,, \quad \rho \;=\; \rho_0 + \rho_1\,. \tag{4.4}$$

We shall assume that $|p_1| \ll |p_0|$, $|\rho_1| \ll |\rho_0|$ and that $|\mathbf{u}_1|$ is small (with $\rho_0\partial\mathbf{u}_1/\partial t$ of the same order as ∇p_1, so that the nonlinear terms ($\mathbf{u}_1\cdot\nabla\mathbf{u}_1$ and $\mathbf{u}_1\cdot\nabla\rho_1$) can be neglected. Then, (4.1) and (4.2) give

$$\rho_0\frac{\partial\mathbf{u}_1}{\partial t} \;=\; -\nabla p_0 - \nabla p_1 + (\rho_0+\rho_1)\,\mathbf{g}\,,$$
$$\nabla\cdot\mathbf{u}_1 \;=\; 0\,, \qquad \frac{\partial\rho_1}{\partial t} + \mathbf{u}_1\cdot\nabla\rho_0 \;=\; 0\,.$$

Given that the basic state satisfies (4.3), this becomes

$$\begin{aligned}\rho_0\frac{\partial\mathbf{u}_1}{\partial t} \;&=\; -\nabla p_1 + \rho_1\mathbf{g}\,,\\ \nabla\cdot\mathbf{u}_1 \;=\; 0\,, \qquad \frac{\partial\rho_1}{\partial t} &+ \mathbf{u}_1\cdot\nabla\rho_0 \;=\; 0\,.\end{aligned} \tag{4.5}$$

Writing these equations in component form gives

$$\begin{aligned}\rho_0\frac{\partial u_1}{\partial t} \;=\; -\frac{\partial p_1}{\partial x}\,, \qquad &\rho_0\frac{\partial v_1}{\partial t} \;=\; -\frac{\partial p_1}{\partial y} - \rho_1 g\,,\\ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \;=\; 0\,, \qquad &\frac{\partial\rho_1}{\partial t} + v_1\frac{d\rho_0}{dy} \;=\; 0\,.\end{aligned} \tag{4.6}$$

The equations (4.6) are linear in the dependent variables u_1, v_1, p_1 and ρ_1 and the coefficients in the equation are functions only of y. Since the coefficients are independent of x and t, we can seek solutions of the form

$$u_1(x,y,t) \;=\; \tilde{u}_1(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \tag{4.7}$$

and similarly for v_1, p_1 and ρ_1. Substituting this into (4.6) gives

$$\begin{aligned} -\rho_0\mathrm{i}\omega\tilde{u}_1 \;&=\; -\mathrm{i}k\tilde{p}_1\,, \qquad & -\rho_0\mathrm{i}\omega\tilde{v}_1 \;&=\; -\frac{d\tilde{p}_1}{dy} - \tilde{\rho}_1 g\,, \\ \mathrm{i}k\tilde{u}_1 + \frac{d\tilde{v}_1}{dy} \;&=\; 0\,, \qquad & -\mathrm{i}\omega\tilde{\rho}_1 + \tilde{v}_1\frac{d\rho_0}{dy} \;&=\; 0\,. \end{aligned} \tag{4.8}$$

Eliminating $\tilde{u}_1$, $\tilde{p}_1$ and $\tilde{\rho}_1$ gives

$$\frac{d^2\tilde{v}_1}{dy^2} + \frac{\rho_0'}{\rho_0}\frac{d\tilde{v}_1}{dy} + k^2\left(\frac{N^2}{\omega^2} - 1\right)\tilde{v}_1 \;=\; 0\,, \tag{4.9}$$

where

$$N^2 \;=\; -g\frac{\rho_0'}{\rho_0}\,, \qquad \text{and} \qquad \rho_0' \;=\; \frac{d\rho_0}{dy}\,. \tag{4.10}$$

When $\rho_0' < 0$ (i.e. light fluid lies over heavier fluid), N is known as the *buoyancy frequency* or Brunt-Väisälä frequency.

Equation (4.9) is a second-order ODE in y. The coefficients are also functions of y through ρ_0'/ρ_0. To illustrate the wave character of this problem, we choose the simplest example where ρ_0'/ρ_0 is a constant $(= -1/H)$ i.e. $\rho_0 = \rho_c \exp(-y/H)$. The length H is known as the *scale height* of the density variation. Then (4.9) becomes (with $N^2 = g/H$)

$$\frac{d^2\tilde{v}_1}{dy^2} - \frac{1}{H}\frac{d\tilde{v}_1}{dy} + k^2\left(\frac{N^2}{\omega^2} - 1\right)\tilde{v}_1 \;=\; 0\,. \tag{4.11}$$

This may be easily solved to give

$$v_1 \;=\; \mathrm{e}^{y/2H}\mathrm{e}^{\mathrm{i}(kx+ly-\omega t)}\,, \tag{4.12}$$

where we have assumed

$$l^2 \;=\; k^2\left(\frac{N^2}{\omega^2} - 1\right) - \frac{1}{4H^2} \;>\; 0\,.$$

Rewriting, this gives

$$\omega^2 \;=\; \frac{N^2k^2}{k^2 + l^2 + (1/4H^2)}\,. \tag{4.13}$$

This relationship between the *frequency* ω and the *wave vector* $\mathbf{k} = (k,l)$ is known as a *dispersion relation.*

Short wavelength limit

Often the wavelength will be short compared with the scale height H, i.e. $(k^2+l^2)^{-1/2} \ll H$. Then (4.13) gives

$$\omega \;\approx\; \pm\frac{Nk}{(k^2+l^2)^{1/2}}\,. \tag{4.14}$$

Then, the group velocity

$$\mathbf{c}_g \;=\; \left(\frac{\partial\omega}{\partial k},\frac{\partial\omega}{\partial l}\right) \;=\; \pm\frac{Nl}{(k^2+l^2)^{3/2}}\big(l,-k\big)\,, \tag{4.15}$$

while the phase velocity $\mathbf{c}$ has magnitude $\omega/(k^2+l^2)^{1/2}$ and is in the direction normal to the wave front $kx+ly=$const., i.e. in the direction $\nabla(kx+ly)=(k,l)$. Hence

$$\mathbf{c} \;=\; \pm\frac{Nk}{(k^2+l^2)^{3/2}}\big(k,l\big)\,. \tag{4.16}$$

An important feature to note for these internal gravity waves is that $\mathbf{c}\cdot\mathbf{c}_g=0$, i.e. that a wave packet is observed to travel perpendicular to that direction of travel of individual wave crests.

[For a discussion of wave phase and group velocities, see Acheson Chapter 3.]

4.3 Surface Waves

In Section 4.2, we considered gravity as the restoring force in a situation where the basic state density varied continously throughout the fluid. We now consider the case where the density variation is concentrated at $y=0$, i.e. where we have one fluid of density ρ_1 occupying the region $y>0$ and another of density ρ_2 occupying $y<0$, with ρ_1 and ρ_2 both independent of y. Then, there is no buoyancy force within either fluid (as each has constant density) and hence no restoring force. If we restrict our attention to *inviscid*, *irrotational* fluids, then

$$\nabla\cdot\mathbf{u} \;=\; 0 \quad\text{and}\quad \mathbf{u} \;=\; \nabla\phi\,, \qquad \text{so}\quad \nabla^2\phi \;=\; 0\,, \tag{4.17}$$

describes the flow inside each layer of fluid. The motion of the system is then driven by the density difference across the interface between the two fluids. The conditions at the interface are then crucial in determining the flow.

In the basic state, the interface is at $y = 0$. A perturbation displaces this to

$$y \ = \ \eta(x,t)\,, \tag{4.18}$$

assuming a 2-dimensional system with $\mathbf{g} = -g\hat{\mathbf{y}}$ and $\mathbf{u} = \big(u(x,y,t), v(x,y,t), 0\big)$ Hence, while the problem inside each layer is straightforward, requiring a solution of (4.17), connecting the solutions in the two layers requires some care because the interface is not fixed. In fact the position of the interface (4.18) must emerge as part of the solution to the whole problem.

To simplify the problem, let us consider the case where the upper fluid is air.

The Free-Surface Condition

Let us define

$$F \ = \ y - \eta(x,t)\,. \tag{4.19}$$

For any fluid parcel on the free surface, $F = 0$. Moreover, since all parcels on the surface remain on the surface, we can say that

$$\frac{DF}{Dt} \ = \ \frac{\partial F}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla} F \ = \ 0 \qquad \text{on } y = \eta\,.$$

Hence

$$-\frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} + v \ = \ 0 \qquad \text{on } y = \eta\,. \tag{4.20}$$

Note: Two special cases of (4.20) are:

(i) A horizontal surface, for which (4.20) becomes $v = \partial\eta/\partial t$.

(ii) A stationary surface, for which (4.20) gives $v/u = \partial\eta/\partial x$.

Pressure Condition at the Free surface

Euler's equation (1.20) for an unsteady flow becomes

$$\frac{\partial}{\partial t}\boldsymbol{\nabla}\phi \ = \ -\boldsymbol{\nabla}\left(\frac{p}{\rho} + \chi + \tfrac{1}{2}\mathbf{u}^2\right)\,, \tag{4.21}$$

for an irrotational flow. Here $\chi = gy$. Integrating gives

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gy + \tfrac{1}{2}\mathbf{u}^2 \ = \ G(t)\,, \tag{4.22}$$

where $G(t)$ is an arbitrary function of time. [Note: taking ∇(4.22) gives (4.21).]

For all practical purposes, the pressure at the free surface is the atmospheric pressure p_0. We are free to choose $G(t)$ since the choice does not affect $\mathbf{u} = \nabla\phi$. Choosing $G(t) = p_0/\rho$, (4.22) becomes

$$\frac{\partial \phi}{\partial t} + gy + \tfrac{1}{2}(u^2 + v^2) \;=\; 0\,, \qquad \text{on } y = \eta\,. \tag{4.23}$$

Summary

The problem for surface water waves consists of solving (4.17) in the fluid, subject to the surface conditions (4.20) and (4.23), i.e.

$$\nabla^2\phi \;=\; 0\,, \tag{4.24a}$$

subject to

$$-\frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} + v \;=\; 0 \qquad \text{on } y = \eta\,, \tag{4.24b}$$

and

$$\frac{\partial \phi}{\partial t} + gy + \tfrac{1}{2}(u^2 + v^2) \;=\; 0\,, \qquad \text{on } y = \eta\,, \tag{4.24c}$$

where $\mathbf{u} = \nabla\phi$ and the surface is $y = \eta(x,t)$.

Linearisation

As with the internal gravity wave problem considered in Section 4.2, we shall simplify the problem (4.24) by considering an infinitessimal perturbation to the interface, permitting a linearisation of the problem about the basic state $\mathbf{u} = \mathbf{0}$, $\eta = 0$. The equation (4.24a) describing the flow is already linear. It is the surface conditions (4.24b) and (4.24c) that are nonlinear.

Consider first (4.24b). Assuming all displacements and velocities are small, such that we can neglect squares and products of u, v and η, (4.24b) becomes

$$v(x,\eta,t) \;=\; \frac{\partial \eta}{\partial t}\,. \tag{4.25}$$

Expanding the left-hand side as a Taylor series about $\eta = 0$ gives

$$v(x,0,t) + \eta\frac{\partial v}{\partial \eta}(x,0,t) + \ldots \;=\; \frac{\partial \eta}{\partial t}\,.$$

Consistent with the earlier linearisation of this equation, we neglect products of small quantities, so since v and η are small, we have

$$v(x,0,t) \;=\; \frac{\partial \eta}{\partial t}\,, \tag{4.26}$$

or

$$v = \frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}, \qquad \text{on } y = 0. \tag{4.27}$$

A similar treatment of (4.24c) gives

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \qquad \text{on } y = 0. \tag{4.28}$$

The linear problem then consists of solving (4.24a) subject to (4.27) and (4.28).

Solution of the Linear Problem

Let us seek a solution of the form

$$\eta = A\cos(kx - \omega t), \tag{4.29}$$

where A is the (small) amplitude of the perturbation to the surface. In a linear theory, A is undetermined.

The choice (4.29) requires

$$\phi = f(y)\sin(kx - \omega t), \tag{4.30}$$

to be consistent with (4.27) and (4.28).

Substituting (4.30) into (4.24a) gives

$$f'' - k^2 f = 0, \tag{4.31}$$

which has general solution

$$f = Ce^{ky} + De^{-ky}, \tag{4.32}$$

where C and D are constants. Without loss of generality we can take $k > 0$.

Solution for Infinite Depth

The problem is simplest for the case of a fluid of infinite depth. (The case of finite depth is dealt with in Question 1 of Problems 4.)

Since $\mathbf{u}$ must be bounded in the limit $y \to -\infty$, we must choose $D = 0$ in (4.32). Then we have

$$\phi = Ce^{ky}\sin(kx - \omega t). \tag{4.33}$$

Now, substituting the expressions (4.33) for ϕ and (4.29) for η into the surface conditions (4.27) and (4.28) gives

$$Ck\sin(kx - \omega t) = \omega A\sin(kx - \omega t),$$

and

$$-\omega C\cos(kx - \omega t) = -gA\cos(kx - \omega t),$$

so that

$$Ck = \omega A, \qquad \text{and} \qquad \omega C = gA.$$

Hence we have

$$\phi = \frac{\omega}{k} A e^{ky} \sin(kx - \omega t), \tag{4.34}$$

with

$$\omega^2 = gk. \tag{4.35}$$

Note that A remains as an undetermined constant. If the surface perturbation is given, then this determines the amplitude of the disturbance for $y < 0$. Equation (4.35) is the dispersion relation for surface water waves in deep water, from which we can derive the wave speed

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}}, \tag{4.36}$$

and the group speed

$$c_g = \frac{d\omega}{dk} = \tfrac{1}{2}\sqrt{\frac{g}{k}}. \tag{4.37}$$

Particle Paths

The solution (4.34) gives us the solution for the flow in $y < 0$ corresponding to the disturbance (4.29) to the surface. Using $\mathbf{u} = \nabla\phi$ gives

$$\begin{aligned} \frac{dx}{dt} &= u = \frac{\partial\phi}{\partial x} = \frac{\omega}{k} A e^{ky} k \cos(kx - \omega t), \\ \frac{dy}{dt} &= v = \frac{\partial\phi}{\partial y} = \frac{\omega}{k} A k e^{ky} \sin(kx - \omega t). \end{aligned} \tag{4.38}$$

If we assume that any particle departs from its mean position $(\overline{x}, \overline{y})$ by only a small amount (x', y') (i.e. $x = \overline{x} + x'$, $y = \overline{y} + y'$), then we may find its position as a function of time by integrating

$$\begin{aligned} \frac{dx'}{dt} &= \omega A e^{k\overline{y}} \cos(k\overline{x} - \omega t), \\ \frac{dy'}{dt} &= \omega A e^{k\overline{y}} \sin(k\overline{x} - \omega t), \end{aligned}$$

to get

$$\begin{aligned} x' &= -A e^{k\overline{y}} \sin(k\overline{x} - \omega t), \\ y' &= A e^{k\overline{y}} \cos(k\overline{x} - \omega t). \end{aligned} \tag{4.39}$$

Note that $x'^2 + y'^2 = A^2 e^{2k\overline{y}}$, so the particle paths are circles with radius decreasing exponentially with depth.

Limitation of Linear Theory

In the above analysis, we neglected the term $u\partial\eta/\partial x$ compared with the term v in (4.24b) to get (4.25). From our solution (4.38), we can see that u and v are comparable in magnitude. Hence, our analysis can only be valid if $|\partial\eta/\partial x| \ll 1$, i.e. if $kA \ll 1$. This can be interpreted as saying that, for the validity of our linear analysis, we require the slope of the surface displacement to be small *or* the surface displacement must be small compared with the wavelength of the disturbance.

Surface Tension

In Section 4.1 we mentioned surface tension as a possible restoring force that could lead to wave motion. In the above analysis of surface waves, for simplicity, we neglected that effect. Here we introduce the modification to the pressure condition at the interface that is required to include the effect of surface tension.

At the interface between two fluids (for example water and air), a *surface tension* force T acts, directed tangentially to the surface.

In our example of a surface $y = \eta(x,t)$, the vertical component of the force is $T\partial\eta/\partial s$ where s is the distance along the surface. For small wave amplitudes, this is approximately $T\partial\eta/\partial x$. Consider a small element, length δx, of the surface. It experiences surface tension forces at both of its ends. The net force on δx is then

$$T\frac{\partial\eta}{\partial x}\bigg|_{x+\delta x} - \; T\frac{\partial\eta}{\partial x}\bigg|_{x} \;\approx\; T\frac{\partial^2\eta}{\partial x^2}\delta x\,. \tag{4.40}$$

This gives the net upward force per unit surface area as

$$T\frac{\partial^2\eta}{\partial x^2}\,.$$

This must be balanced by the difference between the pressure in the upper layer and that in the lower layer at the interface. When the upper layer is air, we then have, instead of $p = p_0$,

$$p \;=\; p_0 - T\frac{\partial^2\eta}{\partial x^2} \qquad \text{at } y \;=\; \eta(x,t)\,, \tag{4.41}$$

where p is the pressure in the fluid at the interface. Equation (4.24c) is then modified to

$$\frac{\partial\phi}{\partial t} - \frac{T}{\rho_0}\frac{\partial^2\eta}{\partial x^2} + gy + \tfrac{1}{2}(u^2+v^2) \;=\; 0\,, \qquad \text{on } y = \eta\,. \tag{4.42}$$

4.4 Instability

In the examples of Sections 4.2 and 4.3, with light fluid above heavy fluid, gravity acts as a restoring force, leading to wave motion. If the heavy fluid overlies the light fluid, gravity acts to amplify any perturbation, leading to instability.

Other effects can be included that can compete with gravity. For example, surface tension can act as a restoring force while gravity is acting to amplify a perturbation. Question 3 of Problems 4 considers the case where both surface tension and gravity are present.

Another important source of energy to drive an instability is the kinetic energy of a flow. So far, we have considered basic states that are stationary. It is equally possible to consider basic states where the basic flow $\mathbf{u}_0$ is nonzero. The analysis goes through very much as before provided $\mathbf{u}_0$ is steady. On Problems 4, Questions 4 and 5 look at non-zero basic flows. Question 5 additionally includes the effect of surface tension.

4.5 Instability of Plane-Parallel Flows

In Section 2, we solved for simple flows of the form $U(y)\hat{\mathbf{x}}$. Here, we consider the stability of such flows, taking

$$\mathbf{u}_0 \;=\; U(y)\hat{\mathbf{x}}\,, \tag{4.43}$$

and for simplicity take the density $\rho = \rho_0 =$ const. The flow is confined in the plane layer $0 < y < d$. We consider a 2-dimensional disturbance, so that

$$\mathbf{u} \;=\; \mathbf{u}_0(y) + \mathbf{u}_1(x, y, t)\,, \quad p \;=\; p_0 + p_1(x, y, t)\,, \tag{4.44}$$

with $\mathbf{u}_1 = (u_1, v_1, 0)$.

The Navier-Stokes equations, neglecting gravity,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} \;=\; -\frac{1}{\rho_0}\boldsymbol{\nabla} p + \nu\nabla^2\mathbf{u}\,, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} \;=\; 0\,, \tag{4.45}$$

become, in component form

$$\begin{aligned}
\frac{\partial u_1}{\partial t} + (U+u_1)\frac{\partial u_1}{\partial x} + v_1\frac{\partial}{\partial y}(U+u_1) \;&=\; -\frac{1}{\rho_0}\left(\frac{\partial p_0}{\partial x} + \frac{\partial p_1}{\partial x}\right) + \nu\nabla^2(U+u_1)\,,\\
\frac{\partial v_1}{\partial t} + (U+u_1)\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_1}{\partial y} \;&=\; -\frac{1}{\rho_0}\left(\frac{\partial p_0}{\partial y} + \frac{\partial p_1}{\partial y}\right) + \nu\nabla^2 v_1\,,\\
\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \;&=\; 0\,.
\end{aligned} \tag{4.46}$$

In the absence of a perturbation we have

$$0 = -\frac{1}{\rho_0}\frac{\partial p_0}{\partial x} + \nu\frac{d^2U}{dy^2}\,, \qquad 0 = -\frac{1}{\rho_0}\frac{\partial p_0}{\partial y}\,. \tag{4.47}$$

Substituting this into (4.46) gives

$$\begin{aligned}
\frac{\partial u_1}{\partial t} + (U+u_1)\frac{\partial u_1}{\partial x} + v_1\frac{\partial}{\partial y}(U+u_1) &= -\frac{1}{\rho_0}\frac{\partial p_1}{\partial x} + \nu\nabla^2 u_1\,,\\
\frac{\partial v_1}{\partial t} + (U+u_1)\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_1}{\partial y} &= -\frac{1}{\rho_0}\frac{\partial p_1}{\partial y} + \nu\nabla^2 v_1\,,\\
\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} &= 0\,.
\end{aligned} \tag{4.48}$$

Now, linearising on the basis of $|u_1| \ll |U|$ gives

$$\begin{aligned}
\frac{\partial u_1}{\partial t} + U\frac{\partial u_1}{\partial x} + v_1\frac{dU}{dy} &= -\frac{1}{\rho_0}\frac{\partial p_1}{\partial x} + \nu\nabla^2 u_1\,,\\
\frac{\partial v_1}{\partial t} + U\frac{\partial v_1}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p_1}{\partial y} + \nu\nabla^2 v_1\,,\\
\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} &= 0\,.
\end{aligned} \tag{4.49}$$

We now proceed as for the example of internal gravity waves, writing

$$u_1(x,y,t) = \tilde{u}_1(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \tag{4.50}$$

and similarly for v_1 and p_1. Then, (4.49) become

$$\begin{aligned}
-\mathrm{i}\omega\tilde{u}_1 + U\mathrm{i}k\tilde{u}_1 + U'\tilde{v}_1 &= -\mathrm{i}k\frac{\tilde{p}_1}{\rho_0} + \nu\left(\frac{d^2}{dy^2} - k^2\right)\tilde{u}_1\,,\\
-\mathrm{i}\omega\tilde{v}_1 + U\mathrm{i}k\tilde{v}_1 &= -\frac{1}{\rho_0}\frac{d\tilde{p}_1}{dy} + \nu\left(\frac{d^2}{dy^2} - k^2\right)\tilde{v}_1\,,\\
\mathrm{i}k\tilde{u}_1 + \frac{d\tilde{v}_1}{dy} &= 0\,.
\end{aligned} \tag{4.51}$$

For the case where viscous effects are negligible, we can eliminate $\tilde{u}_1$ and $\tilde{p}_1$ to get (with $c = \omega/k$)

$$\begin{aligned}
(\mathrm{i}kU - \mathrm{i}\omega)\tilde{v}_1 &= -\frac{1}{\rho_0}\frac{d}{dy}\left[\frac{\mathrm{i}\rho_0}{k}\left((\mathrm{i}kU - \mathrm{i}\omega)\frac{\mathrm{i}}{k}\frac{d\tilde{v}_1}{dy} + U'\tilde{v}_1\right)\right]\\
\Rightarrow \quad k^2(U-c)\tilde{v}_1 &= -\left(-(U-c)\frac{d^2\tilde{v}_1}{dy^2} - U'\frac{d\tilde{v}_1}{dy} + U'\frac{d\tilde{v}_1}{dy} + U''\tilde{v}_1\right)\\
\Rightarrow \quad (U-c)\left(\frac{d^2\tilde{v}_1}{dy^2} - k^2\tilde{v}_1\right) - U''\tilde{v}_1 &= 0\,.
\end{aligned} \tag{4.52}$$

This is Rayleigh's equation (for a constant density fluid). To determine the stability of the flow it must be sloved subject to the boundary conditions

$$\tilde{v}_1(0) \;=\; \tilde{v}_1(d) \;=\; 0\,. \tag{4.53}$$

In general c will be found to be complex $c = c_r + \mathrm{i}c_i$. The real part determines the wave character of a perturbation while the sign of c_i determines whether the perturbation grows or decays. Solutions are described as *neutrally stable* if $c_i = 0$. Then, there is the possibility of a singularity in the equation if $U(y) = c_r$ for some y in $0 < y < d$. In practice, such a singularity is removed by viscous effects, see later.

The Inflection Point Theorem

Using (4.52) and (4.53) we can derive a simple criterion that determines whether or not instability is possible.

Firstly, we assume $c_i \neq 0$. Then we multiply (4.52) through by $\tilde{v}_1^*/(U-c)$ (where a * denotes complex conjugate) and integrate from 0 to d to get

$$\int_0^d \left(\frac{d^2\tilde{v}_1}{dy^2}\tilde{v}_1^* - k^2\tilde{v}_1\tilde{v}_1^* - \frac{U''\tilde{v}_1\tilde{v}_1^*}{U-c} \right) dy \;=\; 0\,.$$

Integrating the first term by parts, and using (4.53) (which also must apply to $\tilde{v}_1^*$), we get

$$\int_0^d \left(\left| \frac{d\tilde{v}_1}{dy} \right|^2 + k^2|\tilde{v}_1|^2 + U''(U - c_r + \mathrm{i}c_i)\frac{|\tilde{v}_1|^2}{|U-c|^2} \right) dy \;=\; 0\,. \tag{4.54}$$

The imaginary part of this requires (since $c_i \neq 0$), U'' must change sign somewhere in the interval $(0, d)$, i.e. for instability, $U(y)$ must have an inflection point.

Experimental results

Experiments have been conducted to test the theoretical prediction of the inflection point theorem. When the basic flow U has an inflection point, instability will be observed. When U does not have an inflection point, instability can still be found.

The explanation for this lies in the role of viscosity. Any experiment must use a viscous fluid (even though the viscosity may be very small).

The Effect of Viscosity - The Orr-Sommerfeld Equation

If we consider moving from an inviscid system to a viscous one, it turns out that viscosity can act in two quite distinct ways (which are not mutually exclusive):

1. The first is the expected role. Viscosity acts to resist motion and so will have a damping effect on any instability. Only if the energy source driving instability is large enough to overcome the viscous damping can a a perturbation grow.

2. The second is unexpected. Viscosity gives the system additional freedom to extract energy from the basic flow U. In the inviscid 2D system the vorticity of each fluid parcel is conserved [see (1.23)]. In the viscous system, vorticity can diffuse. The viscous system is therefore less constrained and can be unstable even when the inviscid system is stable. Mathematically, the viscous system is described by a fourth-order equation (see below) while the inviscid system is described by a second-order one (4.52).

The Orr-Sommmerfeld Equation

If we take the viscous system (4.51) and use the stream function representation

$$\tilde{u}_1 \;=\; \mathcal{U}\frac{\partial\psi}{\partial\tilde{y}} \qquad \tilde{v}_1 \;=\; -\mathcal{U}\frac{\partial\psi}{\partial\tilde{x}} \;=\; -\mathcal{U}\mathrm{i}\tilde{k}\psi\,,$$

where we use the non-dimensionalisation

$$x=\tilde{x}\,d\,,\quad y=\tilde{y}\,d\,,\quad k=\tilde{k}/d\,,\quad c=\tilde{c}\,\mathcal{U}\,,\qquad U=\tilde{U}\,\mathcal{U}$$

then, instead of Rayleigh's equation, we obtain

$$-\frac{1}{\mathrm{i}\tilde{k}R}\left(\frac{d^2}{d\tilde{y}^2}-\tilde{k}^2\right)^2\psi+(\tilde{U}-\tilde{c})\left(\frac{d^2}{d\tilde{y}^2}-\tilde{k}^2\right)\psi-\tilde{U}''\psi \;=\; 0\,, \qquad (4.55)$$

where

$$R \;=\; \frac{\mathcal{U}d}{\nu}$$

is the Reynolds number and $\mathcal{U}$ is a typical magnitude of the basic flow U. Equation (4.55) is the Orr-Sommerfeld equation.

When $R \gg 1$ it can be reduced to Rayleigh's equation, except where $\tilde{U}-\tilde{c}$ is close to zero. A location where $\tilde{U}=\tilde{c}$ is known as a *critical layer*. Solutions of (4.55) must be found numerically and are a severe test of numerical methods when $R >> 1$.

Example - Plane Poiseuille Flow

For plane poiseuille flow (flow between two parralle plates driven by a pressure gradient)

$$\tilde{U} \;=\; y(d-y)\,, \qquad \text{and} \qquad \tilde{U}'' \;=\; \text{const.}\ (\,=-1)\,,$$

hence there is no inflection point in the flow.

The stability diagram looks like

and numerical solution gives $R_c = 5772.2$ and $k_c = 1.0205$.