Field Theory

Answer for 6th Set of Problems

DYNAMICS OF RELATIVISTIC PARTICLES & E-M FIELDS

1. **(a)** The equation governing the motion of the particle is the covariant Lorentz force $dU^\alpha/d\tau = e/(mc)F^{\alpha\beta}U_\beta$, which for the case $\vec{B} = 0$ can be written as

$$\frac{d(\gamma c)}{d\tau} = \frac{\gamma e}{mc}\vec{E}\cdot\vec{v}, \quad \frac{d(\gamma\vec{v})}{d\tau} = \frac{\gamma e}{m}\vec{E}.$$

Decomposing the velocity into components $v_\parallel$ parallel and $v_\perp$ orthogonal to $\vec{E}$, we obtain

$$\begin{aligned}
\frac{d(\gamma c)}{d\tau} &= \frac{eE}{mc}(\gamma v_\parallel) = \eta(\gamma v_\parallel), \\
\frac{d(\gamma v_\parallel)}{d\tau} &= \frac{eE}{mc}(\gamma c) = \eta(\gamma c), \\
\frac{d(\gamma v_\perp)}{d\tau} &= 0,
\end{aligned}$$

where $\eta \equiv eE/mc$. From the last equation it is obvious that $\gamma v_\perp = \alpha$ with some constant α. The most general solution of the first two equations is given by

$$\begin{aligned}
\gamma c &= A\cosh(\eta\tau) + B\sinh(\eta\tau), \\
\gamma v_\parallel &= A\sinh(\eta\tau) + B\cosh(\eta\tau),
\end{aligned}$$

where A and B are constants. Using the initial conditions $v_\parallel = 0$ and $v_\perp = v_0$ at $\tau = 0$, we find

$$\alpha = \gamma_0 v_0, \quad A = \gamma_0 c, \quad B = 0,$$

where $\gamma_0 = 1/\sqrt{1 - v_0^2/c^2}$. Thus we get

$$\begin{aligned}
\gamma &= \gamma_0\cosh(\eta\tau), \\
v_\parallel &= c\tanh(\eta\tau), \\
v_\perp &= \frac{v_0}{\cosh(\eta\tau)}.
\end{aligned}$$

In order to express these results as functions of the laboratory time t, we use the relation $dt = \gamma d\tau$ to find

$$t = \int_0^\tau \gamma d\tau = \int_0^\tau \gamma_0\cosh(\eta\tau)d\tau = \frac{\gamma_0}{\eta}\sinh(\eta\tau).$$

Therefore,

$$\sinh(\eta\tau) = \frac{\eta t}{\gamma_0}, \quad \cosh(\eta\tau) = \sqrt{1 + \frac{\eta^2 t^2}{\gamma_0^2}}, \quad \tanh(\eta\tau) = \frac{\eta t}{\gamma_0}\frac{1}{\sqrt{1 + \eta^2 t^2/\gamma_0^2}},$$

Finally we can write

$$\begin{aligned}
\gamma &= \gamma_0\sqrt{1 + \frac{\eta^2 t^2}{\gamma_0^2}}, \\
v_\parallel &= \frac{\eta c t}{\gamma_0\sqrt{1 + \eta^2 t^2/\gamma_0^2}}, \\
v_\perp &= \frac{v_0}{\sqrt{1 + \eta^2 t^2/\gamma_0^2}}.
\end{aligned}$$

If the coordinate system is choosen such that $\vec{E} = E\vec{e}_z$ and $\vec{v}_0 = v_0\vec{e}_x$, and if the particle is at the origin initially, then the position of the particle is given by

$$\begin{aligned} x &= \int_0^t v_\perp dt = \frac{v_0\gamma_0}{\eta}\sinh^{-1}\left(\frac{\eta t}{\gamma_0}\right), \\ z &= \int_0^t v_\parallel dt = \frac{c\gamma_0}{\eta}\left[\sqrt{1+\frac{\eta^2 t^2}{\gamma_0^2}} - 1\right]. \end{aligned}$$

Further, the velocity v is

$$v^2 = v_\parallel^2 + v_\perp^2 = \frac{\gamma_0^2 v_0^2 + \eta^2 c^2 t^2}{\gamma_0^2 + \eta^2 t^2}.$$

(b) In order to determine the trajectory, we need to eliminate the time-dependence. From the equation for x, we get

$$t = \frac{\gamma_0}{\eta}\sinh\left(\frac{\eta x}{\gamma_0 v_0}\right).$$

Substituting this into the equation for z, we obtain the trajectory

$$z = \frac{c\gamma_0}{\eta}\left[\cosh\left(\frac{\eta x}{\gamma_0 v_0}\right) - 1\right].$$

For times $t \ll \gamma_0/\eta$ (or $x \ll \gamma_0 v_0/\eta$), we can use the approximation $\cosh\chi \approx 1 + \chi^2/2$ for $\chi \ll 1$, and find that the trajectory can be written as

$$z \approx \frac{\eta c}{2\gamma_0 v_0^2}x^2,$$

which is a parabola. This trajectory can also be written as a function of t as

$$x \approx v_0 t, \quad z \approx \frac{eE}{2m\gamma_0}t^2.$$

On the other hand, for times $t \gg \gamma_0/\eta$, the trajectory becomes

$$x \approx \frac{\gamma_0 v_0}{\eta}\ln\left(\frac{2\eta t}{\gamma_0}\right), \quad z \approx ct,$$

where we used the relation $\sinh^{-1}\chi = \ln(\chi + \sqrt{\chi^2+1})$. If t is eliminated from the above expressions, we get

$$z \approx \frac{c\gamma_0}{2\eta}\exp\left(\frac{\eta x}{\gamma_0 v_0}\right),$$

For late times, the particle moves along the z-direction with a speed approaching c, while the velocity in x-direction approaches zero.

2. **(a)** The Darwin Lagrangian for the two particles reads

$$L = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{8c^2}\left(m_1 v_1^4 + m_2 v_2^4\right) - \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_2}{2 r_{12} c^2}\left[\vec{v}_1\cdot\vec{v}_2 + \left(\vec{v}_1\cdot\hat{\vec{r}}\right)\left(\vec{v}_2\cdot\hat{\vec{r}}\right)\right].$$

First we introduce center of mass coordinates by

$$\vec{r} = \vec{x}_1 - \vec{x}_2, \qquad \vec{R} = (m_1\vec{x}_1 + m_2\vec{x}_2)/M,$$

where $M = m_1 + m_2$ is the total mass. Inverting this gives

$$\vec{x}_1 = \vec{R} + m_2/M\vec{r}, \qquad \vec{x}_2 = \vec{R} - m_1/M\vec{r}.$$

As a result, the individual terms in the above Lagrangian become

$$\begin{aligned}
\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2 &= \frac{1}{2}MV^2+\frac{1}{2}\mu v^2,\\
\frac{1}{8c^2}\left(m_1v_1^4+m_2v_2^4\right) &= \frac{1}{8c^2}\left[MV^4+6\mu V^2v^2+4\mu\frac{m_2-m_1}{M}\left(\vec{V}\cdot\vec{v}\right)v^2+\mu\frac{m_1^3+m_2^3}{M^3}v^4\right],\\
\vec{v}_1\cdot\vec{v}_2 &= V^2+\frac{m_2-m_1}{M}\vec{V}\cdot\vec{v}-\frac{\mu}{M}v^2,\\
\left(\vec{v}_1\cdot\hat{\vec{r}}\right)\left(\vec{v}_2\cdot\hat{\vec{r}}\right) &= \left(\vec{V}\cdot\hat{\vec{r}}\right)^2+\frac{m_2-m_1}{M}\left(\vec{V}\cdot\hat{\vec{r}}\right)\left(\vec{v}\cdot\hat{\vec{r}}\right)-\frac{\mu}{M}\left(\vec{v}\cdot\hat{\vec{r}}\right)^2,
\end{aligned}$$

where $\mu=m_1m_2/M$ is the reduced mass and $\vec{V}=\dot{\vec{R}}$ is the center of mass velocity. For $\vec{V}=0$, the Lagrangian becomes

$$L=\frac{1}{2}\mu v^2+\frac{\mu}{8c^2}\frac{m_1^3+m_2^3}{M^3}v^4-\frac{q_1q_2}{r}-\frac{\mu q_1q_2}{2Mrc^2}\left[v^2+\left(\vec{v}\cdot\hat{\vec{r}}\right)^2\right], \tag{1}$$

and the canonical momentum $p_i=\partial L/\partial v_i$ becomes

$$\vec{p}=\mu\vec{v}+\frac{\mu v^2}{2c^2}\frac{m_1^3+m_2^3}{M^3}\vec{v}-\frac{\mu q_1q_2}{Mrc^2}\left[\vec{v}+\left(\vec{v}\cdot\hat{\vec{r}}\right)\hat{\vec{r}}\right]. \tag{2}$$

(b) The Hamiltonian is obtained from the Lagrangian (1) by the relation $H=\vec{p}\cdot\vec{v}-L$. Then

$$\begin{aligned}
H &= \vec{p}\cdot\vec{v}-\frac{\mu v^2}{2}-\frac{\mu v^4}{8c^2}\frac{m_1^3+m_2^3}{M^3}+\frac{q_1q_2}{r}+\frac{\mu q_1q_2}{2Mrc^2}\left[v^2+\left(\vec{v}\cdot\hat{\vec{r}}\right)^2\right]\\
&= \frac{p^2}{2\mu}-\frac{1}{2\mu}\left(\vec{p}-\mu\vec{v}\right)^2-\frac{\mu v^4}{8c^2}\frac{m_1^3+m_2^3}{M^3}+\frac{q_1q_2}{r}+\frac{\mu q_1q_2}{2Mrc^2}\left[v^2+\left(\vec{v}\cdot\hat{\vec{r}}\right)^2\right].
\end{aligned} \tag{3}$$

Since we only consider terms to first order in $1/c^2$, we do not need to completely solve (2) for $\vec{v}$ in terms of $\vec{p}$. Instead, it is sufficient to note that

$$\vec{v}=\frac{1}{\mu}\vec{p}+\mathcal{O}\left(\frac{1}{c^2}\right).$$

Insterting this into (3) gives

$$\begin{aligned}
H &= \frac{p^2}{2\mu}-\frac{p^4}{8c^2}\frac{m_1^3+m_2^3}{M^3\mu^3}+\frac{q_1q_2}{r}+\frac{q_1q_2}{2M\mu rc^2}\left[p^2+\left(\vec{p}\cdot\hat{\vec{r}}\right)^2\right]\\
&= \frac{p^2}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)-\frac{p^4}{8c^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)+\frac{q_1q_2}{r}+\frac{q_1q_2}{2m_1m_2rc^2}\left[p^2+\left(\vec{p}\cdot\hat{\vec{r}}\right)^2\right].
\end{aligned}$$

3. **(a)** Since the expression of the symmetric stress tensor for electromagnetic fields is independent from the source terms J^μ, we consider in this part only the case of the free electromagnetic field. The Proca Lagrangian density for free electromagnetic field is

$$\mathcal{L}=-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}+\frac{1}{8\pi}\mu^2A_\mu A^\mu.$$

Since

$$T^{\mu\nu}=\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\lambda\right)}\partial^\nu A_\lambda-g^{\mu\nu}\mathcal{L},$$

we find

$$T^{\mu\nu}=-\frac{1}{4\pi}F^{\mu\lambda}\partial^\nu A_\lambda+\frac{1}{16\pi}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}-\frac{1}{8\pi}\mu^2g^{\mu\nu}A_\alpha A^\alpha.$$

In order to convert this canonical stress tensor to the symmetric stress tensor, we write $\partial^\nu A_\lambda = F^\nu{}_\lambda + \partial_\lambda A^\nu$. Then

$$\begin{aligned} T^{\mu\nu} &= -\frac{1}{4\pi}\left[F^{\mu\lambda}F^\nu{}_\lambda - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + \frac{1}{2}\mu^2 g^{\mu\nu}A^\alpha A_\alpha\right] - \frac{1}{4\pi}F^{\mu\lambda}\partial_\lambda A^\nu \\ &= -\frac{1}{4\pi}\left[F^{\mu\lambda}F^\nu{}_\lambda - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + \frac{1}{2}\mu^2 g^{\mu\nu}A^\alpha A_\alpha - \left(\partial_\lambda F^{\mu\lambda}\right)A^\nu\right] - \frac{1}{4\pi}\partial_\lambda\left(F^{\mu\lambda}A^\nu\right). \end{aligned}$$

Using the Proca equation of motion, $\partial_\lambda F^{\lambda\mu} + \mu^2 A^\mu = 0$, we can write

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda S^{\lambda\mu\nu},$$

where

$$\begin{aligned} \Theta^{\mu\nu} &= -\frac{1}{4\pi}\left[F^{\mu\lambda}F^\nu{}_\lambda - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - \mu^2\left(A^\mu A^\nu - \frac{1}{2}g^{\mu\nu}A^\alpha A_\alpha\right)\right] \\ &= \frac{1}{4\pi}\left[g^{\mu\gamma}F_{\gamma\lambda}F^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + \mu^2\left(A^\mu A^\nu - \frac{1}{2}g^{\mu\nu}A^\alpha A_\alpha\right)\right] \end{aligned}$$

is the symmetric stress tensor and $S^{\lambda\mu\nu} = (1/4\pi)F^{\lambda\mu}A^\nu$ is antisymmetric in the first two indices.

(b) With the symmetric stress tensor $\Theta^{\mu\nu}$ given in part (a), the 4-divergence of $\Theta^{\mu\nu}$ gives

$$\begin{aligned} \partial_\mu\Theta^{\mu\nu} &= -\frac{1}{4\pi}\Big[\left(\partial_\mu F^{\mu\lambda}\right)F^\nu{}_\lambda + F^{\mu\lambda}\left(\partial_\mu F^\nu{}_\lambda\right) - \frac{1}{2}F_{\rho\lambda}\left(\partial^\nu F^{\rho\lambda}\right) \\ &\qquad\qquad - \mu^2\left\{\left(\partial_\mu A^\mu\right)A^\nu + A^\mu\left(\partial_\mu A^\nu\right) - A^\lambda\left(\partial^\nu A_\lambda\right)\right\}\Big] \\ &= -\frac{1}{4\pi}\left[\left(\partial_\mu F^{\mu\lambda}\right)F^\nu{}_\lambda + \frac{1}{2}F_{\rho\lambda}\left(2\partial^\rho F^{\nu\lambda} - \partial^\nu F^{\rho\lambda}\right) + \mu^2 A^\lambda\left(\partial^\nu A_\lambda - \partial_\lambda A^\nu\right)\right] \\ &= -\frac{1}{4\pi}\left[\left(\partial_\mu F^{\mu\lambda} + \mu^2 A^\lambda\right)F^\nu{}_\lambda + \frac{1}{2}F_{\rho\lambda}\left(\partial^\rho F^{\nu\lambda} + \partial^\lambda F^{\rho\nu} + \partial^\nu F^{\lambda\rho}\right)\right] \\ &= -\frac{1}{c}J^\lambda F^\nu{}_\lambda = \frac{1}{c}J_\lambda F^{\lambda\nu}, \end{aligned}$$

where for the second line we use the fact that $\partial_\mu A^\mu = 0$, which is automatic for the Proca equation, and for last line we also use the Bianchi identity, $3\partial^{[\rho}F^{\nu\lambda]} = 0$ as well as the Proca equation of motion, $\partial_\lambda F^{\lambda\mu} + \mu^2 A^\mu = 4\pi J^\mu/c$.

(c) With the explicit form of the Maxwell tensor we can get the relations

$$F^{\alpha\beta}F_{\alpha\beta} = -2\left(E^2 - B^2\right), \quad A^\alpha A_\alpha = \left(A^0\right)^2 - \vec{A}\cdot\vec{A}.$$

Thus

$$\Theta^{\mu\nu} = -\frac{1}{4\pi}\left[F^{\mu\lambda}F^\nu{}_\lambda + \frac{1}{2}g^{\mu\nu}\left(E^2 - B^2\right) - \mu^2\left\{A^\mu A^\nu - \frac{1}{2}g^{\mu\nu}\left(\left(A^0\right)^2 - \vec{A}\cdot\vec{A}\right)\right\}\right].$$

The time-time component of this is

$$\begin{aligned} \Theta^{00} &= -\frac{1}{4\pi}\left[F^{0i}F^0{}_i + \frac{1}{2}\left(E^2 - B^2\right) - \mu^2\left\{\left(A^0\right)^2 - \frac{1}{2}\left(\left(A^0\right)^2 - \vec{A}\cdot\vec{A}\right)\right\}\right] \\ &= -\frac{1}{4\pi}\left[-\frac{1}{2}\left(E^2 + B^2\right) - \frac{\mu^2}{2}\left\{\left(A^0\right)^2 + \vec{A}\cdot\vec{A}\right\}\right] \\ &= \frac{1}{8\pi}\left[E^2 + B^2 + \mu^2\left(A^0A^0 + \vec{A}\cdot\vec{A}\right)\right]. \end{aligned}$$

Similarly, the time-space components are

$$\begin{aligned}
\Theta^{0i} &= -\frac{1}{4\pi}\left[F^{0}{}_{j}F^{ij} - \mu^2 A^0 A^i\right] \\
&= -\frac{1}{4\pi}\left[E_j\left(-\epsilon^{ijk}B_k\right) - \mu^2 A^0 A^i\right] \\
&= -\frac{1}{4\pi}\left[-\epsilon^{ijk}E_j B_k - \mu^2 A^0 A^i\right] \\
&= \frac{1}{4\pi}\left[\left(\vec{E}\times\vec{B}\right)^i + \mu^2 A^0 A^i\right].
\end{aligned}$$