Classical Field Theory: Maxwell Equations

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1J.D.Jackson, "Classical Electrodynamics", 2nd Edition, Section 6
Electrostatics and Magnetostatics deal with steady-state problems in electricity and in magnetism.

The almost independent nature of electric and magnetic fields phenomena disappears when we consider time-dependent problems.

Time varying magnetic fields give rise to electric fields and vice-versa. We then must speak of electromagnetic fields rather than electric or magnetic fields.

The interconnection between electric and magnetic fields and their essential sameness becomes clear only within the framework of Special Theory of Relativity.
Faraday’s Law of Induction

Faraday (1831), observed that a transient current is induced in a circuit if:

1. A steady current flowing in an adjacent circuit is turned on or off
2. The adjacent circuit with a steady current flowing is moved relative to the first circuit
3. A permanent magnet is thrust into or out of the circuit

The changing magnetic flux induces an electric field around the circuit, the line integral of which is called the electromotive force $\mathcal{E}$ and causes a current flow according to Ohm’s law: $\vec{J} = \sigma \vec{E}$ (\(\sigma\) is the conductivity).

The magnetic induction in the neighboring of the circuit is $\vec{B}$ and the magnetic flux linking the circuit is defined by

$$F = \int_S \vec{B} \cdot \vec{n} \, da \quad (1)$$
The **electromotive force** around the circuit is defined by

\[ \mathcal{E} = \oint_C \vec{E}' \cdot d\vec{\ell} \]  

(2)

where \( \vec{E}' \) is the **electric field** at the element \( d\vec{\ell} \) of the circuit \( C \).

Thus Faraday's observation is summed up in the mathematical law:

\[ \mathcal{E} = -k \frac{dF}{dt} \]  

(3)

That is, the **induced electromotive force** around the circuit is proportional to the **time rate of change of magnetic flux** linking the circuit.

The sign is specified by **Lenz’s law**, which states that the induced current (and accompanying magnetic flux) is in such a direction to oppose the change of flux through the circuit. For SI units \( k = 1 \), Gaussian units \( k = 1/c \).
Faraday’s law for a moving circuit

\[ \oint_C \vec{E}' \cdot d\vec{r} = -k \frac{d}{dt} \int_S \vec{B} \cdot \vec{n} \, da \]  

(4)

This is eqn (3) in terms of integrals. We can observe that:

- The **induced electromotive force** is proportional to the **total time derivative** of the flux.

- The flux can be changed **by changing**: the magnetic induction or the shape or the orientation or the position of the circuit.

If the circuit $C$ is moving with a velocity $\vec{v}$ in some direction the total time derivative in eqn (4) must take into account the motion e.g. convective derivative

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \]

and

\[ \frac{d\vec{B}}{dt} = \frac{\partial \vec{B}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{B} = \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times (\vec{B} \times \vec{v}) + \vec{v} \left( \vec{\nabla} \cdot \vec{B} \right) \]
The flux through the circuit may change because:

▶ the flux changes with time at a point

▶ the translation of the circuit changes the location of the boundary.

Eqn (4) can be written as (why?):

\[
\oint_C \left[ \vec{E}' - k \left( \vec{v} \times \vec{B} \right) \right] \cdot d\vec{l} = -k \int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{n} \, da. \tag{5}
\]

In a comoving coordinate system

\[
\oint_C \vec{E} \cdot d\vec{l} = -k \int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{n} \, da \tag{6}
\]

where \( \vec{E} \) is the electric field in the comoving frame. Thus

\[
\vec{E}' = \vec{E} + k \left( \vec{v} \times \vec{B} \right) \tag{7}
\]

With the present choice of units for charge and current, Galilean covariance requires that \( k = 1/c \) (why?).
Finally, the transformation of the electromotive force integral into a surface integral leads to (how?)

\[ \int_S \left( \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \right) \cdot \vec{n} \, da = 0 \] (8)

Since the circuit \( C \) and the bounding surface \( S \) are arbitrary, the integrant must vanish at all points in space. Thus the differential form of Faraday’s law is:

\[ \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0. \] (9)

Note that for time-independent electrostatic fields : \( \vec{\nabla} \times \vec{E} = 0 \).

**KELVIN-STOKES THEOREM :**

relates the surface integral of the curl of a vector field over a surface \( S \) in Euclidean 3-space to the line integral of the vector field over its boundary

\[ \int_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{\ell} \] (10)
Typically in magnetostatics, the creation of a steady-state configuration of currents and associated magnetic fields involves an initial transient period during which the currents and fields are brought from zero to the final values.

For such time-varying fields there are induced electromotive forces which cause the sources of current to do work.

Since the energy in the field is by definition the total work done to establish it, we must consider these contributions.
Energy in the Magnetic Field

- Suppose that we have a circuit with a constant current \( I \) flowing in it.
- If the flux through the circuit \textit{changes} an electromotive force \( E \) is induced around it.
- In order to keep the current constant, the \textbf{sources of current must do work}.
- To determine the rate we note that the \textit{time rate of change of energy of a particle} with velocity \( \boldsymbol{v} \) acted on by a force \( \boldsymbol{F} \) is

\[
\frac{dE}{dt} = \boldsymbol{v} \cdot \boldsymbol{F}
\]

With a changing flux the added field \( \boldsymbol{E}' \) on each conducting electron of charge \( q \) and drift velocity \( \boldsymbol{\bar{v}} \) gives rise to a change in energy per unit time of \( q \boldsymbol{\bar{v}} \cdot \boldsymbol{E}' \) per electron.

Summing over all the electrons in the circuit, we find that the sources do work to maintain the current at the rate

\[
\frac{dW}{dt} = -I \mathcal{E} = \frac{1}{c} I \frac{dF}{dt}
\]
Thus, if the flux change through a circuit carrying a current $I$ is $\delta F$, the work done by the source is:

\[ \delta W = \frac{1}{c} I \delta F \]

We consider the problem of the work done in establishing a general steady-state distribution of currents and fields.

- We can imagine that the build-up process occurs at an infinitesimal rate so that $\nabla \cdot \vec{J} = 0$ ($\vec{J}$ is the current).

- The current distribution can be broken up into a network of elementary loops, the typical ones of which is an elemental tube of current of cross-sectional area $\Delta \sigma$ following a close path $C$ and spanned by a surface $S$ with normal $\vec{n}$.

**Figure**: Distribution of current density broken up into elementary current loops.
We can express the increment of work done against the induced electromotive force in terms of the change of the magnetic induction through the loop:

\[ \Delta (\delta W) = \frac{J \Delta \sigma}{c} \int_S \mathbf{n} \cdot \delta \mathbf{B} \, da \]  

(11)

the extra \( \Delta \) is because we consider only one elementary circuit.

• If we express \( \mathbf{B} \) in terms of the vector potential \( \mathbf{A} \) i.e. \( \mathbf{B} = \nabla \times \mathbf{A} \) then we have

\[ \Delta (\delta W) = \frac{J \Delta \sigma}{c} \int_S (\nabla \times \delta \mathbf{A}) \cdot \mathbf{n} \, da \]  

(12)

With application of Stokes’s theorem this can be written

\[ \Delta (\delta W) = \frac{J \Delta \sigma}{c} \oint_C \delta \mathbf{A} \cdot d\mathbf{l} \]  

(13)

but \( J \Delta \sigma d\mathbf{l} \equiv \mathbf{J} d^3x \) since \( \mathbf{l} \) is parallel to \( \mathbf{J} \). Evidently the sum over all such elemental loops will be the volume integral.

• Hence the total increment of work done by an external source due to a change \( \delta \mathbf{A} (\mathbf{x}) \) in the vector potential is

\[ \delta W = \frac{1}{c} \int \delta \mathbf{A} \cdot \mathbf{J} \, d^3x \]  

(14)
By using Ampère’s law
\[ \nabla \times \vec{H} = \frac{4\pi}{c} \vec{J} \]
we can get an expression in terms of the magnetic fields. Then
\[ \delta W = \frac{1}{4\pi} \int \delta A \cdot (\nabla \times \vec{H}) \, d^3x \] 
(15)
which transforms to (how?)
\[ \delta W = \frac{1}{4\pi} \int \left[ \vec{H} \cdot (\nabla \times \delta A) + \nabla \cdot (\vec{H} \times \delta A) \right] \, d^3x \] 
(16)
If the field distribution is assumed to be localized, the 2nd integrant vanishes (why?) and we get
\[ \delta W = \frac{1}{4\pi} \int \vec{H} \cdot \delta \vec{B} \, d^3x \] 
(17)
which is the analog of the electrostatic equation for the energy change
\[ \delta W = \frac{1}{4\pi} \int \vec{E} \cdot \delta \vec{D} \, d^3x \] 
(18)
where \( \vec{D} = \vec{E} + 4\pi \vec{P} \) is the electric displacement and \( \vec{P} \) the electric polarization (dipole moment per unit volume).
If we bring the fields from zero to the final values the total magnetic energy will be (why?)

\[
W = \frac{1}{8\pi} \int \vec{H} \cdot \vec{B} \, d^3x
\]  
(19)

which is the magnetic analog of the total electrostatic energy

\[
W = \frac{1}{8\pi} \int \vec{E} \cdot \vec{D} \, d^3x
\]  
(20)

The energy of a system of charges in free space i.e. electrostatic energy is:

\[
W = \frac{1}{2} \int \rho(\vec{x})\Phi(\vec{x}) \, d^3x
\]  
(21)

The magnetic equivalent for this expression i.e. the magnetic energy is

\[
W = \frac{1}{2c} \int \vec{J} \cdot \vec{A} \, d^3x
\]  
(22)
Maxwell Equations

Maxwell’s equations are based on the following empirical facts:

1. The electric charges are sources of the vector field of the dielectric displacement density \( \vec{D} \). Hence, for the flux of the dielectric displacement through a surface enclosing the charge we have

\[
\frac{1}{4\pi} \oint_S \vec{D} \cdot \vec{n} da = Q = \int_V \rho d^3x
\]  

This relation can be derived from Coulomb’s force law.

2. Faraday’s induction law:

\[
\mathcal{E} = \oint_C \vec{E} \cdot d\vec{l} = -\frac{1}{c} \frac{\partial}{\partial t} \oint_S \vec{B} \cdot \vec{n} da
\]  

3. The fact that there are no isolated monopoles implies

\[
\oint_S \vec{B} \cdot \vec{n} da = 0
\]  

4. Ampère’s law:

\[
\oint_C \vec{H} \cdot d\vec{l} = \frac{4\pi}{c} \int_S \vec{J} \cdot \vec{n} da
\]
Maxwell Equations

The basic laws of electricity and magnetism can be summarized in differential form:

Coulomb’s law
\[ \nabla \cdot \vec{D} = 4\pi \rho \]

(27)

Ampère’s law \((\nabla \cdot \vec{J} = 0)\)
\[ \nabla \times \vec{H} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t} \]

(28)

Faraday’s law
\[ \nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \]

(29)

no free magnetic poles
\[ \nabla \cdot \vec{B} = 0 \]

(30)

where \(\vec{E}\) and \(\vec{B}\) are the averaged \(\vec{E}\) and \(\vec{B}\) of the microscopic or vacuum Maxwell equations. The two extra field quantities \(\vec{D}\) and \(\vec{H}\) usually called the electric displacement and magnetic field (\(\vec{B}\) is then called the magnetic induction and \(\vec{M}\) is the macroscopic magnetization)

\[
D_a = E_a + 4\pi \left( P_a - \sum_b \frac{\partial Q'_{ab}}{\partial x_b} + \ldots \right) \rightarrow \vec{D} = \vec{E} + 4\pi \vec{P} + \ldots 
\]

(31)

\[
H_a = B_a - 4\pi (M_a + \ldots) \rightarrow \vec{H} = \vec{B} - 4\pi \vec{M} + \ldots
\]

(32)
The quantities $\vec{P}$, $\vec{M}$, $Q_{ab}$ represent the macroscopically averaged electric dipole, magnetic dipole and electric quadrupole moment densities of the material medium in the presence of applied fields.

Similarly, the charge and current densities $\rho$ and $\vec{J}$ are macroscopic averages of the free charge and current densities in the medium.

The macroscopic Maxwell equations are a set of 8 eqns involving the components of 4 fields $\vec{E}$, $\vec{B}$, $\vec{D}$ and $\vec{H}$.

The 4 homogeneous eqns can be solved formally by expressing $E$ and $B$ in terms of the scalar potential $\Phi$ and the vector potential $\vec{A}$.

The inhomogeneous eqns cannot be solved until the derived fields $\vec{D}$ and $\vec{H}$ are known in terms of $\vec{E}$ and $\vec{B}$. These connections which are implicit in (32) are known as constitute relations, e.g. $\vec{D} = \epsilon \vec{E}$ and $\vec{H} = \vec{E} / \mu$ ($\epsilon$: electric permittivity & $\mu$ magnetic permeability).

All but Faraday’s law were derived from steady-state observations and there is no a priori reason to expect that the static equations hold unchanged for time-dependent fields.
The above equations without the red term in Ampere’s law are inconsistent. While Ampère’s law ($\nabla \cdot \vec{J} = 0$) is valid for steady-state problems, the complete relation is given by the continuity equation for charge and current

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0.$$ (33)

Maxwell replaced $\vec{J}$ in Ampère’s law by its generalization

$$\vec{J} \rightarrow \vec{J} + \frac{1}{4\pi} \frac{\partial \vec{D}}{\partial t}$$ (34)

Maxwell called the added term displacement current, without it there would be no electromagnetic radiation (Can you repeat his steps?). Maxwell’s equations, form the basis of all classical electromagnetic phenomena. When combined with the Lorentz force equation

$$\vec{F} = q \left( \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right)$$ (35)

and Newton’s 2nd law of motion, provide a complete description of the classical dynamics of interacting charged particles and EM fields.
In electrostatics and magnetostatics we have used the scalar potential $\Phi$ and the vector potential $\vec{A}$ to simplify certain equations. Since $\vec{\nabla} \cdot \vec{B} = 0$, we can define $\vec{B}$ in terms of a vector potential $\vec{A}$:

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad (36)$$

Then Faraday’s law $\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0$ can be written

$$\vec{\nabla} \times \left( \vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = 0 \quad (37)$$

Thus the quantity with vanishing curl can be written as the gradient of a scalar potential $\Phi$:

$$\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} \Phi \quad \text{or} \quad \vec{E} = -\vec{\nabla} \Phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \quad (38)$$

The definition of $\vec{B}$ and $\vec{E}$ in terms of the potentials $\vec{A}$ and $\Phi$ satisfies identically the 2 homogeneous Maxwell equations. While $\vec{A}$ and $\Phi$ are determined by the 2 inhomogeneous Maxwell equations.
If we restrict our considerations to the vacuum form of the Maxwell equations the inhomogeneous form of Maxwell equations can be write in terms of the potentials as:

\[ \nabla^2 \Phi + \frac{1}{c} \frac{\partial}{\partial t} \left( \nabla \cdot A \right) = -4\pi \rho \]  
(39)

\[ \nabla^2 \tilde{A} - \frac{1}{c^2} \frac{\partial^2 \tilde{A}}{\partial t^2} - \nabla \left( \nabla \cdot \tilde{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t} \right) = -\frac{4\pi}{c} J \]  
(40)

These eqns are equivalent to Maxwell eqns but they are still coupled. Thanks to the arbitrariness in the definition of the potentials we can choose transformations of the form

\[ \tilde{A} \rightarrow \tilde{A}' = \tilde{A} + \tilde{\nabla} \Lambda \]  
(41)

\[ \Phi \rightarrow \Phi' = \Phi - \frac{1}{c} \frac{\partial \Lambda}{\partial t} \]  
(42)

Thus we can choose a set of potentials \((\tilde{A}, \Phi)\) such that

\[ \nabla \cdot \tilde{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t} = 0 \]  
(43)
This way we uncouple the pair of equations (39) and (40) and leave two inhomogeneous wave equations, one for $\Phi$ and one for $\vec{A}$

\[
\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -4\pi \rho \quad (44)
\]

\[
\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \vec{j} \quad (45)
\]

This set of equations is equivalent in all respects to the Maxwell equations.
Gauge Transformations: Lorenz Gauge

The transformations

\[
\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla} \Lambda \quad (46)
\]

\[
\Phi \rightarrow \Phi' = \Phi - \frac{1}{c} \frac{\partial \Lambda}{\partial t} \quad (47)
\]

are called gauge transformation and the invariance of the fields under these transformations is called gauge invariance.

The relation between \(\vec{A}\) and \(\Phi\):

\[
\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t} = 0 \quad (48)
\]

is called Lorenz condition \(^2\). [Prove that there will always exist potentials satisfying the Lorentz condition].

The Lorenz gauge is commonly used because:

- It leads to the wave equations (44) and (45) which treat \(\Phi\) and \(\vec{A}\) on equal footings
- It is a coordinate independent concept and fits naturally into the considerations of special relativity.

\(^2\) The condition is from the Danish mathematician and physicist Ludvig Valentin Lorenz (1829-1891) and not from the Dutch physicist Hendrik Lorentz (1853-1928).
Gauge Transformations: Coulomb Gauge

In this gauge
\[ \vec{\nabla} \cdot \vec{A} = 0 \]  (49)

From eqn (39) we see that the scalar potential satisfies the Poisson eqn:
\[ \nabla^2 \Phi = -4\pi \rho \]  (50)

with solution
\[ \Phi(\vec{x}, t) = \int \frac{\rho(\vec{x}', t)}{|\vec{x} - \vec{x}'|} d^3x' \]  (51)

The scalar potential is just the instantaneous Coulomb potential due to the charge density \( \rho(\vec{x}, t) \). This is the origin of the name **Coulomb gauge**.

From eqn (40) we find that the vector potential satisfies:
\[ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \vec{J} + \frac{1}{c} \vec{\nabla} \frac{\partial \Phi}{\partial t} \]  (52)
Finally, if we define the transverse (or solenoidal) current:

$$\vec{J}_t = \frac{1}{4\pi} \vec{\nabla} \times \vec{\nabla} \times \int \frac{\vec{J}}{|\vec{x} - \vec{x'}|} d^3x'$$  \hspace{1cm} (53)$$

and longitudinal (or irrotational) current $\vec{J}_l$ for which $\vec{\nabla} \times \vec{J}_l = 0$ which may cancel the contribution of the term with the potential $\Phi$.

Then, the wave equation for $\vec{A}$ can be expressed entirely in terms of the transverse current $\vec{J}_t$ [Can you explain it?]

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \vec{J}_t$$  \hspace{1cm} (54)$$

The Coulomb or transverse gauge is often used when no sources are present. Then $\Phi = 0$, and $\vec{A}$ satisfies the homogeneous wave equation. The fields are given by

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \quad \text{and} \quad \vec{B} = \vec{\nabla} \times \vec{A}$$  \hspace{1cm} (55)$$
Green Functions for the Wave Equations

The wave eqns (44), (45) and (54) all have the same structure,

\[ \nabla^2 \Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = -4\pi f(\vec{x}, t) \]  
(56)

• To solve (56) it is useful to find a Green function (as in electrostatics)
• In order to remove the time dependence we introduce a Fourier transform with respect to frequency
• We suppose that both \( \Psi(\vec{x}, t) \) and \( f(\vec{x}, t) \) have a Fourier integral representation

\[ \Psi(\vec{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(\vec{x}, \omega)e^{-i\omega t} d\omega, \quad f(\vec{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\vec{x}, \omega)e^{-i\omega t} d\omega \]  
(57)

with the inverse transformations,

\[ \Psi(\vec{x}, \omega) = \int_{-\infty}^{\infty} \Psi(\vec{x}, t)e^{i\omega t} dt, \quad f(\vec{x}, \omega) = \int_{-\infty}^{\infty} f(\vec{x}, t)e^{i\omega t} dt \]  
(58)
When the representation \((57)\) are inserted into \((56)\) it is found that the Fourier transform \(\Psi(\vec{x}, \omega)\) satisfies the \textbf{inhomogeneous Helmholtz wave equation} for each value of \(\omega\) and \(k = \omega / c\)

\[
(\nabla^2 + k^2) \Psi(\vec{x}, \omega) = -4\pi f(\vec{x}, \omega) \tag{59}
\]

Equation \((59)\) is an elliptic PDE similar to Poisson eqn to which it reduces for \(k = 0\).
The Green function \(G(\vec{x}, \vec{x}')\) appropriate to \((59)\) satisfies the inhomogeneous equation

\[
(\nabla^2 + k^2) G_k(\vec{x}, \vec{x}') = -4\pi \delta(\vec{x} - \vec{x}') \tag{60}
\]

If there are no boundary surfaces, the Green function can only depend on \(\vec{R} = \vec{x} - \vec{x}'\), and must be spherically symmetric, that is depend only on \(R = |\vec{R}|\). This means that in spherical coordinates \(G_k(R)\) satisfies

\[
\frac{1}{R} \frac{d^2}{dR^2} (RG_k) + k^2 G_k = 4\pi \delta(\vec{R}) \tag{61}
\]
In other words, everywhere except $R = 0$ the quantity $RG_k(R)$ satisfies the homogeneous equation

\[
\frac{d^2}{dR^2} \left( RG_k \right) + k^2 \left( RG_k \right) = 0
\]

with solution:

\[RG_k(R) = Ae^{ikR} + Be^{-ikR}\]

The general solution for the Green function is:

\[G_k(R) = AG_k^{(+)}(R) + BG_k^{(-)}(R)\] (62)

where

\[G_k^{(±)}(R) = \frac{e^{±ikR}}{R}\] (63)

with $A + B = 1$ and the correct normalization condition at $R \to 0$

\[\lim_{kR \to 0} G_k(R) = \frac{1}{R}\] (64)

The first term of (62) represents a **diverging spherical wave** propagating from the origin, while the second term represents a **converging spherical wave**.
To understand the different time behaviors associated with $G_k^{(+)}$ and $G_k^{(-)}$ we need to construct the corresponding time-dependent Green functions that satisfy

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G_k^{(\pm)}(\vec{x}, t; \vec{x}', t') = -4\pi \delta(\vec{x} - \vec{x}')\delta(t - t') \tag{65}
\]

Note that the source term for (59) is $-4\pi \delta(\vec{x} - \vec{x}') e^{i\omega t'}$

Using the Fourier transforms (57) the time-dependent Green functions become (how?)

\[
G^{(\pm)}(R, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\pm ikR}}{R} e^{-i\omega \tau} d\omega \tag{66}
\]

where $\tau = t - t'$ is the relative time.

The integral actually is a $\delta$ function and the Green function becomes:

\[
G^{(\pm)}(R, \tau) = \frac{1}{R} \delta \left( \tau \mp \frac{R}{c} \right) \tag{67}
\]

or

\[
G^{(\pm)}(\vec{x}, t; \vec{x}', t') = \frac{1}{|\vec{x} - \vec{x}'|} \delta \left( t' - \left[ t \mp \frac{|\vec{x} - \vec{x}'|}{c} \right] \right) \tag{68}
\]
The infinite space Green function is thus a function only of the relative distance $R$ and the relative time $\tau$ between the source and the observation point.

The Green function $G^{(+)}$ is called the retarded Green function and $G^{(-)}$ is called the advanced Green function.

Particular integrals of the inhomogeneous wave equation (56) are

$$\psi^{(\pm)}(\vec{x}, t) = \int \int G^{(\pm)}(\vec{x}, t; \vec{x}', t') f(\vec{x}', t') d^3 x' dt'$$

to either of these maybe added solutions of the homogeneous equation in order to specify a definite physical problem.

**EXAMPLES**

$$\Phi(\vec{x}, t) = \int \int \frac{1}{|\vec{x} - \vec{x}'|} \delta \left( t' - \left[ t \mp \frac{|\vec{x} - \vec{x}'|}{c} \right] \right) \rho(\vec{x}', t') d^3 x' dt'$$

$$\vec{A}(\vec{x}, t) = \int \int \frac{1}{|\vec{x} - \vec{x}'|} \delta \left( t' - \left[ t \mp \frac{|\vec{x} - \vec{x}'|}{c} \right] \right) \frac{\vec{J}(\vec{x}', t')}{c} d^3 x' dt'$$
Poynting’s Theorem : Conservation of Energy

For a continuous distribution of charge and current, the total rate of doing work by the fields in a finite volume $V$ is

$$\int_V \vec{J} \cdot \vec{E} d^3x$$

(69)

This represents a conversion of EM energy into mechanical or thermal energy and it must be balanced by a corresponding rate of decrease of energy in the EM field within the volume $V$.

In order to derive this conservation law explicitly we use the Maxwell eqns to express (69) in other terms. We use Ampère law to eliminate $\vec{J}$

$$\int_V \vec{J} \cdot \vec{E} d^3x = \frac{1}{4\pi} \int_V \left[ c\vec{E} \cdot \left( \nabla \times \vec{H} \right) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] d^3x$$

(70)

If we use the vector identity,

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H})$$
together with the Faraday’s law, we get

\[
\int_V \vec{J} \cdot \vec{E} \, d^3x = \frac{-1}{4\pi} \int_V \left[ c \vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{B} \cdot \frac{\partial \vec{H}}{\partial t} \right] \, d^3x \tag{71}
\]

The terms with the time derivatives can be interpreted as the time derivatives of the electrostatic and magnetic energy densities. If we also remember that the sum/integrals

\[
W_E = \frac{1}{8\pi} \int \vec{E} \cdot \vec{D} \, d^3x \quad \text{and} \quad W_B = \frac{1}{8\pi} \int \vec{H} \cdot \vec{B} \, d^3x \tag{72}
\]

represents the total EM energy (even for time varying fields). Then the \textbf{total energy density} is denoted by

\[
u = \frac{1}{8\pi} \left( \vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H} \right) \equiv \frac{1}{2} \left( \epsilon_0 \vec{E}^2 + \frac{1}{\mu_0} \vec{B}^2 \right) \tag{73}
\]

Then equation (71) can be written (how?):

\[
- \int_V \vec{J} \cdot \vec{E} \, d^3x = \int_V \left[ \frac{\partial u}{\partial t} + \frac{c}{4\pi} \vec{\nabla} \cdot (\vec{E} \times \vec{H}) \right] \, d^3x \tag{74}
\]
Since the volume is arbitrary, this can be cast into the form of a **differential continuity equation** or **conservation law**

\[
\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}
\]  

(75)

The vector \( \vec{S} \) represents the **energy flow** and is called **Poynting vector**

\[
\vec{S} = \frac{c}{4\pi} \left( \vec{E} \times \vec{H} \right) \equiv \left( \vec{E} \times \vec{H} \right)
\]

(76)

- **Poynting’s theorem** (Conservation of energy): The physical meaning of the above relations is that the time rate of change of EM energy within a certain volume, plus the energy flowing out through the boundary surfaces of the volume per unit time, **is equal to** the negative of the total work done by the fields on the sources within the volume.
- In other words, **Poynting’s theorem for microscopic field** \((\vec{E}, \vec{B})\) is a statement of conservation of energy of the combined system of particles and fields.
• If we denote the **total energy of the particles** within the volume $V$ as $E_{\text{mech}}$ and assume that no particles move out of the volume we have

$$\frac{dE_{\text{mech}}}{dt} = \int_V \vec{J} \cdot \vec{E} \, d^3x$$  \hspace{1cm} (77)

and the **total field energy** within $V$ as

$$E_{\text{field}} = \int_V u \, d^3x = \frac{1}{8\pi} \int_V \left( \vec{E}^2 + \vec{B}^2 \right) \, d^3x$$  \hspace{1cm} (78)

**Poynting's theorem** expresses the conservation of energy for the combined system as:

$$\frac{dE}{dt} = \frac{d}{dt} (E_{\text{mech}} + E_{\text{field}}) = -\oint_S \vec{n} \cdot \vec{S} \, da$$  \hspace{1cm} (79)
Poynting’s Theorem: Conservation of Momentum

The **conservation of linear momentum** can be similarly considered. The **total EM force** on a charged particle is

\[ \vec{F} = q \left( \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) \]  

(80)

If we denote as \( \vec{P}_{\text{mech}} \) the sum of all the momenta of all the particles in a volume \( V \) we can write the Newton’s 2nd law (how?),

\[ \vec{F}_{\text{tot}} \equiv \frac{d\vec{P}_{\text{mech}}}{dt} = \int_V \left( \rho \vec{E} + \frac{1}{c} \vec{J} \times \vec{B} \right) d^3x \]  

(81)

where \( \vec{J} \equiv \rho \vec{v} \) we converted the sum over particles to an integral over charge & current densities.

We can use Maxwell equations to eliminate \( \rho \) and \( J \) from (81) by using

\[ \rho = \frac{1}{4\pi} \nabla \cdot \vec{E}, \quad \vec{J} = \frac{c}{4\pi} \left( \nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right) \]  

(82)
After some manipulations we can show (how?) that the rate of change of mechanical momentum of eqn (81) can be written

\[
\frac{d\vec{P}_{\text{mech}}}{dt} + \frac{d}{dt} \int_V \frac{1}{4\pi c} \left( \vec{E} \times \vec{B} \right) d^3x = \frac{1}{4\pi} \int_V \left[ \vec{E}(\vec{\nabla} \cdot \vec{E}) - \vec{E} \times (\vec{\nabla} \times \vec{E}) + \vec{B}(\vec{\nabla} \cdot \vec{B}) - \vec{B} \times (\vec{\nabla} \times \vec{B}) \right] d^3x
\]

We can identify the volume integral on the left as the total EM momentum \( \vec{P}_{\text{field}} \) in the volume \( V \)

\[
\vec{P}_{\text{field}} = \frac{1}{4\pi c} \int_V \left( \vec{E} \times \vec{B} \right) d^3x
\]

The integrant can be interpreted as the density of EM momentum

\[
\vec{g} = \frac{1}{4\pi c} \left( \vec{E} \times \vec{B} \right)
\]

The EM momentum density \( \vec{g} \) is proportional to the energy-flux density \( \vec{S} \) with proportionality constant \( c^{-2} \).
Maxwell Stress Tensor

In order to establish that (84) is a conservation law for momentum, we must convert the volume integral on the right into surface integral of something that will be identified as momentum flow. By defining the Maxwell stress tensor $T_{ab}$ as

$$T_{ab} = \frac{1}{4\pi} \left[ E_a E_b + B_a B_b - \frac{1}{2} \delta_{ab} \left( \vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B} \right) \right]$$

(86)

where the $a$-th component can be written as

$$\left[ \vec{E} (\vec{\nabla} \cdot \vec{E}) - \vec{E} \times (\vec{\nabla} \times \vec{E}) \right]_a = \sum_b \frac{\partial}{\partial x_b} \left( E_a E_b - \frac{1}{2} \delta_{ab} \vec{E} \cdot \vec{E} \right)$$

(87)

also

$$\left( \vec{\nabla} \cdot \vec{T} \right)_a = \sum_b \frac{\partial}{\partial x_b} T_{ab}$$

(88)

Then (84) can be written as:

$$\frac{d}{dt} (P_{\text{mech}} + P_{\text{field}})_a = \sum_b \int_V \frac{\partial}{\partial x_b} T_{ab} d^3x$$

(89)
Then via the divergence theorem we get

\[
\frac{d}{dt} (P_{\text{mech}} + P_{\text{field}})_a = \oint_S \sum_b T_{ab} n_b da
\]

(90)

If (90) represents a statement of conservation of momentum, \( \sum_b T_{ab} n_b \) is the \( a \)-th component of the flow per unit area of momentum across the surface \( S \) into the volume \( V \).

In other words it is the force per unit area transmitted across the surface \( S \) and acting on the combined system of particles and fields inside \( V \).
Conservation of Angular Momentum

The derivation of the EM angular momentum shares the same tactical approach as that of the linear momentum.

Let us define the **mechanical angular momentum** of the system as

\[
\vec{L}_{\text{mech}} = \vec{r} \times \vec{p}_{\text{mech}}
\]  

(91)

where \( \vec{p}_{\text{mech}} \) is the **mechanical momentum density**. Then

\[
\frac{d \vec{L}_{\text{mech}}}{dt} = \vec{r} \times \left( \rho \vec{E} + \frac{1}{c} \vec{J} \times \vec{B} \right),
\]  

(92)

and substitution of \( \rho \) and \( \vec{J} \) from Maxwell’s eqns leads to

\[
\frac{d}{dt} \left[ \vec{L}_{\text{mech}} + \frac{1}{4\pi c} \vec{r} \times \left( \vec{E} \times \vec{B} \right) \right]
\]

(93)

\[
= \frac{1}{4\pi} \vec{r} \times \left[ \vec{E} (\vec{\nabla} \cdot \vec{E}) - \vec{E} \times (\vec{\nabla} \times \vec{E}) + \vec{B} (\vec{\nabla} \cdot \vec{B}) - \vec{B} \times (\vec{\nabla} \times \vec{B}) \right]
\]
By using the definition of the Maxwell stress tensor (86), we can simplify eqn (94) considerably

$$\frac{d}{dt} \left( \vec{L}_{\text{mech}} + \vec{L}_{\text{field}} \right) = \vec{r} \times \vec{\nabla} \cdot \vec{T}$$  \hspace{1cm} (94)

where

$$\vec{L}_{\text{field}} = \vec{r} \times \vec{g}$$  \hspace{1cm} (95)

has the interpretation of being the **EM field angular momentum density**. In integral form since

$$\vec{r} \times \vec{\nabla} \cdot \vec{T} = \vec{\nabla} \cdot \left( \vec{r} \times \vec{T} \right) \quad \text{and} \quad \vec{\nabla} \times \vec{r} = 0$$  \hspace{1cm} (96)

we get

$$\frac{d}{dt} \left( \vec{L}_{\text{mech}} + \int_V \vec{L}_{\text{field}} \, d^3x \right) = \int_S \left( \vec{r} \times \vec{T} \right) \cdot \vec{n} \, da$$  \hspace{1cm} (97)

The right-hand side of this equation represents the integrated torque density due to the fields over the boundary surface $S$. 

Classical Field Theory: Maxwell Equations