Radiation by Moving Charges

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\[1\text{J.D. Jackson, ”Classical Electrodynamics”, 3rd Edition, Chapter 14} \]
The Liénard-Wiechert potential describes the electromagnetic effect of a moving charge.

Built directly from Maxwell’s equations, this potential describes the complete, relativistically correct, time-varying electromagnetic field for a point-charge in arbitrary motion.

These classical equations harmonize with the 20th century development of special relativity, but are not corrected for quantum-mechanical effects.

Electromagnetic radiation in the form of waves are a natural result of the solutions to these equations.

These equations were developed in part by Emil Wiechert around 1898 and continued into the early 1900s.
We will study potentials and fields produced by a point charge, for which a trajectory $\vec{x}_0(t')$ has been defined a priori. It is obvious that when a charge $q$ is radiating is giving away momentum and energy, and possibly angular momentum and this emission affects the trajectory. This will be studied later. For the moment, we assume that the particle is moving with a velocity much smaller than $c$.

The density of the moving charge is given by

$$\rho(\vec{x}', t') = q\delta(\vec{x}' - \vec{x}_0[t'])$$  \hspace{1cm} (1)

and since in general the current density $\vec{J}$ is $\rho \vec{v}$, we also have

$$\vec{J}(\vec{x}', t') = q\vec{v}\delta(\vec{x}' - \vec{x}_0[t'])$$, \hspace{0.5cm} \text{where} \hspace{0.5cm} \vec{v}(t') = \frac{d\vec{x}_0}{dt'}  \hspace{1cm} (2)$$

In the Lorentz gauge ($\vec{\nabla} \cdot \vec{A} + (1/c)\partial\Phi/\partial t = 0$) the potential satisfy the wave equations (??) and (??) whose solutions are the retarded functions

$$\Phi(\vec{x}, t) = \int \frac{\rho(\vec{x}', t - |\vec{x} - \vec{x}'|/c)}{|\vec{x} - \vec{x}'|} d^3x'$$  \hspace{1cm} (3)
\[ \vec{A}(\vec{x}, t) = \frac{1}{c} \int \frac{\vec{J}(\vec{x}', t - |\vec{x} - \vec{x}'|/c)}{|\vec{x} - \vec{x}'|} d^3x' \] (4)

It is not difficult to see that these \textit{retarded potentials} take into account the finite propagation speed of the EM disturbances since an effect measured at \(\vec{x}\) and \(t\) was produced at the position of the source at time

\[ \tilde{t} = t - \frac{|\vec{x} - \vec{x}_0(\tilde{t})|}{c} \] (5)

Thus, using our expressions for \(\rho\) and \(\vec{J}\) from eqns (1) and (2) and putting \(\vec{\beta} \equiv \vec{v}/c\),

\[ \Phi(\vec{x}, t) = q \int \frac{\delta(\vec{x}' - \vec{x}_0[t - |\vec{x} - \vec{x}'|/c])}{|\vec{x} - \vec{x}'|} d^3x' \] (6)

\[ \vec{A}(\vec{x}, t) = q \int \frac{\vec{\beta}(t - |\vec{x} - \vec{x}'|/c)\delta(\vec{x}' - \vec{x}_0[t - |\vec{x} - \vec{x}'|/c])}{|\vec{x} - \vec{x}'|} d^3x' \] (7)
Note that for a given space-time point \((\vec{x}, t)\), there exists only one point on the whole trajectory, the retarded coordinate \(\vec{x}_r\) corresponding to the retarded time \(\tilde{t}\) defined in (5) which produces a contribution

\[
\vec{x} = \vec{x}_0(\tilde{t}) = \vec{x}_0(t - |\vec{x} - \vec{x}_0|/c) \tag{8}
\]

Let us also define the vector

\[
\vec{R}(t') = \vec{x} - \vec{x}_0(t') \tag{9}
\]

in the direction \(\vec{n} \equiv \vec{R}/R\). Then

\[
\Phi(\vec{x}, t) = q \int \frac{\delta(\vec{x}' - \vec{x}_0[t - R(t')/c])}{R(t')} \, d^3x' \tag{10}
\]

\[
\vec{A}(\vec{x}, t) = q \int \frac{\vec{\beta}(t - R(t')/c)\delta(\vec{x}' - \vec{x}_0[t - R(t')/c])}{R(t')} \, d^3x' \tag{11}
\]

Because the integration variable \(\vec{x}'\) appears in \(R(t')\) we transform it by introducing a new parameter \(\vec{r}^*\), where

\[
\vec{x}^* = \vec{x}' - \vec{x}_0 \left[t - R(t')/c\right] \tag{12}
\]
The volume elements $d^3x^*$ and $d^3x'$ are related by the Jacobian transformation

$$d^3x^* = Jd^3x' , \quad \text{where} \quad J \equiv \left[ 1 - \vec{n}(t') \cdot \vec{\beta}(t') \right] \quad (13)$$

is the Jacobian (how?). With the new integration variable, the integrals for the potential transform to

$$\Phi(\vec{x}, t) = q \int \frac{\delta(\vec{x}^*) \, d^3x^*}{|\vec{x} - \vec{x}^* - \vec{x}_0(\tilde{t})|(1 - \vec{n} \cdot \vec{\beta})} \quad (14)$$

and

$$\vec{A}(\vec{x}, t) = q \int \frac{\vec{\beta}(\tilde{t}) \, \delta(\vec{x}^*) \, d^3x^*}{|\vec{x} - \vec{x}^* - \vec{x}_0(\tilde{t})|(1 - \vec{n} \cdot \vec{\beta})} \quad (15)$$

which can be evaluated trivially, since the argument of the Dirac delta function restricts $\vec{x}^*$ to a single value

$$\Phi(\vec{x}, t) = \left[ \frac{q}{(1 - \vec{n} \cdot \vec{\beta})|\vec{x} - \tilde{x}|} \right]_{\tilde{t}} = \left[ \frac{q}{(1 - \vec{n} \cdot \vec{\beta})R} \right]_{\tilde{t}} \quad (16)$$

$$\vec{A}(\vec{x}, t) = \left[ \frac{q\vec{\beta}}{(1 - \vec{n} \cdot \vec{\beta})|\vec{x} - \tilde{x}|} \right]_{\tilde{t}} = \left[ \frac{q\vec{\beta}}{(1 - \vec{n} \cdot \vec{\beta})R} \right]_{\tilde{t}} \quad (17)$$
These are the Liénard - Wiechert potentials.

It is worth noticing the presence of the term \((1 - \vec{n} \cdot \vec{\beta})\), which clearly arises from the fact that the velocity of the EM waves is finite, so the retardation effects must be taken into account in determining the fields.
Special Note about the shrinkage factor \((1 - \vec{n} \cdot \vec{\beta})\)

Consider a thin cylinder moving along the \(x\)-axis with velocity \(v\). To calculate the field at \(x\) when the ends of the cylinder are at \((x_1, x_2)\), we need to know the location of the retarded points \(\tilde{x}_1\) and \(\tilde{x}_2\)

\[
\frac{x_1 - \tilde{x}_1}{x - \tilde{x}_1} = \frac{v}{c} \quad \text{and} \quad \frac{x_2 - \tilde{x}_2}{x - \tilde{x}_2} = \frac{v}{c}
\]  

(20)

by setting \(\tilde{L} \equiv \tilde{x}_2 - \tilde{x}_1\) and \(L \equiv x_2 - x_1\) and subtracting we get

\[
\tilde{L} - L = \frac{v}{c} \tilde{L} \quad \rightarrow \quad \tilde{L} = \frac{L}{1 - v/c}
\]

(21)

That is, the effective length \(\tilde{L}\) and the natural length \(L\) differ by the factor \((1 - \vec{x} \cdot \vec{\beta})^{-1} = (1 - v/c)^{-1}\) because the source is moving relative to the observer and its velocity must be taken into account when calculating the retardation effects.
Liénard - Wiechert potentials : radiation fields

The next step after calculating the potentials is to calculate the fields via the relations

\[ \vec{B} = \nabla \times \vec{A} \quad \text{and} \quad \vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \nabla \Phi \]  

(22)

and we write the Liénard - Wiechert potentials in the equivalent form

\[ \Phi(\vec{x}, t) = q \int \frac{\delta(t' - t - R(t')/c)}{R(t')} \, dt' \]  

(23)

\[ \vec{A}(\vec{x}, t) = q \int \frac{\vec{\beta}(t') \delta(t' - t + R(t')/c]}{R(t')} \, dt' \]  

(24)

where \( R(t') \equiv |\vec{x} - \vec{x}_0(t')| \). This can be verified by using the following property of the Dirac delta function (how?)

\[ \int g(x) \delta[f(x)] \, dx = \sum_i \left[ \frac{g(x_i)}{|df/dx|} \right]_{f(x_i)=0} \]  

(25)

which holds for regular functions \( g(x) \) and \( f(x) \) of the integration variable \( x \) where \( x_i \) are the zeros of \( f(x) \).

- The advantage in pursuing this path is that the derivatives in eqn (22) can be carried out before the integration over the delta function.
This procedure simplifies the evaluation of the fields considerably since, we do not need to keep track of the retarded time until the last step. We get for the electric field

\[ \vec{E}(\vec{x}, t) = -q \int \vec{\nabla} \left[ \frac{\delta(t' - t + R(t')/c)}{R(t')} \right] dt' \]

Thus, differentiating the integrand in the first term, we get (HOW?)

\[ \vec{E}(\vec{x}, t) = q \int \left[ \frac{\vec{n}}{R^2} \delta \left( t' - t + \frac{R(t')}{c} \right) - \frac{\vec{n}}{cR} \delta' \left( t' - t + \frac{R(t')}{c} \right) \right] dt' \]

But (HOW?)

\[ \delta' \left( t' - t + \frac{R(t')}{c} \right) = -\frac{\partial}{\partial t} \delta \left( t' - t + \frac{R(t')}{c} \right) \]

\[ \vec{E}(\vec{x}, t) = q \int \frac{\vec{n}}{R^2} \delta \left( t' - t + \frac{R(t')}{c} \right) dt' + \frac{q}{c} \frac{\partial}{\partial t} \int \left( \frac{\vec{n} - \vec{\beta}}{cR(t')} \right) \delta \left( t' - t + \frac{R(t')}{c} \right) dt' \]
We evaluate the integrals using the Dirac delta function expressed in equation (25). But we need to know the derivatives of the delta function's arguments with respect to $t'$. Using the chain rule of differentiation

$$\frac{d}{dt'} \left(t' - t + \frac{R(t')}{c}\right) = \left(1 - \vec{n} \cdot \vec{\beta}\right)\tilde{t}$$

(30)

with which we get the result (HOW?):

$$\vec{E}(\vec{r}, t) = q\left\{\frac{\vec{n}}{(1 - \vec{n} \cdot \vec{\beta})R^2}\right\}_{\tilde{t}} + \frac{q}{c} \frac{\partial}{\partial t} \left\{\frac{\vec{n} - \vec{\beta}}{(1 - \vec{n} \cdot \vec{\beta})R}\right\}_{\tilde{t}}$$

(31)

Since

$$\frac{\partial R}{\partial t} = \left(\frac{\partial R}{\partial t'}\right) \left(\frac{\partial t'}{\partial t}\right) = -\vec{n} \cdot \vec{v} \left(\frac{\partial t'}{\partial t}\right) = c \left(1 - \frac{\partial t'}{\partial t}\right) \Rightarrow \frac{\partial \tilde{t}}{\partial t} = \frac{1}{(1 - \vec{n} \cdot \vec{\beta})}$$

(32)

Thus

$$\frac{\partial}{\partial t} \left\{\frac{\vec{n} - \vec{\beta}}{(1 - \vec{n} \cdot \vec{\beta})R}\right\}_{\tilde{t}} = \frac{1}{(1 - \vec{n} \cdot \vec{\beta})} \frac{\partial}{\partial \tilde{t}} \left\{\frac{\vec{n} - \vec{\beta}}{(1 - \vec{n} \cdot \vec{\beta})R^2}\right\}_{\tilde{t}}$$

(33)
By using the additional pieces

\[ \dot{R}|_{\tilde{t}} = -c \left( \tilde{n} \cdot \tilde{\beta} \right)_{\tilde{t}} \]  

(34)

\[ \dot{\tilde{n}}|_{\tilde{t}} = \frac{c}{R} \left[ \tilde{n}(\tilde{n} \cdot \tilde{\beta}) - \tilde{\beta} \right]_{\tilde{t}} \]  

(35)

\[ \frac{d}{d\tilde{t}} \left( 1 - \tilde{n} \cdot \tilde{\beta} \right)_{\tilde{t}} = - \left( \tilde{n} \cdot \dot{\tilde{\beta}} + \tilde{\beta} \cdot \dot{\tilde{n}} \right) \]  

(36)

and we finally get

\[ \tilde{E}(\tilde{r}, \tilde{t}) = q \left\{ \frac{(\tilde{n} - \tilde{\beta})(1 - \beta^2)}{(1 - \tilde{n} \cdot \tilde{\beta})^3 R^2} + \frac{\tilde{n} \times \left[ (\tilde{n} - \tilde{\beta}) \times \dot{\tilde{\beta}} \right]}{c(1 - \tilde{n} \cdot \tilde{\beta})^3 R} \right\}_{\tilde{t}} \]  

(37)

A similar procedure for \( \tilde{B} \) shows that

\[ \tilde{B}(\tilde{r}, \tilde{t}) = \tilde{\nabla} \times \tilde{A} = \tilde{n}(\tilde{t}) \times \tilde{E} \]  

(38)
Some observations

- When the particle is at rest and unaccelerated with respect to us, the field reduces simply to Coulomb’s law $q\bar{n}/R^2$. Whatever corrections are introduced do not alter the empirical law.

- We also see a clear separation into the near field (which falls off as $1/R^2$) and the radiation field (which falls off as $1/R$).

- Unless the particle is accelerated ($\dot{\beta} \neq 0$), the field falls off rapidly at large distances. But when the radiation field is present, it dominates over the near field far from the source.

- As $\beta \to 1$ with $\dot{\beta} = 0$ the field displays a “bunching” effect. This “bunching” is understood as being a retardation effect, resulting from the finite velocity of EM waves.
Power radiated by an accelerated charge

If the velocity of an accelerated charge is small compared to the speed of light ($\beta \to 0$) then from eqn (37) we get

$$\vec{E} = \frac{q}{c} \left[ \frac{\vec{n} \times (\vec{n} \times \dot{\beta})}{R} \right]_{\text{ret}}$$  \hspace{1cm} (39)

The instantaneous energy flux is given by the Poynting vector

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = \frac{c}{4\pi} |\vec{E}|^2 \vec{n}$$  \hspace{1cm} (40)

The power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{c}{4\pi} R^2 |\vec{E}|^2 = \frac{q^2}{4\pi c} |\vec{n} \times (\vec{n} \times \dot{\beta})|^2$$  \hspace{1cm} (41)

and if $\Theta$ is the angle between the acceleration $\vec{\dot{v}}$ and $\vec{n}$ then the power radiated can be written as

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} |\vec{\dot{v}}|^2 \sin^2 \Theta$$  \hspace{1cm} (42)
The total \textit{instantaneous power radiated} is obtained by integration over the solid angle. Thus

\[
P = \frac{q^2}{4\pi c^3} |\mathbf{\dot{v}}|^2 \int_0^\pi 2\pi \sin^3 \Theta d\Theta = \frac{2}{3} \frac{q^2}{c^3} |\mathbf{\dot{v}}|^2
\]  

(43)

This expression is known as the \textbf{Larmor formula} for a nonrelativistic accelerated charge.

\textbf{NOTE} : From equation (39) is obvious that the radiation is polarized in the plane containing $\mathbf{\dot{v}}$ and $\mathbf{n}$. 

\begin{center}
\begin{tikzpicture}[>=latex]
\draw[->] (0,0) -- (2,2) node[above right] {$\mathbf{\dot{v}}$};
\draw[->] (0,0) -- (2,-2) node[below right] {$\mathbf{n}$};
\draw (0,0) -- (1,1) node[midway,above] {$\Theta$};
\end{tikzpicture}
\end{center}
Relativistic Extension

Larmor's formula (43) has an “easy” relativistic extension so that can be applied to charges with arbitrary velocities

\[ P = \frac{2}{3} \frac{e^2}{m^2 c^3} |\vec{p}|^2 \]  

(44)

where \( m \) is the mass of the charged particle and \( \vec{p} \) its momentum. The Lorentz invariant generalization is

\[ P = -\frac{2}{3} \frac{e^2}{m^2 c^3} \left( \frac{dp_\mu}{d\tau} \frac{dp^\mu}{d\tau} \right) \]  

(45)

where \( d\tau = dt/\gamma \) is the proper time and \( p^\mu \) is the charged particle’s energy-momentum 4-vector. Obviously for small \( \beta \) it reduces to (44)

\[ -\frac{dp_\mu}{d\tau} \frac{dp^\mu}{d\tau} = \left( \frac{d\vec{p}}{d\tau} \right)^2 - \frac{1}{c^2} \left( \frac{dE}{d\tau} \right)^2 = \left( \frac{d\vec{p}}{d\tau} \right)^2 - \beta^2 \left( \frac{dp}{d\tau} \right)^2 \]  

(46)

If (45) is expressed in terms of the velocity & acceleration (\( E = \gamma mc^2 \) & \( \vec{p} = \gamma m \vec{v} \) with \( \gamma = 1/(1 - \beta^2)^{1/2} \)), we obtain the Liénard result (HOW?)

\[ P = \frac{2}{3} \frac{e^2}{c} \gamma^6 \left[ \left( \frac{\vec{v}}{\gamma} \right)^2 - \left( \frac{\vec{\gamma} \times \vec{v}}{\gamma} \right)^2 \right] \]  

(47)
Applications

- In the charged-particle accelerators radiation losses are sometimes the limiting factor in the maximum practical energy attainable.
- For a given applied force the radiated power \( P \) depends inversely on the square of the mass of the particle involved. Thus these radiative effects are largest for electrons.
- In a linear accelerator the motion is 1-D. From (46) we can find that the radiated power is

\[
P = \frac{2}{3} \frac{e^2}{m^2 c^3} \left( \frac{dp}{dt} \right)^2
\]

The rate of change of momentum is equal to the rate of change of the energy of the particle per unit distance. Thus

\[
P = \frac{2}{3} \frac{e^2}{m^2 c^3} \left( \frac{dE}{dx} \right)^2
\]

showing that for linear motion the power radiated depends only on the external forces which determine the rate of change of particle energy with distance, not on the actual energy or momentum of the particle.
The ratio of power radiated to power supplied by external sources is

\[
\frac{P}{dE/dt} = \frac{2}{3} \frac{e^2}{m^2c^3} \frac{1}{v} \frac{dE}{dx} = \frac{2}{3} \frac{(e^2/mc^2)}{mc^2} \frac{dE}{dx} \tag{50}
\]

Which shows that the radiation loss in an electron linear accelerator will be unimportant unless the gain in energy is of the order of \( mc^2 = 0.5 \text{MeV} \) in a distance of \( e^2/mc^2 = 2.8 \times 10^{-13} \text{cm} \), or of the order of \( 2 \times 10^{14} \text{MeV/meter} \). Typically radiation losses are completely negligible in linear accelerators since the gains are less than \( 50 \text{MeV/meter} \).

\[\star\] Can you find out what will happen in circular accelerators like synchrotron or betatron?

In circular accelerators like synchrotron or betatron can change drastically. In this case the momentum \( \vec{p} \) changes rapidly in direction as the particle rotates, but the change in energy per revolution is small. This means that:

\[
\left| \frac{d\vec{p}}{d\tau} \right| = \gamma \omega |\vec{p}| \gg \frac{1}{c} \frac{dE}{d\tau} \tag{51}
\]

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Then the radiated power, eqn (45), can be written approximately

\[ P = \frac{2}{3} \frac{e^2}{m^2 c^3} \gamma^2 \omega^2 |\vec{p}|^2 = \frac{2}{3} \frac{e^2 c}{\rho^2} \beta^4 \gamma^4 \]  

(52)

where \( \omega = \frac{c \beta}{\rho} \), \( \rho \) being the orbit radius.

The radiative loss per revolution is:

\[ \delta E = \frac{2\pi \rho}{c \beta} P = \frac{4\pi}{3} \frac{e^2}{\rho} \beta^3 \gamma^4 \]  

(53)

For high-energy electrons (\( \beta \approx 1 \)) this gets the numerical value

\[ \delta E \text{(MeV)} = 8.85 \times 10^{-2} \left[ E \text{(GeV)} \right]^4 \rho \text{(meters)} \]  

(54)

In a 10GeV electron synchrotron (Cornell with \( \rho \sim 100\text{m} \)) the loss per revolution is 8.85MeV. In LEP (CERN) with beams at 60 GeV (\( \rho \sim 4300\text{m} \)) the losses per orbit are about 300 MeV.
Angular Distribution of Radiation Emitted by an Accelerated Charge

The energy per unit area per unit time measured at an observation point at time \( t \) of radiation emitter by charge at time \( t' = t - R(t'/c) \) is:

\[
\left[ \vec{S} \cdot \vec{n} \right]_{\text{ret}} = \frac{e^2}{4\pi c} \left\{ \frac{1}{R^2} \left| \frac{\vec{n} \times [(\vec{n} - \vec{\beta}) \times \vec{\beta}]}{(1 - \vec{\beta} \cdot \vec{n})^3} \right|^2 \right\}_{\text{ret}}
\]  

(55)

The energy radiated during a finite period of acceleration, say from \( t' = T_1 \) to \( t' = T_2 \) is

\[
E = \int_{T_1 + R(T_1)/c}^{T_2 + R(T_2)/c} \left[ \vec{S} \cdot \vec{n} \right]_{\text{ret}} dt = \int_{t' = T_1}^{t' = T_2} \left( \vec{S} \cdot \vec{n} \right) \frac{dt}{dt'} dt'
\]

(56)

Note that the useful quantity is \((\vec{S} \cdot \vec{n})(dt/dt')\) i.e. the power radiated per unit area in terms of the charge’s own time. Thus we define the power radiated per unit solid angle to be

\[
\frac{dP(t')}{d\Omega} = R^2 \left( \vec{S} \cdot \vec{n} \right) \frac{dt}{dt'} = R^2 \left( \vec{S} \cdot \vec{n} \right) (1 - \vec{\beta} \cdot \vec{n})
\]

(57)
If $\vec{\beta}$ and $\dot{\vec{\beta}}$ are nearly constant (e.g. if the particle is accelerated for short time) then (57) is proportional to the angular distribution of the energy radiated.

For the Poynting vector (55) the angular distribution is

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c} \frac{|\vec{n} \times \{(\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}}\}|^2}{(1 - \vec{n} \cdot \vec{\beta})^5}$$

(58)

The simplest example is linear motion in which $\vec{\beta}$ and $\dot{\vec{\beta}}$ are parallel i.e. $\vec{\beta} \times \dot{\vec{\beta}} = 0$ and (HOW?)

$$\frac{dP(t')}{d\Omega} = \frac{e^2 \gamma^2}{4\pi c^3} \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$

(59)

For $\beta \ll 1$, this is the Larmor result (42). But as $\beta \to 1$, the angular distribution is tipped forward and increases in magnitude.
• The angle $\theta_{\text{max}}$ for which the intensity is maximum is:

$$\cos \theta_{\text{max}} = \frac{1}{3\beta} \left( \sqrt{1 + 15\beta^2} - 1 \right) \quad \text{for} \quad \beta \approx 1 \quad \rightarrow \theta_{\text{max}} \approx \frac{1}{2\gamma} \quad (60)$$

For relativistic particles, $\theta_{\text{max}}$ is very small, thus the angular distribution is confined to a very narrow cone in the direction of motion.

• For small angles the angular distribution (59) can be written

$$\frac{dP(t')}{d\Omega} \approx \frac{8}{\pi} \frac{e^2 \dot{v}^2}{c^3} \gamma^8 \frac{(\gamma \theta)^2}{(1 + \gamma^2 \theta^2)^5} \quad (61)$$

The peak occurs at $\gamma \theta = 1/2$, and the half-power points at $\gamma \theta = 0.23$ and $\gamma \theta = 0.91$. 

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• The root mean square angle of emission of radiation in the relativistic limit is

$$\langle \theta^2 \rangle^{1/2} = \frac{1}{\gamma} = \frac{mc^2}{E}$$  \hspace{1cm} (62)

The total power can be obtained by integrating (59) over all angles

$$P(t') = \frac{2}{3} \frac{e^2}{c^3} \dot{v}^2 \gamma^6$$  \hspace{1cm} (63)

in agreement with (47) and (48). In other words this is a generalization of Larmor’s formula.

• It is instructive to express this in terms of the force acting on the particle. This force is \( \vec{F} = d\vec{p}/dt \) where \( \vec{p} = m\gamma \vec{v} \) is the particle’s relativistic momentum. For linear motion in the \( x \)-direction we have \( p_x = mv\gamma \) and

$$\frac{dp_x}{dt} = m\dot{v}\gamma + m\dot{v}\beta^2\gamma^3 = m\dot{v}\gamma^3$$

and Larmor’s formula can be written as

$$P = \frac{2}{3} \frac{e^2}{c^3} \frac{|\vec{F}|^2}{m^2}$$  \hspace{1cm} (64)

This is the total charge radiated by a charge in instantaneous linear motion.
Angular distr. of radiation from a charge in circular motion

The angular distribution of radiation for a charge in instantaneous circular motion with acceleration $\vec{\beta}$ perpendicular to its velocity $\vec{\beta}$ is another example. We choose a coordinate system such as $\vec{\beta}$ is in the z-direction and $\vec{\dot{\beta}}$ in the x-direction then the general formula (58) reduces to (HOW?)

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^3} \frac{v^2}{(1 - \beta \cos \theta)^3} \left[ 1 - \frac{\sin^2 \theta \cos^2 \phi}{\gamma^2(1 - \beta \cos \theta)^2} \right]$$

(65)

Although, the detailed angular distribution is different from the linear acceleration case the characteristic peaking at forward angles is present. In the relativistic limit ($\gamma \gg 1$) the angular distribution can be written

$$\frac{dP(t')}{d\Omega} \approx \frac{2e^2}{\pi c^3 \gamma^6} \frac{|\vec{v}|^2}{(1 + \gamma^2 \theta^2)^3} \left[ 1 - \frac{4\gamma^2 \theta^2 \cos^2 \phi}{\gamma^2(1 + \gamma^2 \theta^2)^2} \right]$$

(66)
The root mean square angle of emission in this approximation is similar to (62) just as in the 1-dimensional motion. (SHOW IT?)

The total power radiated can be found by integrating (65) over all angles or from (47)

\[ P(t') = \frac{2}{3} \frac{e^2 |\dot{\vec{v}}|^2}{c^3 \gamma^4} \]  \hspace{1cm} (67)

Since, for circular motion, the magnitude of the rate of momentum is equal to the force i.e. \( \gamma m \ddot{\vec{v}} \) we can rewrite (67) as

\[ P_{\text{circular}}(t') = \frac{2}{3} \frac{e^2}{m^2 c^3} \gamma^2 \left( \frac{d\vec{p}}{dt} \right)^2 \]  \hspace{1cm} (68)

If we compare with the corresponding result (48) for rectilinear motion, we find that the radiation emitted with a transverse acceleration is a factor \( \gamma^2 \) larger than with a parallel acceleration.
Radiation from a charge in arbitrary motion

- For a charged particle in arbitrary & extremely relativistic motion the radiation emitted is a superposition of contributions coming from accelerations **parallel** to and **perpendicular** to the velocity.
- But the radiation from the parallel component is negligible by a factor $1/\gamma^2$, compare (48) and (68). Thus we will keep only the perpendicular component alone.
- In other words the radiation emitted by a particle in arbitrary motion is the same emitted by a particle in instantaneous circular motion, with a radius of curvature $\rho$

$$\rho = \frac{v^2}{\dot{v}_\perp} \approx \frac{c^2}{\dot{v}_\perp}$$  \hspace{1cm} (69)$$

where $\dot{v}_\perp$ is the perpendicular component of the acceleration.
- The angular distribution of radiation given by (65) and (66) corresponds to a narrow cone of radiation directed along the instantaneous velocity vector of the charge.
- The radiation will be visible only when the particle’s velocity is directed toward the observer.
Since the angular width of the beam is $1/\gamma$ the particle will travel a distance of the order of

$$d = \frac{\rho}{\gamma}$$

in a time

$$\Delta t = \frac{\rho}{\gamma v}$$

while illuminating the observer.

If we consider that during the illumination the pulse is rectangular, then in the time $\Delta t$ the front edge of the pulse travels a distance

$$D = c \Delta t = \frac{\rho}{\gamma \beta}$$

Since the particle is moving in the same direction with speed $v$ and moves a distance $d$ in time $\Delta t$ the rear edge of the pulse will be a distance

$$L = D - d = \left(\frac{1}{\beta} - 1\right) \frac{\rho}{\gamma} \approx \frac{\rho}{2\gamma^3}$$

behind the front edge as the pulse moves off.
The Fourier decomposition of a finite wave train, we can find that the spectrum of the radiation will contain appreciable frequency components up to a critical frequency,

\[ \omega_c \sim \frac{c}{L} \sim \left( \frac{c}{\rho} \right) \gamma^3 \]  

(71)

- For circular motion the term \( c/\rho \) is the angular frequency of rotation \( \omega_0 \) and even for arbitrary motion plays the role of the fundamental frequency.
- This shows that a relativistic particle emits a broad spectrum of frequencies up to \( \gamma^3 \) times the fundamental frequency.
- **EXAMPLE**: In a 200MeV sychrotron, \( \gamma_{\text{max}} \approx 400 \), while \( \omega_0 \approx 3 \times 10^8 s^{-1} \). The frequency spectrum of emitted radiation extends up to \( \omega \approx 2 \times 10^{16} s^{-1} \).
• The previous qualitative arguments show that for relativistic motion the radiated energy is spread over a wide range of frequencies.
• The estimation can be made precise and quantitative by use of Parseval’s theorem of Fourier analysis.

The general form of the power radiated per unit solid angle is

\[ \frac{dP(t)}{d\Omega} = |\vec{A}(t)|^2 \] (72)

where

\[ \vec{A}(t) = \left( \frac{c}{4\pi} \right)^2 \left[ R \, \vec{E} \right]_{\text{ret}} \] (73)

and \( \vec{E} \) is the electric field defined in (37).

• Notice that here we will use the observer’s time instead of the retarded time since we study the observed spectrum.

The total energy radiated per unit solid angle is the time integral of (72):

\[ \frac{dW}{d\Omega} = \int_{-\infty}^{\infty} |\vec{A}(t)|^2 dt \] (74)

This can be expressed via the Fourier transform’s as an integral over the frequency.
The Fourier transform is:

\[ \tilde{A}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{A}(t)e^{i\omega t} dt \]  

(75)

and its inverse:

\[ \tilde{A}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{A}(\omega)e^{-i\omega t} d\omega \]  

(76)

Then eqn (74) can be written

\[ \frac{dW}{d\Omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \tilde{A}^*(\omega') \cdot \tilde{A}(\omega)e^{i(\omega' - \omega)t} \]  

(77)

If we interchange the order of integration between \( t \) and \( \omega \) we see that the time integral is the Fourier representation of the delta function \( \delta(\omega' - \omega) \). Thus the energy radiated per unit solid angle becomes

\[ \frac{dW}{d\Omega} = \int_{-\infty}^{\infty} |\tilde{A}(\omega)|^2 d\omega \]  

(78)

The equality of equations (74) and (78) is a special case of Parseval's theorem.

NOTE: It is customary to integrate only over positive frequencies, since the sign of the frequency has no physical meaning.
The energy radiated per unit solid angle per unit frequency interval is

\[
\frac{dW}{d\Omega} = \int_{0}^{\infty} \frac{d^2 I(\omega, \vec{n})}{d\omega d\Omega} d\omega
\]  

(79)

where

\[
\frac{d^2 I}{d\omega d\Omega} = |\vec{A}(\omega)|^2 + |\vec{A}(\omega)|^2
\]  

(80)

If \(\vec{A}(t)\) is real, form (75) - (76) it is evident that \(\vec{A}(\omega) = \vec{A}^*(\omega)\). Then

\[
\frac{d^2 I}{d\omega d\Omega} = 2|\vec{A}(\omega)|^2
\]  

(81)

which relates the power radiated as a function of time to the frequency spectrum of the energy radiated.

**NOTE:** We rewrite eqn (37) for future use

\[
\vec{E}(\vec{r}, t) = e \left\{ \frac{\vec{n} - \vec{\beta}}{\gamma^2(1 - \vec{n} \cdot \vec{\beta})^3 R^2} + \frac{\vec{n} \times \left[ (\vec{n} - \vec{\beta}) \times \vec{\beta} \right]}{c(1 - \vec{n} \cdot \vec{\beta})^3 R} \right\}_{\text{ret}}
\]  

(82)
By using (82) we will try to derive a general expression for the energy radiated per unit solid angle per unit frequency interval in terms of an integral over the trajectory of the particle. We must calculate the Fourier transform of (73) by using (82)

$$
\tilde{A}(\omega) = \left( \frac{e^2}{8\pi^2c} \right)^{1/2} \int_{-\infty}^{\infty} e^{i\omega t} \left[ \frac{\vec{n} \times [(\vec{n} - \vec{\beta}) \times \vec{\beta}]}{(1 - \vec{\beta} \cdot \vec{n})^3} \right]_{\text{ret}} dt \tag{83}
$$

where \( \text{ret} \) means evaluated at \( t = t' + R(t')/c \). By changing the integration variable from \( t \) to \( t' \) we get

$$
\tilde{A}(\omega) = \left( \frac{e^2}{8\pi^2c} \right)^{1/2} \int_{-\infty}^{\infty} e^{i\omega (t' + [R(t')/c])} \frac{\vec{n} \times [(\vec{n} - \vec{\beta}) \times \vec{\beta}]}{(1 - \vec{\beta} \cdot \vec{n})^2} dt' \tag{84}
$$

since the observation point is assumed to be far away the unit vector \( \vec{n} \) can be assumed constant in time, while we can use the approximation

$$
R(t') \approx x - \vec{n} \cdot \vec{r}(t') \tag{85}
$$

where \( x \) is the distance from the origin \( O \) to the observation point \( P \), and \( \vec{r}(t') \) is the position of the particle relative to \( O \).
Then (84) becomes:

\[
\vec{A}(\omega) = \left( \frac{e^2}{8\pi^2c} \right)^{1/2} \int_{-\infty}^{\infty} e^{i\omega(t-\vec{n} \cdot \vec{r}(t)/c)} \frac{\vec{n} \times [(\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{(1 - \vec{\beta} \cdot \vec{n})^2} dt \tag{86}
\]

and the energy radiated per unit solid angle per unit frequency interval (81) is

\[
\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{4\pi^2c} \left| \int_{-\infty}^{\infty} e^{i\omega(t-\vec{n} \cdot \vec{r}(t)/c)} \frac{\vec{n} \times [(\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{(1 - \vec{\beta} \cdot \vec{n})^2} dt \right|^2 \tag{87}
\]

For a specified motion \( \vec{r}(t) \) is known, \( \vec{\beta}(t) \) and \( \dot{\vec{\beta}}(t) \) can be computed, and the integral can be evaluated as a function of \( \omega \) and the direction of \( \vec{n} \).

If we study more than one accelerated charged particles, a coherent sum of amplitudes \( \vec{A}_j(\omega) \) (one for each particle) must replace must replace the single amplitude in (87).
If one notices that, the integrand in (86) is a perfect differential (excluding the exponential)

\[
\vec{n} \times \left[ \left( \vec{n} - \vec{\beta} \right) \times \dot{\vec{\beta}} \right] \quad \frac{1}{(1 - \vec{\beta} \cdot \vec{n})^3} = \frac{d}{dt} \left[ \frac{\vec{n} \times (\vec{n} \times \vec{\beta})}{1 - \vec{\beta} \cdot \vec{n}} \right]
\tag{88}
\]

then by integration by parts we get to the following relation for the intensity distribution:

\[
\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \vec{n} \times (\vec{n} \times \vec{\beta}) e^{i\omega(t - \vec{n} \cdot \vec{r}(t)/c)} dt \right|^2
\tag{89}
\]

For a number of charges \( e_j \) in accelerated motion the integrand in (89) becomes

\[
e^\vec{\beta} e^{-i(\omega/c)\vec{n} \cdot \vec{r}(t)} \to \sum_{j=1}^{N} e_j \vec{\beta}_j e^{-i(\omega/c)\vec{n} \cdot \vec{r}_j(t)}
\tag{90}
\]

In the limit of a continuous distribution of charge in motion the sum over \( j \) becomes an integral over the current density \( \vec{J}(\vec{x}, t) \) :

\[
e^\vec{\beta} e^{-i(\omega/c)\vec{n} \cdot \vec{r}(t)} \to \frac{1}{c} \int d^3 \vec{x} \vec{J}(\vec{x}, t) e^{-i(\omega/c)\vec{n} \cdot \vec{x}}
\tag{91}
\]
Then the intensity distribution becomes:

\[
\frac{d^2 I}{d\omega d\Omega} = \frac{\omega^2}{4\pi^2 c^3} \left| \int dt \int d^3 x \ e^{i\omega(t - \vec{n} \cdot \vec{x}/c)} \vec{n} \times [\vec{n} \times \vec{J}(\vec{x}, t)] \right|^2
\]  

(92)

a result that can be obtained from the direct solution of the inhomogeneous wave equation for the vector potential.

Radiation by Moving Charges
What Is Synchrotron Light?

When charged particles are accelerated, they radiate. If electrons are constrained to move in a curved path they will be accelerating toward the inside of the curve and will also radiate what we call synchrotron radiation.

- Synchrotron radiation of this type occurs naturally in the distant reaches of outer space.
- Accelerator-based synchrotron light was seen for the first time at the GE Research Lab (USA) in 1947 in a type of accelerator known as a synchrotron.
- First considered a nuisance because it caused the particles to lose energy, it recognized in the 1960s as light with exceptional properties.
- The light produced at today’s light sources is very bright. In other words, the beam of x-rays or other wavelengths is thin and very intense. Just as laser light is much more intense and concentrated than the beam of light generated by a flashlight.
Synchrotrons are particle accelerators - massive (roughly circular) machines built to accelerate sub-atomic particles to almost the speed of light.

- The accelerator components include an **electron gun**, one or more **injector accelerators** (usually a linear accelerator and a synchrotron but sometimes just a large linear accelerator) to increase the energy of the electrons, and a storage ring where the electrons circulate for many hours.
- In the storage ring, magnets force the electrons into circular paths.
- As the electron path bends, light is emitted tangentially to the curved path and streams down pipes called beamlines to the instruments where scientists conduct their experiments.

**Figure:** Components of a synchrotron light source typically include (1) an electron gun, (2) a linear accelerator, (3) a booster synchrotron, (4) a storage ring, (5) beamlines, and (6) experiment stations.
The storage ring is specifically designed to include special magnetic structures known as insertion devices (undulators and wigglers).

Insertion devices generate specially shaped magnetic fields that drive electrons into an oscillating trajectory for linearly polarized light or sometimes a spiral trajectory for circularly polarized light.

Each bend acts like a source radiating along the axis of the insertion device, hence the light is very intense and in some cases takes on near-laser-like brightness.

They produce synchrotron radiation - an amazing form of light that researchers are shining on molecules, atoms, crystals and innovative new materials in order to understand their structure and behaviour. It gives researchers unparalleled power and precision in probing the fundamental nature of matter.
Figure 14.12  (a) Schematic diagram of alternating-polarity bending magnets for a wiggler or undulator. (b) Sketch of approximately sinusoidal path of electron in the $x$-$z$ plane. The magnet period is $\lambda_0$, the maximum transverse amplitude is $a$, and the maximum angle is $\psi_0$. 
Synchrotron Radiation

To find the distribution of energy in frequency and in angle it is necessary to calculate the integral (89).

Because the duration of the pulse is very short, it is necessary to know the velocity $\vec{\beta}$ and the position $\vec{r}(t)$ over only a small arc of the trajectory.

The origin of time is chosen so that at $t = 0$ the particle is at the origin of coordinates.

Notice that only for very small angles $\theta$ there will be appreciable radiation intensity.

Figure: The trajectory lies on the plane $x - y$ with instantaneous radius of curvature $\rho$. The unit vector $\vec{n}$ can be chosen to lie in the $x - z$ plane, and $\theta$ is the angle with the x-axis.
The vector part of the integrand in eqn (89) can be written

\[ \vec{n} \times (\vec{n} \times \vec{\beta}) = \beta \left[ -\vec{\epsilon}_{\parallel} \sin(vt/\rho) + \vec{\epsilon}_{\perp} \cos(vt/\rho) \sin \theta \right] \]  

(93)

- \( \vec{\epsilon}_{\parallel} = \vec{\epsilon}_2 \) is a unit vector in the \( y \)-direction, corresponding to the polarization in the plane of the orbit.
- \( \vec{\epsilon}_{\perp} = \vec{n} \times \vec{\epsilon}_2 \) is the orthogonal polarization vector corresponding approx. to polarization perpendicular to the orbit plane (for small \( \theta \)).
- The argument of the exponential is

\[ \omega \left( t - \frac{\vec{n} \cdot \vec{r}(t)}{c} \right) = \omega \left[ t - \frac{\rho}{c} \sin \left( \frac{vt}{\rho} \right) \cos \theta \right] \]  

(94)

- Since we are dealing with small angle \( \theta \) and very short time intervals we can make an expansion to both trigonometric functions to obtain

\[ \omega \left( t - \frac{\vec{n} \cdot \vec{r}(t)}{c} \right) \approx \frac{\omega}{2} \left[ \left( \frac{1}{\gamma^2} + \theta^2 \right) t + \frac{c^2}{3\rho^2} t^3 \right] \]  

(95)

where \( \beta \) was set to unity wherever possible.

CHECK THE ABOVE RELATIONS
Thus the radiated energy distribution (89) can be written

\[
\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{4\pi^2 c} \left| -\varepsilon_\parallel A_\parallel(\omega) + \varepsilon_\perp A_\perp(\omega) \right|^2
\]  

(96)

where the two amplitudes are (How?)

\[
A_\parallel(\omega) \approx \frac{c}{\rho} \int_{-\infty}^{\infty} t \exp\left\{ i\omega \left[ \left( \frac{1}{\gamma^2} + \theta^2 \right) t + \frac{c^2 t^3}{3\rho^2} \right] \right\} dt
\]  

(97)

\[
A_\perp(\omega) \approx \theta \int_{-\infty}^{\infty} \exp\left\{ i\omega \left[ \left( \frac{1}{\gamma^2} + \theta^2 \right) t + \frac{c^2 t^3}{3\rho^2} \right] \right\} dt
\]  

(98)

by changing the integration variable

\[
x = \frac{ct}{\rho(1/\gamma^2 + \theta^2)^{1/2}}
\]

and introducing the parameter \( \xi \)

\[
\xi = \frac{\omega \rho}{3c} \left( \frac{1}{\gamma^2} + \theta^2 \right)^{3/2}
\]  

(99)

allows us to transform the integrals into the form
\[ A_{\parallel}(\omega) \approx \frac{\rho}{c} \left( \frac{1}{\gamma^2} + \theta^2 \right) \int_{-\infty}^{\infty} x \exp \left[ i \frac{3}{2} \xi \left( x + \frac{1}{3} x^3 \right) \right] dx \quad (100) \]

\[ A_{\perp}(\omega) \approx \frac{\rho}{c} \theta \left( \frac{1}{\gamma^2} + \theta^2 \right)^{1/2} \int_{-\infty}^{\infty} \exp \left[ i \frac{3}{2} \xi \left( x + \frac{1}{3} x^3 \right) \right] dx \quad (101) \]

These integrals are identifiable as Airy integrals or as modified Bessel functions (FIND OUT MORE)

\[ \int_{0}^{\infty} x \sin \left[ i \frac{3}{2} \xi \left( x + \frac{1}{3} x^3 \right) \right] dx = \frac{1}{\sqrt{3}} K_{2/3}(\xi) \quad (102) \]

\[ \int_{0}^{\infty} \cos \left[ i \frac{3}{2} \xi \left( x + \frac{1}{3} x^3 \right) \right] dx = \frac{1}{\sqrt{3}} K_{1/3}(\xi) \quad (103) \]

The energy radiated per unit frequency interval per unit solid angle is:

\[ \frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{3\pi^2 c} \left( \frac{\omega \rho}{c} \right)^2 \left( \frac{1}{\gamma^2} + \theta^2 \right)^2 \left[ K_{2/3}^2(\xi) + \frac{\theta^2}{1/\gamma^2 + \theta^2} K_{1/3}^2(\xi) \right] \quad (104) \]

- The 1st term corresponds to radiation polarized in the orbital plane.
- The 2nd term term to radiation polarized perpendicular to that plane.
By integration over all frequencies we find the distribution of energy in angle (Can you prove it?)

\[
\frac{dl}{d\Omega} = \int_0^\infty \frac{d^2l}{d\omega d\Omega} d\omega = \frac{7}{16} \frac{e^2}{\rho} \frac{1}{(1/\gamma^2 + \theta^2)^{5/2}} \left[ 1 + \frac{5}{7} \frac{\theta^2}{(1/\gamma^2) + \theta^2} \right]
\]

This shows the characteristic behavior seen in the circular motion case e.g. in equation (66).

This result can be obtained directly, by integrating a slight generalization of the power formula for circular motion, eqn (65), over all times. Again:

- The **1st term** corresponds polarization parallel to the orbital plane.
- The **2nd term** term to perpendicular polarization.

Integration over all angles shows that **seven (7) times** as much energy is radiated with parallel polarization as with perpendicular polarization. In other words:

The radiation from a relativistically moving charge is very strongly, but not completely, polarized in the plane of motion.
• The radiation is largely confined to the plane containing the motion, being more confined the higher the frequency relative to $c/\rho$.
• If $\omega$ gets too large, then $\xi$ will be large at all angles, and then there will be negligible power emitted at those high frequencies.
• The critical frequency beyond which there will be negligible total energy emitted at any angle can be defined by $\xi = 1/2$ and $\theta = 0$ (WHY?). Then we find

$$\omega_c = \frac{3}{2} \gamma^3 \left( \frac{c}{\rho} \right) = \frac{3}{2} \left( \frac{E}{mc^2} \right)^3 \frac{c}{\rho}$$

(106)

this critical frequency agrees with the qualitative estimate (71).
• If the motion is circular, then $c/\rho$ is the fundamental frequency of rotation, $\omega_0$.
• The critical frequency is given by

$$\omega_c = n_c \omega_0 \quad \text{with harmonic number} \quad n_c = \frac{3}{2} \left( \frac{E}{mc^2} \right)^3$$

(107)
For $\gamma \gg 1$ the radiation is predominantly on the orbital plane and we can evaluate via eqn (104) the angular distribution for $\theta = 0$. Thus for $\omega \ll \omega_c$ we find

$$\frac{d^2 I}{d\omega d\Omega} |_{\theta=0} \approx \frac{e^2}{c} \left[ \frac{\Gamma(2/3)}{\pi} \right]^2 \left( \frac{3}{4} \right)^{1/3} \left( \frac{\omega\rho}{c} \right)^{2/3}$$

(108)

For $\omega \gg \omega_c$

$$\frac{d^2 I}{d\omega d\Omega} |_{\theta=0} \approx \frac{3}{4\pi} \frac{e^2}{c} \gamma^2 \frac{\omega}{\omega_c} e^{-\omega/\omega_c}$$

(109)

These limiting cases show that the spectrum at $\theta = 0$ increases with frequency roughly as $\omega^{2/3}$ well below the critical frequency, reaches a maximum in the neighborhood of $\omega_c$, and then drops exponentially to 0 above that frequency.

- The spread in angle at a fixed frequency can be estimated by determining the angle $\theta_c$ at which $\xi(\theta_c) \approx \xi(0) + 1$.
- In the low frequency range ($\omega \ll \omega_c$), $\xi(0) \approx 0$ so $\xi(\theta_c) \approx 1$ which gives

$$\theta_c \approx \left( \frac{3c}{\omega\rho} \right)^{1/3} = \frac{1}{\gamma} \left( \frac{2\omega_c}{\omega} \right)^{1/3}$$

(110)

We note that the low frequency components are emitted at much wider angles than the average, $\langle \theta^2 \rangle^{1/2} \sim \gamma^{-1}$.  

Radiation by Moving Charges
In the **high frequency limit** \((\omega > \omega_c)\), \(\xi(0) \gg 1\) and the intensity falls off in angle as:

\[
\frac{d^2 I}{d\omega d\Omega} \approx \left. \frac{d^2 I}{d\omega d\Omega} \right|_{\theta=0} \cdot e^{-3\omega \gamma^2 \theta^2 / 2\omega_0}
\]  

(111)

Thus the critical angle defined by the \(1/e\) point is

\[
\theta_c \approx \frac{1}{\gamma} \left( \frac{2\omega_c}{3\omega} \right)^{1/2}
\]  

(112)

This shows that the high-frequency components are confined to an angular range **much smaller** than average.

Differential frequency spectrum as a function of angle. For frequencies comparable to the critical frequency \(\omega_c\), the radiation is confined to angles of order \(1/\gamma\) For much smaller (larger) frequencies the angular spread is larger (smaller).
The frequency distribution of the total energy emitted as the particle passes by can be found by integrating (104) over angles

\[
\frac{dl}{d\omega} = 2\pi \int_{-\pi/2}^{\pi/2} \frac{d^2l}{d\omega d\Omega} \cos \theta d\theta \approx 2\pi \int_{-\infty}^{\infty} \frac{d^2l}{d\omega d\Omega} d\theta
\]  

(113)

• For the low-frequency range we can use (95) at $\theta = 0$ and (108) at $\theta_c$, to get

\[
\frac{dl}{d\omega} \sim 2\pi \theta_c \frac{d^2l}{d\omega d\Omega} \bigg|_{\theta=0} \sim \frac{e^2}{c} \left(\frac{\omega \rho}{c}\right)^{1/3}
\]

(114)

showing the the spectrum increases as $\omega^{1/3}$ for $\omega \ll \omega_c$. This gives a very broad flat spectrum at frequencies below $\omega_c$.

• For the high-frequency limit $\omega \gg \omega_c$ we can integrate (111) over angles to get:

\[
\frac{dl}{d\omega} \approx \sqrt{\frac{3\pi}{2}} \frac{e^2}{c} \gamma \left(\frac{\omega}{\omega_c}\right)^{1/2} e^{-\omega/\omega_c}
\]

(115)
A proper integration of over angles yields the expression,

\[ \frac{dl}{d\omega} \approx \sqrt{3} \frac{e^2}{c} \gamma \frac{\omega}{\omega_c} \int_{\omega/\omega_c}^{\omega} K_{5/3}(x) dx \]  

(116)

In the limit \( \omega \ll \omega_c \), this reduces to the form (114) with numerical coefficient 13/4, while for \( \omega \gg \omega_c \) it is equal to (115).
Bellow the behavior of $dl/d\omega$ as function of the frequency. The peak intensity is of the order of $e^2\gamma/c$ and the total energy is of the order of $e^2\gamma\omega_c/c = 3e^2\gamma^4/\rho$. This is in agreement with the value $4\pi e^2\gamma^4/3\rho$ for the radiative loss per revolution (53) in circular accelerators.

Normalized synchrotron radiation spectrum

$$\frac{1}{l} \frac{dl}{dy} = \frac{9\sqrt{3}}{8\pi} y \int K_{5/3}(x)dx$$

where $y = \omega/\omega_c$ and $l = 4\pi e^2\gamma^4/3\rho$. 

Radiation by Moving Charges
The radiation represented by (104) and (116) is called **synchrotron radiation** because it was first observed in electron synchrotrons (1948).

- For periodic circular motion the spectrum is actually **discrete**, being composed of frequencies that are integral multipoles of the fundamental frequency \( \omega_0 = \frac{c}{\rho} \).
- Thus we should talk about the angular distribution of power radiated in the \( n \)th multiple of \( \omega_0 \) instead of the energy radiated per unit frequency interval per passage of the particle. Thus we can write (WHY?)

\[
\frac{dP_n}{d\Omega} = \frac{1}{2\pi} \left( \frac{c}{\rho} \right)^2 \frac{d^2I}{d\omega d\Omega} \bigg|_{\omega=n\omega_0} \quad (117)
\]

\[
P_n = \frac{1}{2\pi} \left( \frac{c}{\rho} \right)^2 \frac{dI}{d\omega} \bigg|_{\omega=n\omega_0} \quad (118)
\]

- These results have been compared with experiment at various energy synchrotrons. The angular, polarization and frequency distributions are all in good agreement with theory.
- Because of the broad frequency distribution shown in previous Figure, covering the visible, ultraviolet and x-ray regions, synchrotron radiation is a useful tool for studies in condensed matter and biology.
Fourier transform of the electric field produced by a charged particle in circular motion. The plots reveal that the number of relevant harmonics of the fundamental frequency $\omega_0$ increases with $\gamma$. And the dominant harmonic is shifted to higher frequencies. (A. $\gamma = 1$, $\beta = 0$, $\omega_c = 1$, B. $\gamma = 1.2$, $\beta = 0.55$, $\omega_c = 1.7$, C. $\gamma = 1.4$, $\beta = 0.7$, $\omega_c = 2.7$, D. $\gamma = 1.6$, $\beta = 0.78$, $\omega_c = 4.1$,
Thomson Scattering of Radiation

When a plane wave of monochromatic EM radiation hits a free particle of charge \( e \) and mass \( m \) the particle will be accelerated and so emit radiation. The radiation will be emitted in directions other than the propagation direction of the incident wave, but (for non-relativistic motion of the particle) it will have the same frequency as the incident radiation. According to eqn (41) the instantaneous power radiated into polarization state \( \vec{\epsilon} \) by a particle is (How?)

\[
\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} \left| \vec{\epsilon}^* \cdot \vec{V} \right|^2
\]  \hspace{1cm} (119)

If the propagation vector \( \vec{k}_0 \) and its the polarization vector \( \vec{\epsilon}_0 \) can be written

\[
\vec{E}(x, t) = \vec{\epsilon}_0 E_0 e^{i\vec{k}_0 \cdot \vec{x} - i\omega t}
\]
Then from the force eqn \((\vec{F} = q\vec{E})\) the acceleration will be

\[
\dot{\vec{v}}(t) = \vec{\epsilon}_0 \frac{e}{m} E_0 e^{ik_0 \cdot \vec{x} - i\omega t}
\]  \hspace{1cm} (120)

If we assume that the charge moves a negligible part of a wavelength during one cycle of oscillation, the time average of \(|\dot{\vec{v}}|^2\) is \(\frac{1}{2} \Re(\dot{\vec{v}} \cdot \dot{\vec{v}}^*)\)

Then the averaged power per unit solid angle can be expressed as

\[
\langle \frac{dP}{d\Omega} \rangle = \frac{c}{8\pi} |E_0|^2 \left( \frac{e^2}{mc^2} \right)^2 |\vec{\epsilon}^* \cdot \vec{\epsilon}_0|^2
\]  \hspace{1cm} (121)

And since the phenomenon is practically scattering then it is convenient to used the **scattering cross section** as

\[
\frac{d\sigma}{d\Omega} = \frac{\text{Energy radiated/unit time/unit solid angle}}{\text{Incident energy flux in energy/unit area/unit time}}
\]  \hspace{1cm} (122)

The incident energy flux is the time averaging Poynting vector for the plane wave i.e. \(c|E_0|^2/8\pi\). Thus from eqn (121) we get the differential scattering cross section

\[
\frac{d\sigma}{d\Omega} = \left( \frac{e^2}{mc^2} \right)^2 |\vec{\epsilon}^* \cdot \vec{\epsilon}_0|^2
\]  \hspace{1cm} (123)
The scattering geometry with a choice of polarization vectors for the outgoing wave is shown in the Figure. The polarization vector $\vec{\epsilon}_1$ is in the plane containing $\vec{n}$ and $\vec{k}_0$; $\vec{\epsilon}_2$ is perpendicular to it.

In terms of unit vectors parallel to the coordinate axes, $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$ are:

$$
\vec{\epsilon}_1 = \cos \theta (\vec{\epsilon}_x \cos \phi + \vec{\epsilon}_y \sin \phi) - \vec{\epsilon}_z \sin \theta
$$

$$
\vec{\epsilon}_2 = -\vec{\epsilon}_x \sin \phi + \vec{\epsilon}_y \cos \phi
$$

For an incident linearly polarized wave with polarization parallel to the x-axis, the angular distribution is $(\cos^2 \theta \cos^2 \phi + \sin^2 \phi)$. For polarization parallel to the y-axis it is $(\cos^2 \theta \sin^2 \phi + \cos^2 \phi)$. For unpolarized incident radiation the scattering cross section is

$$
\frac{d\sigma}{d\Omega} = \left( \frac{e^2}{mc^2} \right)^2 \frac{1}{2} (1 + \cos^2 \theta)
$$

This is called the Thomson formula for scattering of radiation by a free charge, and is appropriate for the scattering of x-rays by electrons or gamma rays by protons.
The total scattering cross section called the **Thomson cross section**

\[
\sigma_T = \frac{8\pi}{3} \left( \frac{e^2}{mc^2} \right)^2
\]  

(125)

The Thomson cross section for electrons is \(0.665 \times 10^{-24}\) cm\(^2\). The unit of length \(e^2/mc^2 = 2.82 \times 10^{-13}\) cm is called **classical electron radius**.

- This classical Thomson formula is valid only for low frequencies where the momentum of the incident photon can be ignored.
- When the photon momentum \(\hbar \omega/c\) becomes comparable to or larger than \(mc\) modifications occur.
- The most important is that the energy or momentum of the scattered photon is less than the incident energy because the charged particle recoils during the collision.

The outgoing to the incident wave number is given by **Compton formula**

\[
k'/k = \left[ 1 + \frac{\hbar \omega}{mc^2} (1 - \cos \theta) \right]^{-1}
\]  

(126)

In quantum mechanics the scattering of photons by spinless point particles of charge \(e\) and mass \(m\) yields the cross section:

\[
\frac{d\sigma}{d\Omega} = \left( \frac{e^2}{mc^2} \right)^2 \left( \frac{k'}{k} \right)^2 |\vec{\epsilon}^* \cdot \vec{\epsilon}_0|^2
\]  

(127)

**Radiation by Moving Charges**