Special Theory of Relativity

June 1, 2010[1]

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[1] J.D.Jackson, "Classical Electrodynamics", 3rd Edition, Chapter 11

Introduction

Einstein's theory of special relativity is based on the assumption (which might be a deep-rooted superstition in physics) that **all** physical laws should be invariant under transformation between inertial systems.

The demand that Maxwell's equations should be invariant under transformations, and the failure of Galilean transformations to do it led to the **Lorentz transformations** ($\vec{\beta} = \vec{v}/c$, $\gamma = (1-\beta^2)^{-1/2}$)

$$\begin{aligned} x_0 &= \gamma(x_0' - \beta x_1') \\ x_1 &= \gamma(x_1' - \beta x_0') \\ x_2 &= x_2' \\ x_3 &= x_3' \end{aligned} \qquad (1) \qquad\qquad \begin{aligned} x_0 &= \gamma(x_0' + \beta x_1') \\ x_1 &= \gamma(x_1' + \beta x_0') \\ x_2 &= x_2' \\ x_3 &= x_3' \end{aligned} \qquad (2)$$

under which for example the equations of a spherical wave

$$c^2 t^2 - \left(x^2 + y^2 + z^2\right) = 0 \tag{3}$$

propagating with fixed velocity c are invariant.

Lorentz transformations in general demand that the norm

$$s^2 = x_0^2 - \left(x_1^2 + x_2^2 + x_3^2\right) \tag{4}$$

is invariant.

1st Postulate : The laws of nature and the results of all experiments performed in a given frame of reference are independent of the translational motion of the system as a whole
2nd Postulate : The speed of light is finite and independent of the motion of the source
From the 1st postulate it follows that the mathematical equations expressing the laws of nature must be **covariant**, that is, invariant in form, under the Lorentz transformations.
These demands call for rules on the ways that the **scalars**, **4-vectors** and **4-tensors** will transform in a spacetime whose norm is defined by (4).

SPACETIME

The space-time continuum is defined in terms of a 4-dimensional space with coordinates x^0, x^1, x^2, x^3.

Tensors

If we assume that there is a well defined transformation that yields from the coordinates x^0, x^1, x^2, x^3 a new set of coordinates x'^0, x'^1, x'^2, x'^3 according to the rule

$$x'^\alpha = x'^\alpha(x^0, x^1, x^2, x^3) \quad (\alpha = 0, 1, 2, 3) \tag{5}$$

Here we will defined the tensors under their transformation properties.

A **scalar** (tensor of rank 0) is a single quantity whose value is not changed under the transformation. for example the interval s^2 in (4) is a scalar.

Vectors are tensors of rank 1, and we distinguish two kinds.

The **contravariant vector** A^α whose components transformed according to the rule

$$A'^\alpha = \sum_{\beta=0}^{3} \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta \equiv \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta \tag{6}$$

where the partial derivatives are calculated from (5). Explicitly we have 4 equations of the form:

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^0} A^0 + \frac{\partial x'^\alpha}{\partial x^1} A^1 + \frac{\partial x'^\alpha}{\partial x^2} A^2 + \frac{\partial x'^\alpha}{\partial x^3} A^3 \tag{7}$$

The **covariant vector** B_α is defined by the rule

$$B'_\alpha = \sum_{\beta=0}^{3} \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta \equiv \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta \tag{8}$$

where the partial derivatives are calculated from the inverse of (5).
The **contravariant tensor of rank 2** $F^{\alpha\beta}$ consists of 16 quantities (components) that transform according to

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta} \tag{9}$$

A **covariant tensor of rank 2** $G_{\alpha\beta}$ transforms as

$$G'_{\alpha\beta} = \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} G_{\gamma\delta} \tag{10}$$

The **mixed tensor of rank 2** $H^\alpha{}_\beta$ transforms as

$$H'^\alpha{}_\beta = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x^\delta}{\partial x'^\beta} H^\gamma{}_\delta \tag{11}$$

The generalization to arbitrary rank tensors is quite obvious extension of the above relations.

The **inner** or **scalar** product of two vectors is defined as the product of the components of a covariant and a contravariant vector

$$\mathbf{B}\cdot\mathbf{A}\equiv B_\alpha A^\alpha \tag{12}$$

with this definition the scalar product is an invariant or scalar under the transfomation (5):

$$\begin{aligned}\mathbf{B}'\cdot\mathbf{A}' &= B'_\alpha A'^\alpha = \frac{\partial x^\beta}{\partial x'^\alpha}B_\beta\frac{\partial x'^\alpha}{\partial x^\gamma}A^\gamma = \frac{\partial x^\beta}{\partial x^\gamma}B_\beta A^\gamma \\ &= \delta^\beta{}_\gamma B_\beta A^\gamma = B_\gamma A^\gamma = \mathbf{B}\cdot\mathbf{A}\end{aligned} \tag{13}$$

The geometry of the space-time of STR is defined by the invariant interval s^2 defined in (4), which in differential form can be written as

$$(ds)^2=(dx^0)^2-(dx^1)^2-(dx^2)^2-(dx^3)^2 \tag{14}$$

This **norm** or **metric** is a special case of the general differential length element

$$ds^2=g_{\alpha\beta}dx^\alpha dx^\beta \tag{15}$$

where $g_{\alpha\beta}=g_{\beta\alpha}$ is called the **metric tensor**.

For the flat space-time of STR the metric tensor is diagonal with elements

$$g_{00} = 1\,, \quad g_{11} = g_{22} = g_{33} = -1 \tag{16}$$

The contravariant tensor $g^{\alpha\beta}$ is defined as the normalized cofactor of $g_{\alpha\beta}$. For the flat spacetime of STR they are the same

$$g^{\alpha\beta} = g_{\alpha\beta} \tag{17}$$

The **contraction** of the covariant and contravariant metric tensors defines the **Kronecker delta** in 4-dimensions

$$g_{\alpha\gamma} g^{\gamma\beta} = \delta_{\alpha}{}^{\beta} \tag{18}$$

where $\delta_{\alpha}^{\beta} = 0$ if $\alpha \neq \beta$ and $\delta_{\alpha}^{\alpha} = 1$.
From the definition of the scalar product (12) and (15) we can easily conclude that

$$x_{\alpha} = g_{\alpha\beta} x^{\beta} \tag{19}$$

and its inverse

$$x^{\alpha} = g^{\alpha\beta} x_{\beta} \tag{20}$$

This is a more general procedure for lowering and raising indeces

$$F^{\ldots\alpha\ldots}_{\ldots} = g^{\alpha\beta} F^{\ldots}_{\ldots\beta\ldots} \quad \text{and} \quad G^{\ldots}_{\ldots\alpha\ldots} = g_{\alpha\beta} G^{\ldots\beta\ldots}_{\ldots} \tag{21}$$

From the definition of the flat spacetime metric tensor we can easily prove that:

$$A^\alpha = (A^0, \vec{A}), \quad A_\alpha = (A^0, -\vec{A}) \tag{22}$$

The scalar product (12) of two vectors is

$$\mathbf{B} \cdot \mathbf{A} \equiv B_\alpha A^\alpha = B^0 A^0 - \vec{B} \cdot \vec{A}$$

From the transformation property

$$\frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial}{\partial x^\beta}$$

we conclude that the differentiation with respect to a contravariant component of the coordinate vector transforms as the component of a covariant vector. Thus we employ the notation

$$\partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right), \quad \partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right) \tag{23}$$

The 4-divergence of a 4-vector A is the invariant

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla} \cdot \vec{A} \tag{24}$$

an equation familiar in form from continuity of charge and current density.

The 4-dimensional Laplacian operator is defined to be the invariant contraction

$$\Box \equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x^{0\,2}} - \nabla^2 \tag{25}$$

which is of course the operator of the wave equation in vacuum.

The previous examples show how the covariance of a physical law emerges provided suitable Lorentz transformation properties are attributed to the quantities entering the equation.

Invariance of Electric Charge; Covariance in Electrodynamics

- The invariance of the equations of electrodynamics under Lorentz transforms was shown by Lorentz and Poincaré before the formulation of the STR.
- The invariance in form or **covariance** of the Maxwell and Lorentz force equations implies that the various quantities ρ, $\vec{J}$, $\vec{E}$, $\vec{B}$ that enter into the equations transform in a well defined way under Lorentz transformations.

Consider first the Lorentz force equation for a charged particle

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \tag{26}$$

we know that $\vec{p}$ transforms as the space part of energy and momentum

$$p^\alpha = (p_0, \vec{p}) = m\left(U_0, \vec{U}\right)$$

where $p_0 = E/c$ and U^a is the 4-velocity

$$U_0 \equiv \frac{dx_0}{d\tau} = \frac{dx_0}{dt}\frac{dt}{d\tau} = \gamma c\ ,\ \ \vec{U} \equiv \frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{dt}\frac{dt}{d\tau} = \gamma\vec{u} \tag{27}$$

If we use the **proper time of the particle** which is a Lorentz invariant quantity defined as

$$d\tau = \frac{1}{c}ds = dt\sqrt{1-\beta^2} = \frac{1}{\gamma}dt \tag{28}$$

for the differentiation of (26) we can write

$$\frac{d\vec{p}}{d\tau} = \frac{q}{c}\left(U_0\vec{E} + \vec{U} \times \vec{B}\right) \tag{29}$$

the left hand side is the space part of a 4-vector. The corresponding time component equation is the rate of change of the energy of the particle

$$\frac{dp_0}{dt} = \frac{q}{c}\vec{U} \cdot \vec{E} \quad \Leftarrow \quad \frac{dE_{\text{mech}}}{dt} = \int_V \vec{J} \cdot \vec{E} d^3x \tag{30}$$

The right-hand sides of the previous two equations involve three factors, the charge q, the 4-velocity and the electromagnetic fields.
If the transformation properties of two of the three factors are known and Lorentz covariance is demanded, then the transformation properties of the 3rd factor can be established.

The experimental invariance of electric charge and the requirement of Lorentz covariance of the Lorentz force eqn (29) and (30) determines the Lorentz transformation properties of the EM field.

For example, the requirement from (30) that $\vec{U} \cdot \vec{E}$ be the time component of a 4-vector establishes that the components of $\vec{E}$ are the time-space parts of a 2nd rank tensor $F^{\alpha\beta}$ such that

$$\vec{U} \cdot \vec{E} = F^{0\beta} U_\beta$$

We will consider Maxwell equations and we begin with the charge density $\rho(\vec{x}, t)$ and current density $\vec{J}(\vec{x}, t)$ and the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \tag{31}$$

It is natural to postulate that ρ and $\vec{J}$ together form a 4-vector J^α :

$$J^\alpha = \left(c\rho, \vec{J} \right) \tag{32}$$

and the continuity equation takes the covariant form:

$$\partial_\alpha J^\alpha = 0 \tag{33}$$

where the covariant differential operator ∂_α is given by (23).

If we consider the Lorentz gauge

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0 \tag{34}$$

then the wave equations for the vector and scalar potential are

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} &= \frac{4\pi}{c}\vec{J} \\ \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi &= 4\pi\rho \end{aligned} \tag{35}$$

Notice that the differential operator in (35) is the invariant 4-D Laplacian (25) while the right hand side are the components of the 4-vector (32). Obviously, Lorentz covariance requires that the potentials Φ and $\vec{A}$ form a 4-vector potential

$$A^\alpha = \left(\Phi, \vec{A}\right) \tag{36}$$

Then the **wave equation** (35) and the **Lorentz condition** (34) take the covariant forms

$$\Box A^\alpha = \frac{4\pi}{c} J^\alpha , \quad \partial_\alpha A^\alpha = 0 \tag{37}$$

The fields $\vec{E}$ and $\vec{B}$ are expressed in terms of the potentials as

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi\,, \quad \vec{B} = \vec{\nabla} \times \vec{A} \tag{38}$$

where, for example, the x-component of $\vec{E}$ and $\vec{B}$ are explicitly

$$\begin{aligned} E_x &= -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x} = -\left(\partial^0 A^1 - \partial^1 A^0\right) \\ B_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\left(\partial^2 A^3 - \partial^3 A^2\right) \end{aligned} \tag{39}$$

These equations imply that the 6 in total components of the electric and magnetic fields are the elements of a 2nd-rank, antisymmetric field-strength tensor

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{40}$$

explicitly in matrix form

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{41}$$

In the covariant form is:

$$F_{\alpha\beta} = g_{\alpha\gamma} g_{\delta\beta} F^{\gamma\delta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{42}$$

The elements of $F_{\alpha\beta}$ are obtained from $F^{\alpha\beta}$ by putting $\vec{E} \rightarrow -\vec{E}$.

- Notice that

$$F_{\mu\nu} F^{\mu\nu} = 2\left(B^2 - \frac{1}{c^2} E^2\right) = \text{invariant} \tag{43}$$

and the **Lorentz force equation** becomes

$$\frac{dp_\alpha}{d\tau} = qF_{\alpha\beta} u^\beta \quad \text{or} \quad \frac{dp_\alpha}{dt} = qF_{\alpha\beta} \frac{dx^\beta}{dt} \tag{44}$$

The inhomogeneous Maxwell equations are

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho\,, \quad \vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

in terms of $F^{\alpha\beta}$ and J^{α} they take the covariant form (HOW?)

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta \tag{45}$$

Similarly the homogeneous Maxwell equations are

$$\vec{\nabla} \cdot \vec{B} = 0\,, \quad \vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

take the form (HOW?)

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{46}$$

With the above definitions of the various quantities and the reformulation of the wave and Maxwell equations the covariance of the equations of EM is established.
Finally, the Lorentz force (29) and rate of change of energy (30) can be set in manifestly covariant form

$$\frac{dp^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} U_\beta \tag{47}$$

Dual Field-Strength Tensor

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \tag{48}$$

where

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{for} \quad \alpha = 0, \beta = 1, \gamma = 2, \delta = 3 \\ & \text{and for any even permutation} \\ -1 & \text{for any odd permutation} \\ 0 & \text{if any two indices are equal} \end{cases} \tag{49}$$

The elements of $\mathcal{F}^{\alpha\beta}$ are obtained from $F^{\alpha\beta}$ by putting $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$. The homogeneous Maxwell equations can be written in terms of the dual field-strength tensor (prove it) as

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \tag{50}$$

Transformation of Electromagnetic Fields

Since both $\vec{E}$ and $\vec{B}$ are the elements of a 2nd-rank tensor $F^{\alpha\beta}$, their values in one inertial frame can be expressed in terms of the values in another inertial frame, according to

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta} \tag{51}$$

If the one system travels along the direction of x_1 with speed $c\beta$ the explicit transformations are (HOW?)

$$\begin{aligned} E_1' &= E_1 & B_1' &= B_1 \\ E_2' &= \gamma(E_2 - \beta B_3) & B_2' &= \gamma(B_2 + \beta E_3) \\ E_3' &= \gamma(E_3 + \beta B_2) & B_3' &= \gamma(B_3 - \beta E_2) \end{aligned} \tag{52}$$

This suggest that for a general Lorentz transformation between two systems moving with a speed v relative to each other the transformation of the fields can be written (HOW):

$$
\begin{aligned}
\vec{E}' &= \gamma\left(\vec{E}+\vec{\beta}\times\vec{B}\right)-\frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right) \\
\vec{B}' &= \gamma\left(\vec{B}-\vec{\beta}\times\vec{E}\right)-\frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{B}\right)
\end{aligned}
\tag{53}
$$

- These transformations show that $\vec{E}$ and $\vec{B}$ have no independent existence.
- A purely electric or magnetic field in one coordinate system will appear as a mixture of electric and magnetic fields in another coordinate frame.
- Thus one should properly speak of the electromagnetic field $F^{\alpha\beta}$ rather than $\vec{E}$ and $\vec{B}$ separately.

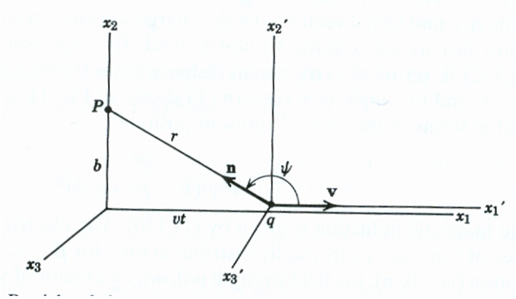
Finally, if no magnetic field exists in a frame K' the inverse of (53) shows that in the frame K the magnetic field $\vec{B}$ and the electric field $\vec{E}$ are linked by the simple relation

$$\vec{B}=\vec{\beta}\times\vec{E} \tag{54}$$

note that $\vec{E}$ is the transformed field from K' to K.

Transformation of Electromagnetic Fields: Example

We will study the fields seen by an observer in the system K when a point charge q moves in a straight line with velocity $\vec{v}$.
The charge is at rest in the system K' and the transformation of the fields is given by the inverse of (53) or (53)



The observer is at the point P. In the frame K' the observer's point P, where the fields are to be evaluated, has coordinates $x_1' = -vt'$, $x_2' = b$, $x_3' = 0$ and is at a distance $r' = \sqrt{b^2 + (vt)^2}$.

In the rest frame K' of the charge the electric and magnetic fields at the observation point are (WHY?)

$$E_1' = -\frac{qvt'}{r'^3} \quad E_2' = \frac{qb}{r'^3} \quad E_3' = 0$$
$$B_1' = 0 \quad B_2' = 0 \quad B_3' = 0$$

In terms of the coordinates of K the nonzero field components are

$$E_1' = -\frac{q\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}}\,, \quad E_2' = \frac{qb}{(b^2+\gamma^2v^2t^2)^{3/2}} \tag{55}$$

Then using the inverse of (53) we find the transformed fields in the system K:

$$\begin{aligned} E_1 &= E_1' = -\frac{q\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}} \\ E_2 &= \gamma E_2' = \frac{\gamma qb}{(b^2+\gamma^2v^2t^2)^{3/2}} && (56) \\ B_3 &= \gamma\beta E_2' = \beta E_2 && (57) \end{aligned}$$

with all the other components vanishing.

- Notice the magnetic induction in the direction x_3.
- The magnetic field becomes nearly equal to the transverse electric field E_2 as $\beta \to 1$.

- At low velocities ($\gamma \approx 1$) the magnetic induction is

$$\vec{B} \approx \frac{q}{c} \frac{\vec{v} \times \vec{r}}{r^3}$$

which is the approximate Ampére-Biot-Savart expression for the magnetic field of a moving charge.

- At high velocities ($\gamma \gg 1$) we see that the transverse electric field E_2 becomes equal to γ times its non-relativistic value.
- At high velocities ($\gamma \gg 1$) the duration of appreciable field strengths at point P is decreased.

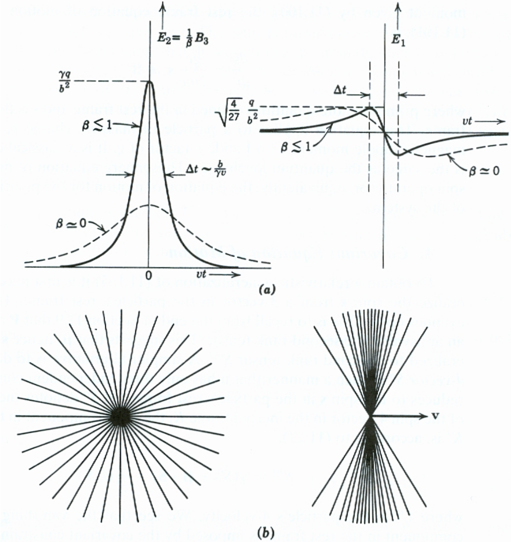


Figure: Fields of a uniformly moving charged oarticle (a) Fields at the observation point P as function of time. (b) Lines of electric force for a particle at rest and in motion ($\gamma = 3$).