Dynamics of Relativistic Particles and EM Fields

October 7, 2008[1]

[1] J.D.Jackson, "Classical Electrodynamics", 3rd Edition, Chapter 12

Lagrangian Hamiltonian for a Relativistic Charged Particle

The equations of motion

$$\frac{d\vec{p}}{dt} = e\left[\vec{E} + \frac{\vec{u}}{c} \times \vec{B}\right] \tag{1}$$

$$\frac{dE}{dt} = e\vec{u} \cdot \vec{E} \tag{2}$$

for a particle with charge e in external fields $\vec{E}$ and $\vec{B}$ can be written in the covariant form (**??**)

$$\frac{dU^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} U_\beta \tag{3}$$

where m is the mass, τ is the proper time, and $U^\alpha = (\gamma c, \gamma \vec{u}) = p^\alpha / m$ is the 4-velocity of the particle.

- The Lagrangian treatment of mechanics is based on the principle of of least action or Hamilton's principle.
- The system is described by generalized coordinates $q_i(t)$ & velocities $\dot{q}_i(t)$.
- The **Lagrangian** L is a functional of $q_i(t)$ and $\dot{q}_u(t)$ and perhaps the time.

- The **action** A is the time integral of L along a possible path of the system.
- The **principle of least action** states that the motion of a mechanical system is such that in going from a configuration a at time t_1 to a configuration b in time t_2 the action

$$A = \int_{t_1}^{t_2} L\left[q_i(t), \dot{q}_i(t), t\right] dt \tag{4}$$

is an **extremum**.

By considering small variations of coordinates and velocities away from the actual path and requiring $\delta A = 0$ one obtains the **Euler-Lagrange** equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{5}$$

Relativistic Lagrangian (Elementary)

From the 1st postulate of STR the action integral must be a Lorentz scalar, because the equations of motion described by the extremum condition $\delta A = 0$. Then if we set in (4) $dt = \gamma d\tau$ we get

$$A = \int_{\tau_1}^{\tau_2} \gamma L d\tau \tag{6}$$

since the proper time is Lorentz invariant the condition that A is invariant requires that γL is also Lorentz invariant.
The Lagrangian for a free particle can be a function of the **velocity** (the only invariant function of the velocity is $\eta_{\alpha\beta} U^\alpha U^\beta = U_\alpha U^\alpha = c^2$) and the **mass** of the particle but not its position.

$$L_{\mathrm{free}} = -mc^2 \sqrt{1 - \frac{u^2}{c^2}} \tag{7}$$

and through (5) the free-particle equation of motion (remember that $\vec{p} = \gamma m \vec{u}$)

$$\frac{d}{dt}(\gamma m \vec{u}) = 0 \tag{8}$$

Since the non-relativistic Lagrangian is $T - V$ and $V = e\Phi$ the interaction part of the relativistic Lagrangian must reduce in the non-relativistic limit to

$$L_{\rm int} \rightarrow L_{\rm int}^{NR} = -e\Phi \quad (9)$$

A wise guess is:

$$L_{\rm int} = -\frac{e}{\gamma c} U_\alpha A^\alpha \quad (10)$$

which may comes from the demand that $\gamma L_{\rm int}$ is :

- linear in the charge of the particle
- linear in the EM potentials
- translationally invariant
- a function no higher than the 1st time derivative of the particle coordinates.

Another writing is:

$$L_{\rm int} = -e\Phi + \frac{e}{c}\vec{u} \cdot \vec{A} \quad (11)$$

and the combination of (7) and (11) yields the complete Lagrangian

$$L = -mc^2\sqrt{1 - \frac{u^2}{c^2}} + \frac{e}{c}\vec{u} \cdot \vec{A} - e\Phi \quad (12)$$

(Verify that leads to the Lorentz force equation)

The **canonical momentum** $\vec{P}$ conjugate to the position coordinate $\vec{x}$ is obtained by the definition

$$P_i \equiv \frac{\partial L}{\partial u_i} = \gamma m u_i + \frac{e}{c} A_i \tag{13}$$

Thus the conjugate momentum is

$$\vec{P} = \vec{p} + \frac{e}{c}\vec{A} \tag{14}$$

where $\vec{p} = \gamma m \vec{u}$ is the ordinary kinetic momentum.
The Hamiltonian H is a function of the coordinate $\vec{x}$ and its conjugate momentum $\vec{P}$ and is a constant of motion if the Lagrangian is not an explicit function of time, in terms of the Lagrangian is :

$$H = \vec{P} \cdot \vec{u} - L \tag{15}$$

by eliminating $\vec{u}$ in favor of $\vec{P}$ and $\vec{x}$ we find (HOW?) that

$$\vec{u} = \frac{c\vec{P} - e\vec{A}}{\sqrt{\left(\vec{P} - \frac{e\vec{A}}{c}\right)^2 + m^2c^2}} \tag{16}$$

This equation together with (12)

$$H = \sqrt{\left(c\vec{P} - e\vec{A}\right)^2 + m^2c^4} + e\Phi \tag{17}$$

(Verify that from this Lagrangian you can get the Lorentz equation)
Equation (17) is an expression for the total energy W of the particle. Actually, it differs by the potential energy term $e\Phi$ and by the replacement $\vec{p} \rightarrow [\vec{P} - (e/c)\vec{A}]$.
These two modifications are actually a conseqency of considering 4-vectors. Notice that

$$(W - e\Phi)^2 - \left(c\vec{P} - e\vec{A}\right)^2 = \left(mc^2\right)^2 \tag{18}$$

is just the 4-vector scalar product

$$p_\alpha p^\alpha = (mc)^2 \tag{19}$$

where

$$p^\alpha \equiv \left(\frac{E}{c}, \vec{p}\right) = \left(\frac{1}{c}(W - e\Phi), \vec{P} - \frac{e}{c}\vec{A}\right) \tag{20}$$

Thus the total energy W/c acts as the time component of a canonically conjugate 4-momentum P^α of which $\vec{P}$ is given by (14).

Relativistic Lagrangian (Covariant Treatment)

If ones wants to use of proper covariant description, has to abandon the vectorial writing $\vec{x}$, $\vec{u}$ and to replace them with the 4-vectors x^α, U^α. Then the free particle Lagrangian (7) will be written as

$$L_{\text{free}} = -\frac{mc}{\gamma}\sqrt{U_\alpha U^\alpha} \tag{21}$$

remember that $U_\alpha U^\alpha = c^2$. The action integral will be:

$$A = -mc\int_{\tau_1}^{\tau_2}\sqrt{U_\alpha U^\alpha}\,d\tau \tag{22}$$

This invariant form can be the starting point for a variational calculation leading to the equation of motion $dU^\alpha/d\tau = 0$. One can further make use of the constraint

$$U_\alpha U^\alpha = c^2 \tag{23}$$

or the equivalent one:

$$U_\alpha \frac{dU^\alpha}{d\tau} = 0 \tag{24}$$

The integrant in (22) is:

$$\sqrt{U_\alpha U^\alpha} d\tau = \sqrt{\frac{dx_\alpha}{d\tau}\frac{dx^\alpha}{d\tau}} d\tau = \sqrt{g^{\alpha\beta} dx_\alpha dx_\beta}$$

i.e. the infinitesimal length element in 4-space. Thus the action integral can be written as

$$A = -mc \int_{s_1}^{s_2} \sqrt{g^{\alpha\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}} ds \tag{25}$$

where the 4-vector coordinate of the particle is $x^\alpha(s)$, where s is a parameter monotonically increasing with τ, but otherwise arbitrary.

- The action integral is an integral along the world line of the particle
- The principle of least action is a statement that the actual path is the longest path, the **geodesic**. We should keep in mind that

$$\sqrt{g^{\alpha\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}} ds = c d\tau \tag{26}$$

and then a straightforward variational calculation with (25) leads to

$$mc \frac{d}{ds}\left[\frac{dx^\alpha/ds}{\left(\frac{dx_\beta}{ds}\frac{dx^\beta}{ds}\right)^{1/2}}\right] = 0 \tag{27}$$

or

$$m\frac{d^2x^\alpha}{d\tau^2} = 0 \tag{28}$$

as expected for a free particle motion.
For a charged particle in an external field the form of the Lagrangian (11) suggests that the manifesltly covariant form of the action integral is

$$A = -\int_{s_1}^{s_2}\left[mc\int_{s_1}^{s_2}\sqrt{g^{\alpha\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}} + \frac{e}{c}\frac{dx_\alpha}{ds}A^\alpha(x)\right]ds \tag{29}$$

Hamilton's principle yields the Euler-Lagrange equations:

$$\frac{d}{ds}\left[\frac{\partial\tilde{L}}{\partial(dx_\alpha/ds)}\right] - \partial^\alpha\tilde{L} = 0 \tag{30}$$

where the Lagrangian is:

$$\tilde{L} = -\left[mc\sqrt{g^{\alpha\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}} + \frac{e}{c}\frac{dx_\alpha}{ds}A^\alpha(x)\right] \tag{31}$$

Then (30) becomes

$$m\frac{d^2x^\alpha}{d\tau^2} + \frac{e}{c}\frac{dA^\alpha(x)}{d\tau} - \frac{e}{c}\frac{dx_\beta}{d\tau}\partial^\alpha A^\beta(x) = 0$$

Since $dA^\alpha/d\tau = (dx_\beta/d\tau)\partial^\beta A^\alpha$ this equation can be written as

$$m\frac{d^2x^\alpha}{d\tau^2} = \frac{e}{c}\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)\frac{dx_\beta}{d\tau} \tag{32}$$

which is the covariant equation of motion (3).

Relativistic Hamiltonian (Covariant Treatment)

The transition to conjugate momenta and a Hamiltonian is simple enough. The conjugate 4-momentum is defined by

$$P^\alpha = -\frac{\partial \tilde{L}}{\partial(dx_\alpha/ds)} = mU^\alpha + \frac{e}{c}A^\alpha \tag{33}$$

which is in agreement with (4). A Hamiltonian can be determined

$$\tilde{H} = P_\alpha U^\alpha + \tilde{L} \tag{34}$$

the by eliminating U^α by means of (33) leads to the expression

$$\tilde{H} = \frac{1}{m}\left(P_\alpha - \frac{eA_\alpha}{c}\right)\left(P^\alpha - \frac{eA^\alpha}{c}\right) - c\sqrt{\left(P_\alpha - \frac{eA_\alpha}{c}\right)\left(P^\alpha - \frac{eA^\alpha}{c}\right)} \tag{35}$$

Then by using the constraint

$$\left(P^\alpha - \frac{eA^\alpha}{c}\right)\left(P_\alpha - \frac{eA_\alpha}{c}\right) = m^2c^2$$

we get Hamilton's equations:

$$
\begin{aligned}
\frac{dx^\alpha}{d\tau} &= \frac{\partial \tilde{H}}{\partial P_\alpha} = \frac{1}{m}\left(P^\alpha - \frac{eA^\alpha}{c}\right) \\
&\text{and} \\
\frac{dP^\alpha}{d\tau} &= -\frac{\partial \tilde{H}}{\partial x_\alpha} = \frac{e}{mc}\left(P_\beta - \frac{eA_\beta}{c}\right)\partial^\alpha A^\beta
\end{aligned}
\tag{36}
$$

These two equation can be shown to be equivalent to the Euler-Lagrange equation (32).

Motion in a Uniform, Static Magnetic Field

We consider the motion of charged particles in a uniform and static magnetic field. The equations (1) and (2) are

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\vec{v} \times \vec{B}\,, \quad \frac{dE}{dt} = 0 \tag{37}$$

where $\vec{v}$ is the particle's velocity. Since the energy is constant in time, the magnitude of the velocity is constant and so is γ.
Then the first equation can be written

$$\frac{d\vec{v}}{dt} = \vec{v} \times \vec{\omega}_B \tag{38}$$

where

$$\vec{\omega}_B = \frac{e\vec{B}}{\gamma mc} = \frac{ec\vec{B}}{E} \tag{39}$$

is the **gyration** or **precession** frequency.
The motion is a circular motion perpendicular to $\vec{B}$ and a uniform translation parallel to $\vec{B}$.

The solution for the **velocity** is (HOW?)

$$\vec{v}(t) = v_{\|}\vec{\epsilon}_3 + \omega_B a(\vec{\epsilon}_1 - i\vec{\epsilon}_2)e^{-i\omega_B t} \quad (40)$$

$\vec{\epsilon}_3$: is a unit vector parallel to the field
$\vec{\epsilon}_1$ and $\vec{\epsilon}_2$: are the other two orthogonal unit vectors
$v_{\|}$: is the velocity component along the field, and
a : the gyration radius
One can see that (40) represents a counterclockwise rotation for positive charge e
Further integration leads to the **displacement** of the particles

$$\vec{x}(t) = \vec{X}_0 + v_{\|}t\vec{\epsilon}_3 + ia(\vec{\epsilon}_1 - i\vec{\epsilon}_2)e^{-i\omega_B t} \quad (41)$$

The path is a **helix** of radius a and **pitch angle** $\alpha = \tan^{-1}(v_{\|}/\omega_B a)$.
The magnitude of the gyration radius a depends on the magnetic induction $\vec{B}$ and the transverse momentum $\vec{p}_\perp$ of the particle

$$cp_\perp = eBa$$

This relation allows for the determination of particle momenta. For particle with charge equal to electron charge the momentum can be written numerically as

$$p_\perp(\mathrm{MeV/c}) = 3 \times 10^{-4} Ba(\text{Gauss-cm}) = 300 Ba(\text{tesla-m}) \quad (42)$$

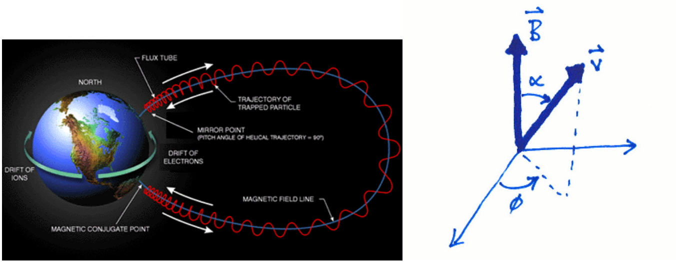

Figure: This three basic motions of charged particles in a magnetic field: gyro, bounce between mirror points, and drift. The pitch angle α between the directions of the magnetic field $\vec{B}$ and the electron velocity $\vec{v}$.

The angle between the direction of the magnetic field and a particle's spiral trajectory is referred to as the **"pitch angle"**, which in a non-uniform magnetic field changes as the ratio between the perpendicular and parallel components of the particle velocity changes. Pitch angle is important because it is a key factor in determining whether a charged particle will be lost to the Earth's atmosphere or not.

Motion in Combined, Uniform, Static E- and B- Field

We will consider a charged particle moving in a combination of electric and magnetic fields $\vec{E}$ and $\vec{B}$, both uniform and static, and for this study they will be considered perpendicular.
From the energy equation (2) we notice that the particle's energy is not constant in time. Consequently we can obtain a simple equation for the velocity, as was done for a static magnetic field.
An appropriate Lorentz transformation can simplify the equations of motion, here we consider a coordinate frame K' moving with velocity $\vec{u}$ with respect to original frame K. Then the Lorentz force equation for a particle in K' is

$$\frac{d\vec{p}'}{dt'} = e\left(\vec{E}' + \frac{\vec{v}' \times \vec{B}'}{c}\right)$$

The fields $\vec{E}'$ and $\vec{B}'$ can be estimated from relations of the previous chapter.

Motion in Combined, Uniform, Static E- and B- Field

CASE: $|\vec{E}| < |\vec{B}|$

If we chose $\vec{u}$ to be perpendicular to the orthogonal vectors $\vec{E}'$ and $\vec{B}'$ i.e.

$$\vec{u} = c\frac{\vec{E} \times \vec{B}}{B^2} \tag{43}$$

we find that the fields in K' (HOW?)

$$\begin{aligned} \vec{E}'_{\parallel} &= 0\,, \qquad \vec{E}'_{\perp} = \gamma\left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B}\right) = 0 \\ \vec{B}'_{\parallel} &= 0\,, \qquad \vec{B}'_{\perp} = \frac{1}{c}\vec{B} = \left(\frac{B^2 - E^2}{B^2}\right)^{1/2} \vec{B} \end{aligned} \tag{44}$$

In the frame K' the only field acting is a static magnetic field $\vec{B}'$ which points in the same direction as $\vec{B}$ but is weaker by a factor $1/\gamma$. Thus the motion in K' is the same as in the previous section, namely spiraling around the lines of force.

As viewed from the original frame, this gyration is accompanied by a uniform "drift" $\vec{u}$ perpendicular to $\vec{E}$ and $\vec{B}$.
The direction of the drift is independent of the sign of the charge of the particle.

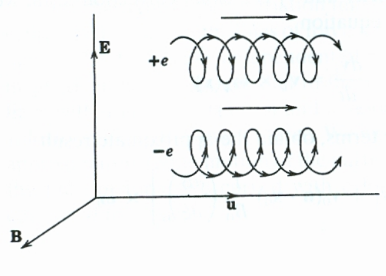

Figure: $\vec{E} \times \vec{B}$ drift of charged particles in perpendicular fields

CASE: $|\vec{E}| > |\vec{B}|$

The electric field is so strong that the particle is continually accelerated in the direction of $\vec{E}$ and its average energy continues to increase with time. If we consider a Lorentz transformation to a system K'' moving relative to the first with velocity

$$\vec{u}' = c\frac{\vec{E} \times \vec{B}}{E^2} \tag{45}$$

we get in the K''

$$\begin{aligned} \vec{E}''_{\parallel} &= 0\,, \qquad \vec{E}''_{\perp} = \frac{1}{\gamma}\vec{E} = \left(\frac{E^2 - B^2}{E^2}\right)^{1/2}\vec{E} \\ \vec{B}''_{\parallel} &= 0\,, \qquad \vec{E}''_{\perp} = \gamma'\left(\vec{B} - \frac{\vec{u}'}{c} \times \vec{E}\right) = 0 \end{aligned} \tag{46}$$

Thus the particle, in the system K'', is acted on by a purely electrostatic field which causes hyperbolic motion with ever-increasing velocity.

Lowest Order Relativistic Corrections to the Lagrangian...

The interaction Lagrangian was given by (11). In its simpler form the non-relativistic Lagrangian for two charged particles

$$L_{\rm int}^{NR} = \frac{q_1 q_2}{r} \tag{47}$$

including lowest order relativistic effects is

$$L_{\rm int} = \frac{q_1 q_2}{r}\left\{-1 + \frac{1}{2c^2}\left[\vec{v}_1\cdot\vec{v}_2 + \frac{(\vec{v}_1\cdot\vec{r})(\vec{v}_2\cdot\vec{r})}{r^2}\right]\right\} \tag{48}$$

For a system of interacting charged particles the complete Darwin Lagrangian correct to order $1/c^2$ can be written by expanding the free-particle Lagrangian (7) for each particle and summing up all the interaction terms of the form (48)

$$\begin{aligned} L_{\rm Darwin} &= \frac{1}{2}\sum_i m_i v_i^2 + \frac{1}{8c^2}\sum_i m_i v_i^4 - \frac{1}{2}\sum_{i,j}^{\sim}\frac{q_i q_j}{r_{ij}} \\ &+ \frac{1}{4c^2}\sum_{i,j}^{\sim}\frac{q_i q_j}{r_{ij}}\left[\vec{v}_i\cdot\vec{v}_j + (\vec{v}_i\cdot\hat{\vec{r}}_{ij})(\vec{v}_j\cdot\hat{\vec{r}}_{ij})\right] \end{aligned} \tag{49}$$

where $r_{ij} = |\vec{x}_i - \vec{x}_j|$, and $\hat{\vec{r}}_{ij}$ is the unit vector in the direction $\vec{x}_i - \vec{x}_j$ and the "tilde" ($\sim$) in the summation indicates omission of the self-energy terms $i = j$.

Lagrangian for the Electromagnetic Field

We now examine a Lagrangian description of the EM field in interaction with specified external sources of charge and current.
The Lagrangian approach to to continuous fields is similar to the approach used for discrete point particles.
The finite number of coordinates $q_i(t)$ and $\dot{q}_i(t)$ are replaced by an infinite number of degrees of freedom.
The coordinate q_i is replaced by a continuous field $\phi_k(x)$ with a discrete index $(k = 1, 2, \ldots, n)$ and a continuous index (x^α), i.e.

$$\begin{aligned} i \to x^\alpha\,, k\,, \quad q_i \quad &\to \quad \phi_k(x) \quad \dot{q}_i \to \partial^\alpha \phi_k(x) \\ L = \sum_i L_i(q_i, \dot{q}_1) \quad &\to \quad \int \mathcal{L}(\phi_k, \partial^\alpha \phi_k) d^3x \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \quad &\to \quad \partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta \phi_k)} = \frac{\partial \mathcal{L}}{\partial \phi_k} \end{aligned} \tag{50}$$

where $\mathcal{L}$ is the **Lagrangian density**, corresponding to a definite point in space-time and equivalent to the individual terms in a discrete particle Lagrangian like (49). For the EM-field the "coordinates" are A^α and the "velocities" $\partial^\beta A^\alpha$.

The action integral take sthe form

$$A = \int\int \mathcal{L}\, d^3x\, dt = \int \mathcal{L}\, d^4x \tag{51}$$

and it will be Lorentz-invariant if the Lagrangian density $\mathcal{L}$ is a Lorentz scalar. In analogy with the situation with discrete particles, we expect the free-field Lagrangian at least to be quadratic in the velocities, that is, $\partial^\beta A^\alpha$ or $F^{\alpha\beta}$. The only Lorentz invariant quadratic forms are $F_{\alpha\beta}F^{\alpha\beta}$ and $F_{\alpha\beta}\mathcal{F}^{\alpha\beta}$ (the last term is pseudoscalar under inversion).
Thus the $\mathcal{L}_{\rm free}$ will be a multipole of $F_{\alpha\beta}F^{\alpha\beta}$ and the $\mathcal{L}_{\rm int}$ according to (10) will be a multipole of $J_\alpha A^\alpha$. Thus the EM Lagrangian density is:

$$\mathcal{L} = -\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{c}J_\alpha A^\alpha \tag{52}$$

If we want to use it for the Euler-Lagrange equations given in (50) we get

$$\mathcal{L} = -\frac{1}{16\pi}g_{\lambda\mu}g_{\nu\sigma}\left(\partial^\mu A^\sigma - \partial^\sigma A^\mu\right)\left(\partial^\lambda A^\nu - \partial^\nu A^\lambda\right) - \frac{1}{c}J_\alpha A^\alpha \tag{53}$$

The term $\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)}$ in the Euler-Lagrange equations becomes (how?)

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = -\frac{1}{4\pi} F_{\beta\alpha} = \frac{1}{4\pi} F_{\alpha\beta} \tag{54}$$

The other part of the Euler-Lagrange equations is

$$\frac{\partial \mathcal{L}}{\partial A^\alpha} = -\frac{1}{c} J_\alpha \tag{55}$$

Thus the equations of motion for the EM field are

$$\frac{1}{4\pi} \partial^\beta F_{\beta\alpha} = \frac{1}{c} J_\alpha \tag{56}$$

which is a covariant form of the inhomogeneous Maxwell equations (**??**). The conservation of the source current density cen be obtained from (56)

$$\frac{1}{4\pi} \partial^\alpha \partial^\beta F_{\beta\alpha} = \frac{1}{c} \partial^\alpha J_\alpha$$

The left hand side has a differential operator which is symmetric in α and β, while $F_{\alpha\beta}$ is antisymmetric. Again the contraction vanishes (why?) and we have:

$$\partial^\alpha J_\alpha = 0\,. \tag{57}$$

Proca Lagrangian; Photon Mass Effect

If we assume that the photon is not massless then the Lagrangian (52) has to be modified by the addition of a " mass" term, this is called Proca Lagrangian

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{\mu^2}{8\pi} A_\alpha A^\alpha - \frac{1}{c} J_\alpha A^\alpha \tag{58}$$

The parameter μ has dimensions of inverse length and is the reciprocal Compton wavelength of the photon ($\mu = m_\gamma c/\hbar$). Instead of (56) the Proca equations of motion are

$$\partial^\beta F_{\beta\alpha} + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha \tag{59}$$

with the same homogeneous equations $\partial_\alpha \mathcal{F}^{\alpha\beta} = 0$ as in Maxwell theory. In contrast to the Maxwell equations the potentials have real physical (observable) significance through the mass term. In the Lorentz gauge (59) can be written

$$\Box A_\alpha + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha \tag{60}$$

In the static limit takes the form

$$\nabla^2 A_\alpha - \mu^2 A_\alpha = -\frac{4\pi}{c} J_\alpha \tag{61}$$

If the source is a point charge q at rest in the origin then the only non-vanishing component is $A_0 = \Phi$. the solution will be the spherically symmetric Yukawa potential

$$\Phi(x) = \frac{q}{r} e^{-\mu r} \tag{62}$$

i.e. we observe an exponential falloff of the static potentials and fields, with $1/e$ distance equal to $1/\mu$.

Notice that the exponential factor alters the character of the Earth's (and other planets) magnetic fields sufficiently to permit us to set quite stringent limits on the photon mass from geomagnetic data.

$$\vec{B}(\vec{x}) = \left[3\hat{\vec{r}}(\hat{\vec{r}} \cdot \vec{m}) - \vec{m}\right] \left(1 + \mu r + \frac{\mu^2 r^2}{3}\right) \frac{e^{-\mu r}}{r^3} - \frac{2}{3}\mu^2 \vec{m} \frac{e^{-\mu r}}{r^3}$$

The result shows that the Earth's magnetic field will appear as a dipole angular distribution plus an added constant magnetic field antiparallel to $\vec{m}$. Measurements show that this " external" field is less than 0.004 times the dipole field at the magnetic equator which leads to $\mu < 4 \times 10^{-10}\text{cm}^{-1}$ or $m_\gamma < 8 \times 10^{-49}$g.

Conservation Laws : Canonical Stress Tensor

In particle mechanics the transition to the Hamilton formulation and conservation of energy is made by first defining the canonical momentum variables

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

and then introducing the Hamiltonian

$$H = \sum_i p_i \dot{q}_i - L \tag{63}$$

where $dH/dt = 0$ if $\partial L/\partial t = 0$.
For fields the Hamiltonian is the volume integral over the 3-D space of Hamiltonian density $\mathcal{H}$ i.e.

$$H = \int \mathcal{H} d^3x \,.$$

It is necessary that the Hamiltonian density $\mathcal{H}$ transform as the `tt` component of a 2nd-rank tensor.

If the Lagrangian density for some fields is a function of the field variables $\phi_k(x)$ and $\partial^\alpha \phi_k(x)$ with $k = 1, 2, \ldots, n$ the Hamiltonian density is defined in analogy to (63) as

$$\mathcal{H} = \sum_k \frac{\partial \mathcal{L}}{\partial(\partial \phi_k / \partial t)} \frac{\partial \phi_k}{\partial t} - \mathcal{L} \tag{64}$$

The first factor in the sum is the field momentum canonically conjugate to $\phi_k(x)$ and $\partial^\alpha \phi_k(x)$ is equivalent to the velocity $\dot{q}_i$.
The Lorentz transformation properties of $\mathcal{H}$ suggest that the covariant generalization of the Hamiltonian density is the canonical stress tensor:

$$T^{\alpha\beta} = \sum_k \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \partial^\beta \phi_k - g^{\alpha\beta} \mathcal{L} \tag{65}$$

For the free EM field Lagrangian:

$$\mathcal{L}_{\rm em} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$$

the canonical stress tensor is

$$T^{\alpha\beta} = \frac{\partial \mathcal{L}_{\rm em}}{\partial(\partial_\alpha A^\lambda)} \partial^\beta A^\lambda - g^{\alpha\beta} \mathcal{L}_{\rm em}$$

By using equation (54) we find

$$T^{\alpha\beta} = -\frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} \partial^{\beta} A^{\lambda} - g^{\alpha\beta} \mathcal{L}_{\rm em} \qquad (66)$$

With $\mathcal{L} = \left(\vec{E}^2 - \vec{B}^2\right)/8\pi$ and (**??**) we find (how?)

$$\begin{aligned} T^{00} &= \frac{1}{8\pi}\left(\vec{E}^2 + \vec{B}^2\right) + \frac{1}{4\pi}\vec{\nabla}\cdot\left(\Phi\vec{E}\right) \\ T^{0i} &= \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)_i + \frac{1}{4\pi}\vec{\nabla}\cdot(A_i\vec{E}) \\ T^{i0} &= \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)_i + \frac{1}{4\pi}\left[\left(\vec{\nabla}\times\Phi\vec{B}\right)_i - \frac{\partial}{\partial x_0}(\Phi E_i)\right] \end{aligned} \qquad (67)$$

If the fields are localized in some finite region of space the integrals over all 3-space at fixed time in some inertial frame of the components T^{00} and T^{0i} can be interpreted as the total energy and c times the total momentum of the EM fields in that frame:

$$\begin{aligned} \int T^{00} d^3x &= \frac{1}{8\pi}\int\left(\vec{E}^2 + \vec{B}^2\right) d^3x = E_{\rm field} \\ \int T^{i0} d^3x &= \frac{1}{4\pi}\int\left(\vec{E}\times\vec{B}\right)_i d^3x = cP_{\rm field} \end{aligned} \qquad (68)$$

The previous definitions of field energy and momentum densities suggests that there should be a covariant generalization of the differential conservation law (**??**) of Poynting's theorem. That is:

$$\partial_\alpha T^{\alpha\beta} = 0 \tag{69}$$

Consider

$$\begin{aligned}\partial_\alpha T^{\alpha\beta} &= \sum_k \partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \partial^\beta \phi_k\right] - \partial^\beta \mathcal{L} \\ &= \sum_k \left[\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \partial^\beta \phi_k + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \partial_\alpha \partial^\beta \phi_k\right] - \partial^\beta \mathcal{L}\end{aligned}$$

but because of the equation of motion (50) the first term can be transformed so that

$$\partial_\alpha T^{\alpha\beta} = \sum_k \left[\frac{\partial \mathcal{L}}{\partial \phi_k} \partial^\beta \phi_k + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \partial^\beta (\partial_\alpha \phi_k)\right] - \partial^\beta \mathcal{L}$$

since $\mathcal{L} = \mathcal{L}(\phi_k, \partial^\alpha \phi_k)$ the term in the square bracket is an implicit differentiation, hence

$$\partial_\alpha T^{\alpha\beta} = \partial^\beta \mathcal{L}(\phi_k, \partial^\alpha \phi_k) - \partial^\beta \mathcal{L} = 0$$

The conservation law (or continuity equation) (69) yields the conservation of total energy and momentum upon integration over all of 3-space at fixed time

$$0 = \int \partial_\alpha T^{\alpha\beta} d^3x = \partial_0 \int T^{0\beta} d^3x + \int \partial_i T^{i\beta} d^3x$$

If the fields are localized the 2nd integral (divergence) gives no contribution. Then with the identification (68) we get

$$\frac{d}{dt} E_{\mathrm{field}} = 0 \,, \quad \frac{d}{dt} \vec{P}_{\mathrm{field}} = 0 \qquad (70)$$

The results are valid for an observer at rest in the frame in which the fields are specified.

Conservation Laws : Symmetric Stress Tensor

The canonical stress tensor $T^{\alpha\beta}$ has a number of deficiencies for example lack of symmetry ! (see T^{0i} and T^{i0}). This affects the consideration of the angular momentum of the field

$$\vec{L}_{\text{field}} = \frac{1}{4\pi c} \int \vec{x} \times (\vec{E} \times \vec{B}) d^3x$$

The angular momentum density is expressed in terms of a 3rd-rank tensor

$$M^{\alpha\beta\gamma} = T^{\alpha\beta} x^\gamma - T^{\alpha\gamma} x^\beta \tag{71}$$

Then as (69) implies conservation of energy and momentum (70) we expect the vanishing of the 4-divergence

$$\partial_\alpha M^{\alpha\beta\gamma} = 0 \tag{72}$$

to imply conservation of the total angular momentum of the field. Direct evaluation gives

$$0 = (\partial_\alpha T^{\alpha\beta}) x^\gamma + T^{\gamma\beta} - (\partial_\alpha T^{\alpha\gamma}) x^\beta - T^{\beta\gamma}$$

then by using (69) one can see that the conservation of angular momentum requires that $T^{\alpha\beta}$ be symmetric. Finally, the involvement of the potentials in (66) makes it not gauge invariant.

We will construct a symmetric, traceless, gauge-invariant stress tensor $\Theta^{\alpha\beta}$ from the canonical stress tensor $T^{\alpha\beta}$ of (66).
We substitute $\partial^\beta A^\lambda = -F^{\lambda\beta} + \partial^\lambda A^\beta$ and obtain

$$T^{\alpha\beta} = \frac{1}{4\pi}\left[g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right] - \frac{1}{4\pi}g^{\alpha\mu}F_{\mu\lambda}\partial^\lambda A^\beta \qquad (73)$$

The first term in (73) is symmetric and gauge invariant while the last term of (73), with the help of the source-free Maxwell equations, can be written (how?)

$$\begin{aligned} T_D^{\alpha\beta} &\equiv -\frac{1}{4\pi}g^{\alpha\mu}F_{\mu\lambda}\partial^\lambda A^\beta = \frac{1}{4\pi}F^{\lambda\alpha}\partial_\lambda A^\beta \\ &= \frac{1}{4\pi}\left(F^{\lambda\alpha}\partial_\lambda A^\beta + A^\beta\partial_\lambda F^{\lambda\alpha}\right) = \frac{1}{4\pi}\partial_\lambda\left(F^{\lambda\alpha}A^\beta\right) \qquad (74) \end{aligned}$$

with the following properties: (why?)

$$(i)\, \partial_\alpha T_D^{\alpha\beta} = 0\,, \quad (ii)\int T_D^{0\beta} d^3x = 0$$

If we define the symmetric stress tensor $\Theta^{\alpha\beta}$

$$\Theta^{\alpha\beta} = T^{\alpha\beta} - T_D^{\alpha\beta} = \frac{1}{4\pi}\left[g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right] \qquad (75)$$

then the differential conservation law (69) will hold if it holds for $T^{\alpha\beta}$.

The explicit components of $\Theta^{\alpha\beta}$ are:

$$\begin{aligned} \Theta^{00} &= \frac{1}{8\pi}\left(E^2 + B^2\right) \\ \Theta^{0i} &= \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)^i \\ \Theta^{ij} &= -\frac{1}{4\pi}\left[E^iE^j + B^iB^j - \frac{1}{2}\delta^{ij}\left(E^2+B^2\right)\right] \end{aligned} \tag{76}$$

The tensor $\Theta^{\alpha\beta}$ can be written in the schematic matrix form

$$\Theta^{\alpha\beta} = \left(\begin{array}{c|c} u & c\vec{g} \\ \hline c\vec{g} & -T_{ij}^{(M)} \end{array}\right) \tag{77}$$

where the `time-time` component is the energy density (**??**) the `time-space` component is the momentum density (**??**) while the `space-space` components are the negative of the Maxwell stress tensor (**??**).

The various covariant or mixed forms of the stress tensor are

$$\Theta_{\alpha\beta} = \left(\begin{array}{c|c} u & -c\vec{g} \\ \hline -c\vec{g} & -T_{ij}^{(M)} \end{array}\right) \quad \Theta^{\alpha}{}_{\beta} = \left(\begin{array}{c|c} u & -c\vec{g} \\ \hline c\vec{g} & T_{ij}^{(M)} \end{array}\right)$$

$$\Theta_{\alpha}{}^{\beta} = \left(\begin{array}{c|c} u & c\vec{g} \\ \hline -c\vec{g} & T_{ij}^{(M)} \end{array}\right)$$

The differential conservation law

$$\partial_\alpha \Theta^{\alpha\beta} = 0 \tag{78}$$

embodies Poynting's theorem and conservation of momentum for free fields. For example, for $\beta = 0$ we have

$$0 = \partial_\alpha \Theta^{\alpha 0} = \frac{1}{c}\left(\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S}\right)$$

where $\vec{S} = c^2\vec{g}$ is the Poynting vector, this is the source-free form of (**??**). Similarly for $\beta = i$

$$0 = \partial_\alpha \Theta^{\alpha i} = \frac{\partial g_i}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j} T_{ij}^{(M)}$$

which is equivalent to (**??**).

The conservation of the field angular momentum defined by

$$\Theta^{\alpha\beta\gamma} = \Theta^{\alpha\beta} x^{\gamma} - \Theta^{\alpha\gamma} x^{\beta} \tag{79}$$

is assured by (78) and the symmetry of $\Theta^{\alpha\beta}$.

Conservation Laws for EM fields interacting with Charged Particles

When external forces are present the Lagrangian for the Maxwell equations is (52), the symmetric stress tensor for the EM field retains its form (75) but the coupling to the external force makes its divergence non-vanishing.

$$\begin{aligned}\partial_\alpha \Theta^{\alpha\beta} &= \frac{1}{4\pi}\left[\partial^\mu\left(F_{\mu\lambda}F^{\lambda\beta}\right) + \frac{1}{4}\partial^\beta\left(F_{\mu\lambda}F^{\mu\lambda}\right)\right] \\ &= \frac{1}{4\pi}\left[(\partial^\mu F_{\mu\lambda})F^{\lambda\beta} + F_{\mu\lambda}\partial^\mu F^{\lambda\beta} + \frac{1}{2}F_{\mu\lambda}\partial^\beta F^{\mu\lambda}\right]\end{aligned}$$

the 1st term can be transformed via (56) and we get

$$\partial_\alpha \Theta^{\alpha\beta} + \frac{1}{c}F^{\beta\lambda}J_\lambda = \frac{1}{8\pi}F_{\mu\lambda}\left(\partial^\mu F^{\lambda\beta} + \partial^\mu F^{\lambda\beta} + \partial^\beta F^{\mu\lambda}\right)$$

the last two terms (in blue) can be transformed via the homogeneous Maxwell equation $\partial^\mu F^{\lambda\beta} + \partial^\beta F^{\mu\lambda} + \partial^\lambda F^{\beta\mu} = 0$ by $-\partial^\lambda F^{\beta\mu} = \partial^\lambda F^{\mu\beta}$

$$\partial_\alpha \Theta^{\alpha\beta} + \frac{1}{c}F^{\beta\lambda}J_\lambda = \frac{1}{8\pi}F_{\mu\lambda}\left(\partial^\mu F^{\lambda\beta} + \partial^\lambda F^{\mu\beta}\right)$$

The right-hand side is zero (why?) and thus the divergence of the stress tensor is

$$\partial_\alpha \Theta^{\alpha\beta} = -\frac{1}{c} F^{\beta\lambda} J_\lambda \tag{80}$$

The time and space components of this equation are

$$\frac{1}{c}\left(\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S}\right) = -\frac{1}{c}\vec{J}\cdot\vec{E} \tag{81}$$

and

$$\frac{\partial g_i}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j} T_{ij}^{(M)} = -\left[\rho E_i + \frac{1}{c}\left(\vec{J}\times\vec{B}\right)_i\right] \tag{82}$$

These are the conservation of energy and momentum equations for EM fields interacting with sources described by $J^\alpha = (c\rho, \vec{J})$.
The negative of the right hand side term in (80) is called the **Lorentz force density**,

$$f^\beta \equiv \frac{1}{c} F^{\beta\lambda} J_\lambda = \left(\frac{1}{c}\vec{J}\cdot\vec{E},\ \rho\vec{E} + \frac{1}{c}\vec{J}\times\vec{B}\right) \tag{83}$$

If the sources are a number of charged particles then the volume integral of f^β leads through the Lorents force equation (1) to the time rate of change of the sum of the energies or momenta of all particles:

$$\int f^\beta d^3x = \frac{dP^\beta_{\mathrm{particles}}}{dt}$$

The conservation of the 4-momentum for the combined system of particle and fields:

$$\int d^3x \left(\partial_\alpha \Theta^{\alpha\beta} + f^\beta\right) = \frac{d}{dt}\left(P^\beta_{\mathrm{field}} + P^\beta_{\mathrm{particles}}\right) = 0 \qquad (84)$$