Field Theory

5th Set of Problems

THEORY OF SPECIAL RELATIVITY

1. (a) Express the Lorentz scalars $F^{\alpha\beta}F_{\alpha\beta}$, $\mathcal{F}^{\alpha\beta}F_{\alpha\beta}$, and $\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta}$ in terms of $\vec{E}$ and $\vec{B}$.
(b) Is it possible to have an electromagnetic field that appears as a purely electric field in one inertial frame and as a purely magnetic field in some other inertial frame ?
(c) What are the criteria imposed on $\vec{E}$ and $\vec{B}$ such that there is an inertial frame in which there is no electric field ?
(d) For macroscopic media, $\vec{E}$, $\vec{B}$ form the field tensor $F^{\alpha\beta}$ and $\vec{D}$, $\vec{H}$ the tensor $G^{\alpha\beta}$. What further invariants can be formed ? What are their explicit expressions in terms of the 3-vector fields ?

2. In the rest frame of a conducting medium the current density satisfies Ohm's law, $\vec{J}' = \sigma\vec{E}'$, where σ is the conductivity and primes denote quantities in the rest frame.
(a) Taking into account the possibility of convection current as well as conduction current, show that the covariant generalization of Ohm's law is
$$J^\alpha - \frac{1}{c^2}\left(U_\beta J^\beta\right)U^\alpha = \frac{\sigma}{c}F^{\alpha\beta}U_\beta,$$
where U^α is the 4-velocity of the medium.
(b) Show that if the medium has a velocity $\vec{v} = c\vec{\beta}$ with respect to some inertial frame that the 3-vector corrent in that frame is
$$\vec{J} = \gamma\sigma\left[\vec{E} + \vec{\beta}\times\vec{B} - \vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right)\right] + \rho\vec{v},$$
where ρ is the charge density observed in that frame.
(c) If the medium is uncharged in its rest frame ($\rho' = 0$), what is the charge density and the expression for $\vec{J}$ in the frame of part (b) ?

3. The electric and magnetic fields of a particle of charge q moving in a straight line with speed $v = \beta c$ become more and more concentrated as $\beta \to 1$. Choose axes so that the charge moves along the z axis in the positive direction, passing the origin at $t = 0$. Let the spatial coordinates of the observation point be (x, y, z) and define the transverse vector $\vec{r}_\perp$, with components x and y. Consider the fields and the source in the limit of $\beta = 1$.
(a) Show that the fields can be written as
$$\vec{E} = 2q\frac{\vec{r}_\perp}{r_\perp^2}\delta(ct - z); \qquad \vec{B} = 2q\frac{\hat{\vec{v}}\times\vec{r}_\perp}{r_\perp^2}\delta(ct - z),$$
where $\hat{\vec{v}}$ is a unit vector in the direction of the particle's velocity.
(b) Show by substitution into the Maxwell equations that these fields are consistent with a 4-vector source density,
$$J^\alpha = qcv^\alpha\delta^{(2)}(\vec{r}_\perp)\delta(ct - z),$$
where the 4-vector $v^\alpha = \left(c, \hat{\vec{v}}\right)$.
(c) Show that the fields of part (a) are derivable from either of the following 4-vector potentials,
$$A^0 = A^z = -2q\delta(ct - z)\ln(\lambda r_\perp); \qquad \vec{A}_\perp = 0,$$
or
$$A^0 = 0 = A^z; \qquad \vec{A}_\perp = -2q\Theta(ct - z)\vec{\nabla}_\perp\ln(\lambda r_\perp),$$
where λ is an irrelevant parameter setting the scale of the logarithm.
Show that the two potentials differ by a gauge transformation and find the gauge function, χ.