Field Theory

6th Set of Problems

DYNAMICS OF RELATIVISTIC PARTICLES & E-M FIELDS

1. A particle with mass m and charge e moves in a uniform, static, electric filed $\vec{E}_0$.
(a) Solve for the velocity and position of the particle as explicit function of time, assuming that the initial velocity $\vec{v}_0$ was perpendicular to the electric fields.
(b) Eliminate the time to obtain the trajectory of the particle in space. Discuss the shape of the path for short and long times (define "short" and "long" times).

2. (a) Specialize the Darwin Lagrangian

$$L_{\text{Darwin}} = \frac{1}{2}\sum_i m_i v_i^2 + \frac{1}{8c^2}\sum_i m_i v_i^4 - \frac{1}{2}\tilde{\sum_{i,j}} \frac{q_i q_j}{r_{ij}} + \frac{1}{4c^2}\tilde{\sum_{i,j}} \frac{q_i q_j}{r_{ij}} \left[\vec{v}_i \cdot \vec{v}_j + (\vec{v}_i \cdot \hat{\vec{r}}_{ij})(\vec{v}_j \cdot \hat{\vec{r}}_{ij})\right]$$

to the interaction of two charged particles (m_1, q_1) and (m_2, q_2) where $r_{ij} = |\vec{x}_i - \vec{x}_j|$ and $\hat{\vec{r}}_{ij}$ is the unit vector in the direction $\vec{x}_i - \vec{x}_j$ and the "tilde" $(\sim)$ in the summation indicates omission of the self-energy terms $i = j$. Introduce reduced particle coordinates, $\vec{r} = \vec{x}_1 - \vec{x}_2$, $\vec{v} = \vec{v}_1 - \vec{v}_2$ and also center of mass coordinates. Write out the Lagrangian in the reference frame in which the velocity of the center of mass vanishes and evaluate the canonial momentum components, $p_x = \partial L / \partial v_x$, etc.
(b) Calculate the Hamiltonian to first order in $1/c^2$ and show that it is

$$H = \frac{p^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \frac{q_1 q_2}{r} - \frac{p^4}{8c^2}\left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right) + \frac{q_1 q_2}{2 m_1 m_2 c^2}\left(\frac{p^2 + \left(\vec{p} \cdot \hat{\vec{r}}\right)^2}{r}\right).$$

3. (a) Starting with the Proca Lagrangian density

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{\mu^2}{8\pi} A_\alpha A^\alpha - \frac{1}{c} J_\alpha A^\alpha \tag{1}$$

and following the same prodecure as for the electromagnetic fields, show that the symmetric part of the stress-energy-momentum tensor for the Proca *fields* is

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\left[g^{\alpha\gamma} F_{\gamma\lambda} F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\lambda\nu} F^{\lambda\nu} + \mu^2 \left(A^\alpha A^\beta - \frac{1}{2} g^{\alpha\beta} A_\lambda A^\lambda\right)\right].$$

(b) For these fields in interaction with the external source J^β, show that the differential conservation laws take the same form as for the electromagnetic fields, namely,

$$\partial_\alpha \Theta^{\alpha\beta} = \frac{J_\lambda F^{\lambda\beta}}{c}.$$

(c) Show explicitly that the time-time and space-time components of $\Theta^{\alpha\beta}$ are

$$\begin{aligned}
\Theta^{00} &= \frac{1}{8\pi}\left[E^2 + B^2 + \mu^2\left(A^0 A^0 + \vec{A} \cdot \vec{A}\right)\right], \\
\Theta^{i0} &= \frac{1}{4\pi}\left[\left(\vec{E} \times \vec{B}\right)^i + \mu^2 A^i A^0\right].
\end{aligned}$$