Field Theory

7th Set of Problems

GENERAL THEORY OF RELATIVITY

1. (a) From the 2-dimensional line element ds^2 calculate the components g_{ik} of the metric tensor for the sphere, the cylinder barrel and a rotational paraboloid.

   (b) For the sphere and the cylinder barrel calculate the Γ^i_{kl} from the metric.

2. Prove that if vector a^λ is tangent in a point P of a geodesic curve it will be tangent to this curved if parallel transported along the curve.

3. Let $\xi^0 = t$, ξ^1, ξ^2, ξ^3 be the coordinates of an inertial frame, i.e. $ds^2 = dt^2 - (d\xi^1)^2 - (d\xi^2)^2 - (d\xi^3)^2$. Let further x^0, x^1, x^2, x^3 be the coordinates of a rotating frame that are subject to the following transformation

$$\begin{aligned} \xi^0 &= x^0, \\ \xi^1 &= x^1 \cos(\omega t) - x^2 \sin(\omega t), \\ \xi^2 &= x^1 \sin(\omega t) + x^2 \cos(\omega t), \\ \xi^3 &= x^3, \end{aligned}$$

   where ω is constant and $\omega^2((x^1)^2 + (x^2)^2) \ll 1$.

   (a) Calculate the Christoffel symbols through

$$\Gamma^\mu_{\alpha\beta} = \frac{\partial^2 \xi^\rho}{\partial x^\alpha \partial x^\beta} \frac{\partial x^\mu}{\partial \xi^\rho}.$$

   (b) Set up the equations of motion in the rotating coordinate system as

$$\frac{d^2 x^k}{dt^2} + \Gamma^k_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0$$

   and interpret the occurring terms.

   (c) Calculate the components $g_{\mu\nu}(x)$, $g^{\mu\nu}(x)$ of the metric tensor in the rotating system.

4. Prove that

$$\phi^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h \quad \Rightarrow \quad h^{\mu\nu} = \phi^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\phi$$

5. Prove that Einstein equations can be written as :

$$R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

6. Show that Kepler's 3rd law is valid for circular orbits around a Schwarzschild black-hole.

7. What will be the orbital period of an observer in circular motion around a Schwarzschild black-hole at distance $r = 12M$ as it measured a) by his clock, b) by a clock at infinity and c) by a standing clock at $r = 12M$?