

## 1st Set of Examples <sup>1</sup>

1. If  $F^{\alpha\beta}$  is antisymmetric on its two indices. Show that

$$F_{\mu,\nu}^{\alpha} F_{\alpha}^{\nu} = -F_{\mu\alpha,\beta} F^{\alpha\beta}$$

### SOLUTION

We have to use the metric tensor  $\eta_{\mu\nu}$  in order to lower the indices. So we get:

$$F_{\mu,\beta}^{\alpha} F_{\alpha}^{\beta} = (F_{\mu\gamma} \eta^{\gamma\alpha})_{,\beta} (F^{\beta\sigma} \eta_{\sigma\alpha}) = (F_{\mu\gamma})_{,\beta} F^{\beta\sigma} (\eta^{\gamma\alpha} \eta_{\sigma\alpha}) + F_{\mu\gamma} \eta^{\gamma\alpha}_{,\beta} (F^{\beta\sigma} \eta_{\sigma\alpha}) \quad (1)$$

Since  $\eta_{\mu\nu}$  is constant,  $\eta_{\mu\nu,\beta} = 0$  and we get:

$$F_{\mu\gamma,\beta} F^{\beta\sigma} (\eta^{\gamma\alpha} \eta_{\sigma\alpha}) = F_{\mu\gamma,\beta} F^{\beta\sigma} \delta_{\sigma}^{\gamma} = F_{\mu\gamma,\beta} F^{\beta\gamma} = -F_{\mu\gamma,\beta} F^{\gamma\beta} \quad (2)$$

where in the last passage we have used the antisymmetry of the tensor  $F_{\mu\nu}$ . Considering that  $\gamma$  is a dummy index, we can relabel it so that:

$$F_{\mu,\beta}^{\alpha} F_{\alpha}^{\beta} = -F_{\mu\alpha,\beta} F^{\alpha\beta} \quad (3)$$

2. In a coordinate system with coordinates  $x^{\mu}$ , the invariant line element is  $ds^2 = \eta_{\alpha\beta} dx^{\alpha} dx^{\beta}$ . If the coordinates are transformed  $x^{\mu} \rightarrow \bar{x}^{\mu}$ , show that the line element is  $ds^2 = \bar{g}_{\mu\nu} d\bar{x}^{\mu} d\bar{x}^{\nu}$ , and express  $\bar{g}_{\mu\nu}$  in terms of the partial derivatives  $\partial x^{\mu} / \partial \bar{x}^{\nu}$ . For two arbitrary vectors  $\mathbf{U}$  and  $\mathbf{V}$ , show that

$$\mathbf{U} \cdot \mathbf{V} = U^{\alpha} V^{\beta} \eta_{\alpha\beta} = \bar{U}^{\alpha} \bar{V}^{\beta} \bar{g}_{\alpha\beta}$$

### SOLUTION

We start by writing the differentials in terms of the new coordinate:

$$\begin{aligned} ds^2 &= \eta_{\alpha\beta} dx^{\alpha} dx^{\beta} \\ &= \eta_{\alpha\beta} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} d\bar{x}^{\mu} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} d\bar{x}^{\nu} \\ &= \left( \eta_{\alpha\beta} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \right) d\bar{x}^{\mu} d\bar{x}^{\nu} \end{aligned} \quad (4)$$

If we write the line element as  $ds^2 = \bar{g}_{\mu\nu} d\bar{x}^{\mu} d\bar{x}^{\nu}$  and compare this expression with the last passage of the equation above, we can identify:

$$\bar{g}_{\mu\nu} = \left( \eta_{\alpha\beta} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \right) \quad (5)$$

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<sup>1</sup>The problems have been chosen by the *Problem Book in Relativity and Gravitation* By Alan P. Lightman, Richard H. Price, William H. Press, Saul A. Teukolsky, Published by Princeton University Press, 1975

Now consider a vector  $U^\alpha$ . The transformation of coordinate  $x^\mu \rightarrow \bar{x}^\mu$  for  $U^\alpha$  is:

$$U^\alpha = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \bar{U}^\mu \quad (6)$$

So for the scalar product  $\mathbf{U} \cdot \mathbf{V}$  we get:

$$\begin{aligned} \mathbf{U} \cdot \mathbf{V} &= U^\alpha V^\beta \eta_{\alpha\beta} \\ &= \left( \bar{U}^\mu \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \right) \left( \bar{V}^\nu \frac{\partial x^\beta}{\partial \bar{x}^\nu} \right) \eta_{\alpha\beta} \\ &= \bar{U}^\mu \bar{V}^\nu \left( \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \eta_{\alpha\beta} \right) \\ &= \bar{U}^\mu \bar{V}^\nu \bar{g}_{\mu\nu} \end{aligned} \quad (7)$$

3. If  $\Lambda_\beta^\alpha$  and  $\bar{\Lambda}_\beta^\alpha$  are two matrices which transform the components of a tensor from one coordinate basis to another, show that the matrix  $\Lambda_\gamma^\alpha \bar{\Lambda}_\beta^\gamma$  is also a coordinate transformation.

**SOLUTION**

Take two coordinate transformations as:

$$\bar{x}^\mu = \bar{x}^\mu(x^\nu) \quad \Lambda_\beta^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^\beta} \quad (8)$$

and

$$\tilde{x}^\mu = \tilde{x}^\mu(x^\nu) \quad \tilde{\Lambda}_\beta^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} \quad (9)$$

The product matrix is given by:

$$\Lambda_\gamma^\alpha \tilde{\Lambda}_\beta^\gamma = \frac{\partial \bar{x}^\alpha}{\partial x^\gamma} \frac{\partial \tilde{x}^\gamma}{\partial x^\beta} \quad (10)$$

The expression above can seem to differ from a usual coordinate transformation. We will show that this is not the case. Take another transformation of coordinate such as:

$$\bar{x}^\mu = \bar{x}^\mu(x^\nu) = \bar{x}^\mu[\tilde{x}^\gamma(x^\nu)] \quad (11)$$

then

$$\bar{\Lambda}_\beta^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^\beta} = \frac{\partial \bar{x}^\alpha}{\partial \tilde{x}^\gamma} \frac{\partial \tilde{x}^\gamma}{\partial x^\beta} \quad (12)$$

This is the same expression as (10) also if it differs for the  $\tilde{x}^\gamma$  in the denominator: the partial derivatives are always taken with respect to the argument variable, and it is possible to call these variables with different 'names' without that the meaning of the operation changes. So the expression (12) is a transformation of coordinate as well as (10).

4. Show that the second rank tensor  $F$  which is antisymmetric in one coordinate frame ( $F_{\mu\nu} = -F_{\nu\mu}$ ) is antisymmetric in all frames. Show that the contravariant components are also antisymmetric ( $F^{\mu\nu} = -F^{\nu\mu}$ ). Show that symmetry is also coordinate invariant.

**SOLUTION**

We start transforming the tensor  $F_{\mu\nu}$ :

$$\bar{F}_{\mu\nu} = \Lambda_{\mu}^{\alpha}\Lambda_{\nu}^{\beta}F_{\alpha\beta} = -\Lambda_{\mu}^{\alpha}\Lambda_{\nu}^{\beta}F_{\beta\alpha}. \quad (13)$$

where in the last passage we have used the antisymmetry propriety of  $F_{\mu\nu}$ . Now if we consider that  $\alpha$  and  $\beta$  are two dummy indices, we can relabel them, for example naming  $\alpha \rightarrow \beta$  and  $\beta \rightarrow \alpha$  so that:

$$\bar{F}_{\mu\nu} = -\Lambda_{\mu}^{\alpha}\Lambda_{\nu}^{\beta}F_{\beta\alpha} = -\Lambda_{\mu}^{\beta}\Lambda_{\nu}^{\alpha}F_{\alpha\beta} = -\bar{F}_{\nu\mu} \quad (14)$$

So if a tensor is antisymmetric in one coordinate frame, it is antisymmetric in all the coordinate frame. The same applied to an antisymmetric contravariant tensor:

$$F^{\mu\nu} = g^{\alpha\mu}g^{\beta\nu}F_{\beta\alpha} = -g^{\alpha\mu}g^{\beta\nu}F_{\alpha\beta} = -g^{\beta\mu}g^{\alpha\nu}F_{\beta\alpha} = -F^{\nu\mu} \quad (15)$$

where we have used the antisymmetry propriety of  $F^{\mu\nu}$  and the fact that  $\alpha$  and  $\beta$  are dummy indices and, consequently, we can relabel it.

5. Let  $A_{\mu\nu}$  be an antisymmetric tensor so that  $A_{\mu\nu} = -A_{\nu\mu}$  and let  $S^{\mu\nu}$  be a symmetric tensor so that  $S^{\mu\nu} = S^{\nu\mu}$ . Show that

$$A_{\mu\nu}S^{\mu\nu} = 0$$

. For any arbitrary tensor  $V_{\mu\nu}$  establish the following two identities:

$$V^{\mu\nu}A_{\mu\nu} = \frac{1}{2}(V^{\mu\nu} - V^{\nu\mu})A_{\mu\nu} \quad V^{\mu\nu}S_{\mu\nu} = \frac{1}{2}(V^{\mu\nu} + V^{\nu\mu})S_{\mu\nu}$$

If  $A_{\mu\nu}$  is antisymmetric, then  $A_{\mu\nu}S^{\mu\nu} = -A_{\nu\mu}S^{\mu\nu} = -A_{\nu\mu}S^{\nu\mu}$ . Because  $\mu$  and  $\nu$  are dummy indices, we can relabel it and obtain:

$$A_{\mu\nu}S^{\mu\nu} = -A_{\nu\mu}S^{\nu\mu} = -A_{\mu\nu}S^{\mu\nu}$$

so that  $A_{\mu\nu}S^{\mu\nu} = 0$ , i.e. the product of a symmetric tensor times an antisymmetric one is equal to zero.

**SOLUTION**

Each tensor can be written like the sum of a symmetric part  $\bar{V}_{\mu\nu} = \frac{1}{2}(V_{\mu\nu} + V_{\nu\mu})$  and an antisymmetric part  $\tilde{V}_{\mu\nu} = \frac{1}{2}(V_{\mu\nu} - V_{\nu\mu})$  so that a  $V_{\mu\nu} = \bar{V}_{\mu\nu} + \tilde{V}_{\mu\nu} = \frac{1}{2}(V_{\mu\nu} + V_{\nu\mu} + V_{\mu\nu} - V_{\nu\mu}) = V_{\mu\nu}$ . So we have:

$$V^{\mu\nu}A_{\mu\nu} = \frac{1}{2}(\bar{V}_{\mu\nu}A_{\mu\nu} + \tilde{V}_{\mu\nu}A_{\mu\nu}) = \frac{1}{2}\tilde{V}_{\mu\nu}A_{\mu\nu} = \frac{1}{2}(V_{\mu\nu} - V_{\nu\mu})A_{\mu\nu} \quad (16)$$

since we have already show that the scalar product of a symmetric tensor with an anti-symmetric one is equal to zero. In the same way we can show that:

$$V^{\mu\nu}S_{\mu\nu} = \frac{1}{2}(\bar{V}_{\mu\nu}S_{\mu\nu} + \tilde{V}_{\mu\nu}S_{\mu\nu}) == \frac{1}{2}\bar{V}_{\mu\nu}S_{\mu\nu} = \frac{1}{2}(V_{\mu\nu} + V_{\nu\mu})S_{\mu\nu} \quad (17)$$

6. (a) In a n-dimensional metric space, how many independent components are there for an r-rank tensor  $T^{\alpha\beta\dots}$  with no symmetries?

(b) How many independents component are there if the tensor is symmetric on s of its indices?

(c) How many independents component are there if the tensor is antisymmet-ric on a of its indices?

### SOLUTION

(a) In the case of no symmetries the number of independent components is  $n^r$  (where r is the rank of the tensor).

(b) If we have s symmetric indices, then we have to calculate how many inequivalent way we have to choose them (including the repetitions) in a set n. The number is given by:

$$\frac{(n + s - 1)!}{(n - 1)!s!}$$

while the remaning r-s indices can be choosen in  $n^{r-s}$  ways so that the number of inde-pendent component is:

$$n^{r-s} \frac{(n + s - 1)!}{(n - 1)!s!}$$

(c) If we have a antisymmetric indices, then we have to calculate how many inequivalent way we have to choose them (including the repetitions) in a set n. The number is given by:

$$\frac{n!}{(n - a)!a!}$$

while the remaning r-a indices can be choosen in  $n^{r-a}$  ways so that the number of inde-pendent component is:

$$n^{r-a} \frac{n!}{(n - a)!a!}$$

Note that for  $a = n$  we have just a possibility to choose the a indices, while for  $a > n$  there is no possibility: this means that all the components must be zero.

7. If **F** is antisymmetric, **T** is symmetric and **V** is an arbitrary tensor, give explicit formulation for the following :  $V_{[\mu\nu]}$ ,  $F_{[\mu\nu]}$ ,  $F_{(m\nu)}$ ,  $T_{[\mu\nu]}$ ,  $T_{(\mu\nu)}$ ,  $V_{[\mu\nu\gamma]}$ ,  $T_{(\mu\nu,\gamma)}$ ,  $F_{[\mu\nu,\gamma]}$ . Show that  $F_{\mu\nu} = -A_{[\mu,\nu]} + A_{[\nu,\mu]}$  implies

$$F_{\mu\nu,\gamma} + F_{\gamma\mu,\nu} + F_{\nu\gamma,\mu} = 0$$

## SOLUTION

(a)

$$V_{[\mu\nu]} = \frac{1}{2}(V_{\mu\nu} - V_{\nu\mu})$$

$$F_{[\mu\nu]} = F_{\mu\nu}$$

$$F_{(\mu\nu)} = 0$$

( $F_{\mu\nu}$  is antisymmetric)

$$T_{(\mu\nu)} = T_{\mu\nu}$$

$$T_{[\mu\nu]} = 0$$

( $T_{\mu\nu}$  is symmetric)

$$V_{[\mu\nu\gamma]} = \frac{1}{6}(V_{\mu\nu\gamma} - V_{\mu\gamma\nu} + V_{\gamma\mu\nu} - V_{\gamma\nu\mu} + V_{\nu\gamma\mu} - V_{\nu\mu\gamma})$$

$$F_{[\mu\nu,\gamma]} = \frac{1}{3}(F_{\mu\nu,\gamma} + F_{\gamma\mu,\nu} + F_{\nu\gamma,\mu})$$

$$T_{[\mu\nu,\gamma]} = \frac{1}{3}(T_{\mu\nu,\gamma} + T_{\gamma\mu,\nu} + T_{\nu\gamma,\mu})$$

(b)  $F_{\mu\nu}$  is an antisymmetric tensor so:

$$F_{\mu\nu,\gamma} + F_{\gamma\mu,\nu} + F_{\nu\gamma,\mu} = 3F_{[\mu\nu,\gamma]}$$

but  $F_{\mu\nu} = -A_{[\mu,\nu]}$  and consequently  $F_{[\mu\nu,\gamma]} = -A_{[[\mu,\nu],\gamma]}$

But we have:

$$\begin{aligned} A_{[[\mu,\nu],\gamma]} &= \frac{1}{2}(A_{[\mu,\nu,\gamma]} - A_{[\nu,\mu,\gamma]}) = \frac{1}{2} \left[ \frac{1}{6}(A_{\mu,\nu,\gamma} - A_{\nu,\mu,\gamma} + A_{\gamma,\mu,\nu} - A_{\mu,\gamma,\nu} + A_{\nu,\gamma,\mu} - A_{\gamma,\nu,\mu}) \right. \\ &\quad \left. - \frac{1}{6}(A_{\nu,\mu,\gamma} - A_{\mu,\nu,\gamma} + A_{\gamma,\nu,\mu} - A_{\nu,\gamma,\mu} + A_{\mu,\gamma,\nu} - A_{\gamma,\mu,\nu}) \right] \\ &= \frac{1}{2} \left[ \frac{2}{6}(A_{\mu,\nu,\gamma} - A_{\nu,\mu,\gamma} + A_{\gamma,\mu,\nu} - A_{\mu,\gamma,\nu} + A_{\nu,\gamma,\mu} - A_{\gamma,\nu,\mu}) \right] \\ &= A_{[\mu,\nu,\gamma]} \end{aligned} \tag{18}$$

and since  $A_{\mu,\nu,\gamma} = A_{\mu,\gamma,\nu}$ , it follows

$$F_{[\mu\nu,\gamma]} = -A_{[[\mu,\nu],\gamma]} = -A_{[\mu,\nu,\gamma]} = 0$$

and so:

$$F_{\mu\nu,\gamma} + F_{\gamma\mu,\nu} + F_{\nu\gamma,\mu} = 3F_{[\mu\nu,\gamma]} = 0$$

8. Show that the Kronecker delta  $\delta^\mu_\nu$  is a tensor.

**SOLUTION**

We have just to show that  $\delta^\mu_\nu$  transform like a tensor. We have for a transformation of coordinate  $x^\mu \rightarrow \bar{x}^\gamma(x^\mu)$ :

$$\bar{\delta}^\mu_\nu = \frac{\partial \bar{x}^\mu}{\partial x^\gamma} \frac{\partial x^\sigma}{\partial \bar{x}^\nu} \delta^\gamma_\sigma = \frac{\partial \bar{x}^\mu}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial \bar{x}^\nu} = \bar{\delta}^\mu_\nu \quad (19)$$

where the last equality follows from the fact that  $\frac{\partial x^\sigma}{\partial \bar{x}^\mu}$  and  $\frac{\partial \bar{x}^\mu}{\partial x^\sigma}$  are thematrix inverse of each other. So  $\delta^\mu_\nu$  transforms like a tensor.

9. Prove that, except for scaling by a constant, there is an unique tensor  $\epsilon_{\alpha\beta\gamma\delta}$  which is totally antisymmetric on all its 4 indices. The usual choice is to take  $\epsilon_{0123} = 1$  in Minkowsky coordinates. What are the components of  $\epsilon$  in a generic coordinate system, with metric  $g_{\mu\nu}$ ?

**SOLUTION**

For a totally antisymmetric vector with rank r and a antisymmetric components in a n-folds, we have already shown that the number of independent components is given by:

$$n^{r-a} \frac{n!}{(n-a)!a!}$$

For  $\epsilon_{\alpha\beta\gamma\delta}$  we have  $n = a = 4$  so that there is just one possibility to choose the component, i.e. once time that  $\epsilon_{0123}$  is given, the tensor is fixed in an unique way. In Minkowski coordinates, we have:

$$\epsilon_{0123} = -\epsilon_{1023} = \epsilon_{1032} = \dots = -1$$

In a generic frame:

$$\bar{\epsilon}_{\alpha\beta\gamma\delta} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \frac{\partial x^\gamma}{\partial \bar{x}^\lambda} \frac{\partial x^\delta}{\partial \bar{x}^\sigma} \epsilon_{\mu\nu\lambda\sigma} = \det\left[\frac{\partial x^a}{\partial \bar{x}^a}\right] \epsilon_{\alpha\beta\gamma\delta} \quad (20)$$

Now we have to find the relation between the expression above and the metric tensor  $g_{\mu\nu}$ . We have already shown that:

$$\bar{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \eta_{\alpha\beta} \quad (21)$$

and so:

$$\det[\bar{g}_{\mu\nu}] = \left|\det\left[\frac{\partial x^a}{\partial \bar{x}^a}\right]\right|^2 \det[\eta_{\alpha\beta}] = -\left|\det\left[\frac{\partial x^a}{\partial \bar{x}^a}\right]\right|^2 \quad (22)$$

It follows from the expression above that:

$$\det\left[\frac{\partial x^a}{\partial \bar{x}^a}\right] = [-\det(\bar{g}_{\mu\nu})]^{1/2} \quad (23)$$

Substituting this last expression in (20) we get:

$$\bar{\epsilon}_{\alpha\beta\gamma\delta} = [-\det(\bar{g}_{\mu\nu})]^{1/2} \epsilon_{\alpha\beta\gamma\delta} \quad (24)$$

10. In an orthonormal frame, show that

$$\epsilon_{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta}$$

What is the analogous relation in a general frame, with metric tensor  $g_{\mu\nu}$ ?

**SOLUTION**

In an orthonormal frame we get:

$$\epsilon^{\alpha\beta\gamma\delta} = \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\gamma\lambda}\eta^{\delta\sigma}\epsilon_{\mu\nu\lambda\sigma} \quad (25)$$

Since that in this frame only the elements on the diagonal are different from zero we get:

$$\epsilon^{0123} = \eta^{00}\eta^{11}\eta^{22}\eta^{33}\epsilon_{0123} = -\epsilon_{0123} \quad (26)$$

because  $\eta^{00} = -1$  and  $\eta^{ii} = 1$ . So in this frame:

$$\epsilon^{\alpha\beta\gamma\delta} = -\epsilon_{\alpha\beta\gamma\delta} \quad (27)$$

In order to find the expression in a generic frame, we use the equation (24) and so we get:

$$\epsilon_{\alpha\beta\gamma\delta} = [-\det(\bar{g}_{\mu\nu})]^{1/2}\epsilon_{\alpha\beta\gamma\delta} = -[-\det(\bar{g}_{\mu\nu})]^{1/2}\epsilon^{\alpha\beta\gamma\delta} \quad (28)$$