Newtonian Gravity

Poisson equation

\[ \nabla^2 U(\vec{x}) = 4\pi G \rho(\vec{x}) \rightarrow U(\vec{x}) = -G \int d^3 \vec{x}' \frac{\rho(\vec{x})}{|\vec{x} - \vec{x}'|} \]

For a spherically symmetric mass distribution of radius \( R \)

\[ U(r) = -\frac{1}{r} \int_0^R r'^2 \rho(r') dr' \quad \text{for} \quad r > R \]

\[ U(r) = -\frac{1}{r} \int_0^r r'^2 \rho(r') dr' - \int_r^R r' \rho(r') dr' \quad \text{for} \quad r < R \]
For a non-spherical distribution the term \(1/|\vec{x} - \vec{x}'|\) can be expanded as

\[
\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{r} + \sum_k \frac{x^k x'^k}{r^3} + \frac{1}{2} \sum_k \sum_l \left(3x'^k x'^l - r^2 \delta^l_k\right) \frac{x^k x^l}{r^5} + \ldots
\]

\[
U(\vec{x}) = -\frac{GM}{r} - \frac{G}{r^3} \sum_k x^k D^k - \frac{G}{2} \sum_{k l} Q^{k l} \frac{x^k x^l}{r^5} + \ldots
\]

**Gravitational Multipoles**

\[
M = \int \rho(\vec{x}') d^3 x' \quad \text{Mass}
\]

\[
D^k = \int x'^k \rho(\vec{x}') d^3 x' \quad \text{Mass Dipole moment}^{a}
\]

\[
Q^{k l} = \int \left(3x'^k x'^l - r^2 \delta^l_k\right) \rho(\vec{x}') d^3 x' \quad \text{Mass Quadrupole tensor}^{b}
\]

\(^{a}\)If the center of mass is chosen to coincide with the origin of the coordinates then \(D^k = 0\) (no mass dipole).

\(^{b}\)If \(Q^{k l} \neq 0\) the potential will contain a term proportional to \(\sim 1/r^3\) and the gravitational force will deviate from the inverse square law by a term \(\sim 1/r^4\).
The Earth’s polar and equatorial diameters differ by $3/1000$ (Sun $10^{-5}$). This deviation produces a quadrupole term in the gravitational potential, which causes perturbations in the elliptical Kepler orbits of satellites. Usually, we define the dimensionless parameter

$$J_2 = -\frac{Q^{33}}{2M_{\odot}R_{\odot}^2}$$

as a convenient measure of the oblateness of a nearly spherical body. For the Sun the oblateness due to rotation gives $J_2 \approx 10^{-7}$. The main perturbation is the precession of Kepler’s ellipse and this can be used for precise determinations of the multipole moments and the mass distribution in the Earth.

What about the Sun?
In GR gravitational phenomena arise not from forces and fields, but from the curvature of the 4-dim spacetime. The starting point for this consideration is the **Equivalence Principle** which states that the Gravitational and the Inertial masses are equal. The equality $m_G = m_I$ is one of the most accurately tested principles in physics! Now it is known experimentally that: $\gamma = \frac{m_G - m_I}{m_G} < 10^{-13}$

**Experimental verification**

- Galileo (1610), Newton (1680) $\gamma < 10^{-3}$
- Bessel (19th century) $\gamma < 2 \times 10^{-5}$
- Eötvös (1890) & (1908) $\gamma < 3 \times 10^{-9}$
- Dicke et al. (1964) $\gamma < 3 \times 10^{-11}$
- Braginsky et al. (1971) $\gamma < 9 \times 10^{-13}$
- Kuroda and Mio (1989) $\gamma < 8 \times 10^{-10}$
- Adelberger et al. (1990) $\gamma < 1 \times 10^{-11}$
- Su et al. (1994) $\gamma < 1 \times 10^{-12}$ (torsional)
- Williams et al. (‘96), Anderson & Williams (‘01) $\gamma < 1 \times 10^{-13}$
Equivalence Principle II

- **Weak Equivalence Principle**: The motion of a neutral test body released at a given point in space-time is independent of its composition.

- **Strong Equivalence Principle**: The results of all local experiments in a frame in free fall are independent of the motion. The results are the same for all such frames at all places and all times. The results of local experiments in free fall are consistent with STR.

Einstein’s Theory of Gravity
Equivalence Principle : Dicke’s Experiment

The experiment is based on measuring the effect of the gravitational field on two masses of different material in a torsional pendulum.

\[
\frac{GMm^{(E)}_G}{R^2} = \frac{m^{(E)}_Gv^2}{R} \implies v^2 = \frac{GM}{R} \left( \frac{m_G}{m_i} \right)^{(E)}
\]

The forces acting on both masses are:

\[
F^{(j)} + \frac{GMm^{(j)}_G}{R^2} = \frac{m^{(j)}_Gv^2}{R} \implies F^{(j)} = \frac{GMm^{(j)}_G}{R^2} \left[ \left( \frac{m_G}{m_i} \right)^{(E)} - \left( \frac{m_G}{m_i} \right)^{(j)} \right]
\]

and the total torque applied is:

\[
L = (F^{(1)} - F^{(2)}) \ell = \frac{GM}{R^2} \left[ m^{(1)}_G - m^{(2)}_G \right] \ell
\]

here we assumed that: \( m^{(1)}_i = m^{(2)}_i = m \).
Towards a New Theory for Gravity

Because of the success of Newton’s theory of gravity, our new theory should obey two demands:

- In an appropriate first (weak field) approximation the new theory should reduce to the Newtonian one.
- Beyond this approximation the new theory should predict small deviations from newtonian theory which must be verified by experiments/observations.

Einstein’s equivalence principle:

**Gravitational and Inertial forces/accelerations are equivalent and they cannot be distinguished by any physical experiment.**

This statement has 3 implications:

1. **Gravitational forces/accelerations are described in the same way as the inertial ones.**
2. **When gravitational accelerations are present the space cannot be flat.**
3. **Consequence:** if gravity is present there cannot exist inertial frames.
1. **Gravitational forces/accelerations are described in the same way as the inertial ones.**

- This means that the motion of a freely moving particle, observed from an inertial frame, will be described by $\frac{d^2x^\mu}{dt^2} = 0$ while from a non-inertial frame its movement will be described by the geodesic equation

  \[ \frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} = 0. \]

- The 2nd term appeared due to the use of a non-inertial frame, i.e. the **inertial accelerations** will be described by the Christoffel symbols.
- But according to Einstein the **gravitational accelerations** as well will be described by the Christoffel symbols.
- This leads to the following conclusion: The metric tensor should play the role of the gravitational potential since $\Gamma^\mu_{\rho\sigma}$ is a function of the metric tensor and its derivatives.
2. **When gravitational accelerations are present the space cannot be flat**

Since the Christoffel symbols are non-zero and the Riemann tensor is not zero as well. In other words, the presence of the gravitational field forces the space to be curved, i.e. there is a direct link between the presence of a gravitational field and the geometry of the space.

3. **Consequence:** if gravity is present there cannot exist inertial frames.

If it was possible then one would have been able to discriminate among inertial and gravitational accelerations which against the “generalized equivalence principle” of Einstein. The absence of “special” coordinate frames (like the inertial) and their substitution from “general” (non-inertial) coordinate systems lead in naming Einstein’s theory for gravity “General Theory of Relativity”.
Einstein’s Equations

- Since the source of the gravitational field is a tensor \((T_{\mu\nu})\) the field should be also described by a 2nd order tensor e.g. \(F_{\mu\nu}\).
- Since the role of the gravitational potential is played by the metric tensor then \(F_{\mu\nu}\) should be a function of the metric tensor \(g_{\mu\nu}\) and its 1st and 2nd order derivatives.
- Moreover, the law of energy-momentum conservation implies that \(T_{\mu\nu}{}_{;\mu} = 0\) which suggests that \(F_{\mu\nu}{}_{;\mu} = 0\).
- Then since \(F_{\mu\nu}\) should be a linear function of the 2nd derivative of \(g_{\mu\nu}\) we come to the following form of the field equations (how & why?):

\[
F_{\mu\nu} = R_{\mu\nu} + ag_{\mu\nu} R + bg_{\mu\nu} = \kappa T_{\mu\nu}
\]

where \(\kappa = \frac{8\pi G}{c^4}\). Then since \(F_{\mu\nu}{}_{;\mu} = 0\) there should be

\[
(R_{\mu\nu} + ag_{\mu\nu} R + bg_{\mu\nu});_{\mu} = 0
\]

which is possible only for \(a = -1/2\). Thus the final form of Einstein’s equations is:

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}.
\]

where \(\Lambda = \frac{8\pi G}{c^2} \rho_v\) is the so called cosmological constant.
We will show that Einstein’s equations can be written as

\[ R_{\mu\nu} = -\kappa \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \]  

(5)

This is true because if we multiply

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu} \]

with \( g^{\rho\nu} \) they can be written as:

\[ 2R^{\rho}_{\phantom{\rho}\nu} - \delta^{\rho}_{\phantom{\rho}\nu} R = -2\kappa T^{\rho}_{\phantom{\rho}\nu} \]

and by contracting \( \rho \) and \( \nu \) we get \( 2R - 4R = -2\kappa T \) i.e. \( R = \kappa T \). Thus:

\[ R_{\mu\nu} = -\kappa T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = -\kappa \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) . \]
Newtonian Limit

In the absence of strong gravitational fields and for small velocities both Einstein & geodesic equations reduce to the Newtonian ones.

- Geodesic equations:

\[
\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = -\frac{d^2t/ds^2}{(dt/ds)^2} \frac{dx^\mu}{dt} \Rightarrow \frac{d^2x^j}{dt^2} \approx g_{00,j} \quad (6)
\]

If \( g_{00} \approx \eta_{00} + h_{00} = 1 + h_{00} = 1 + 2 \frac{U}{c^2} \) then \( \frac{d^2x^k}{dt^2} \approx -\frac{\partial U}{\partial x^k} \)

- Einstein equations

\[
R_{\mu\nu} = -\kappa \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \Rightarrow \nabla^2 U = \frac{1}{2} \kappa \rho \quad (7)
\]

where \( \kappa = 8\pi G \) or \( \kappa = 8\pi G/c^4 \).

Here we have used the following approximations:

\[
\Gamma^j_{00} \approx \frac{1}{2} g_{00,j} \quad \text{and} \quad R_{00} \approx \Gamma^j_{00,j} \approx \nabla^2 U
\]
Newtonian Limit: The geodesic equations

The geodesic equations are typically written with respect to the proper time \( \tau \) or the proper length \( s \). In Newtonian theory the absolute time and the proper time are identical thus the equations need to be written with respect to the coordinate time \( t \), which is not and affine parameter! Thus we will use the form of the geodesic equations (for non affine parameters) presented in Chapter 1 i.e. (8)

\[
\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} = - \frac{d^2\sigma/ds^2}{(d\sigma/ds)^2} \frac{dx^\mu}{d\sigma} \tag{8}
\]

and if we select \( \sigma = x^0 = t \) the above relation will be written:

\[
\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = - \frac{d^2t/ds^2}{(dt/ds)^2} \frac{dx^\mu}{dt} \tag{9}
\]

The 1st of the above equations (the one for \( x^0 \)) simplified because \( dx^0/dt = dt/dt = 1 \) and \( d^2x^0/dt^2 = 0 \) and thus

\[
\Gamma^0_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = - \frac{d^2t/ds^2}{(dt/ds)^2} \tag{10}
\]
This can be substituted in (9) and the remaining 3 equations will have the form for the coordinates $x^k$ for $(k = 1, 2, 3)$ become:

$$
\frac{d^2x^k}{dt^2} + \left( \Gamma^k_{\alpha\beta} - \Gamma^0_{\alpha\beta} \frac{dx^k}{dt} \right) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0.
$$

(11)

Then we will use the approximation that all velocities are much smaller than the speed of light i.e. $u^k = \frac{dx^k}{dt} \ll 1$ (in geometrical units we assume that $c = 1$). Thus $\Gamma^k_{\alpha\beta} \gg \Gamma^0_{\alpha\beta} \frac{dx^k}{dt}$ and the previous equation will be approximately:

$$
\frac{d^2x^k}{dt^2} \approx -\Gamma^k_{00}
$$

(12)

In other words the Christoffel symbol $\Gamma^k_{00}$ corresponds to the Newtonian force per unit of mass.

In a space which is “slightly curved” i.e. where

$$
g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}
$$

with $\eta_{\mu\nu} \gg h_{\mu\nu}$.

Under this assumption the Christoffel $\Gamma^k_{00}$, gets the form

$$
\Gamma^k_{00} = \frac{1}{2} g^{k\lambda} (2g_{\lambda 0,0} - g_{00,\lambda}) \approx -\frac{1}{2} \eta^{k\lambda} g_{00,\lambda} = \frac{1}{2} \delta^{kj} g_{00,j} = \frac{1}{2} g_{00,k}
$$

(13)
Thus in the Newtonian limit the geodesic equations reduced to:

\[ \frac{d^2 x^k}{dt^2} \approx g_{00,k}. \] (14)

Which reminds the Newtonian relation

\[ \frac{d^2 x^k}{dt^2} \approx \frac{\partial U}{\partial x^k} \] (15)

which suggests that \( g_{00} \) has the form

\[ g_{00} \approx \eta_{00} + h_{00} = 1 + h_{00} = 1 + 2U \] (16)

here we set \( U = 2h_{00} \).
Newtonian Limit: Einstein equations

We will show that Einstein’s equations in the Newtonian limit reduce to the well known Poisson equation of Newtonian gravity. For small concentrations of masses the dominant component of the energy momentum tensor is the \( T_{00} \). Thus the dominant component of the Einstein’s equations is

\[
R_{00} = -\kappa \left( T_{00} - \frac{1}{2} g_{00} T \right) \approx -\kappa \left( T_{00} - \frac{1}{2} \eta_{00} T \right) \approx -\frac{1}{2} \kappa T_{00}
\]

\[
= -\frac{1}{2} \kappa \rho
\]

where we assumed that \( T = g^{\mu\nu} T_{\mu\nu} \approx \eta^{\mu\nu} T_{\mu\nu} \approx \eta^{00} T_{00} = T_{00} \).

The 00 component of the Ricci tensor is given by the relation

\[
R_{00} = \Gamma_{00,\mu}^{\mu} - \Gamma_{0\mu,0}^{\mu} + \Gamma_{00}^{\mu} \Gamma_{\nu \mu}^{\nu} - \Gamma_{0\nu}^{\mu} \Gamma_{0\mu}^{\nu} \approx \Gamma_{00,\mu}^{\mu} \approx \Gamma_{00,j}^{j}
\]

But as we have shown \( \Gamma_{00}^{j} \approx g_{00,j}/2 \) and thus:

\[
R_{00} \approx \Gamma_{00,j}^{j} \approx \frac{1}{2} \frac{\partial^2 g_{00}}{\partial x^j \partial x^j} = \frac{1}{2} \nabla^2 g_{00} \approx \nabla^2 U
\]

leading to:

Einstein’s Theory of Gravity
\[ \nabla^2 U = \frac{1}{2} \kappa \rho \] \hspace{1cm} (20)

From which by comparing with Poisson’s equation (1) we get the value of the coupling constant \( \kappa \) that is:

\[ \kappa = 8\pi G . \] \hspace{1cm} (21)
A **static spacetime** is one for which a timelike coordinate $x^0$ has the following properties:

(I) all metric components $g_{\mu\nu}$ are independent of $x^0$

(II) the line element $ds^2$ is invariant under the transformation $x^0 \rightarrow -x^0$.

Note that the 1st property does not imply the 2nd (e.g. the time reversal on a rotating star changes the sense of rotation, but the metric components are constant in time).

• A spacetime that satisfies (I) but not (II) is called **stationary**.

• The line element $ds^2$ of a static metric depends only on **rotational invariants** of the spacelike coordinates $x^i$ and their differentials, i.e. the metric is **isotropic**.
The only rotational invariants of the spacelike coordinates $x^i$ and their differentials are

$$\vec{x} \cdot \vec{x} \equiv r^2, \quad \vec{x} \cdot d\vec{x} \equiv r dr, \quad d\vec{x} \cdot d\vec{x} \equiv dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

Thus the more general form of a spatially isotropic metric is:

$$ds^2 = A(t, r)dt^2 - B(t, r)dt(\vec{x} \cdot d\vec{x}) - C(t, r)(\vec{x} \cdot d\vec{x})^2 - D(t, r)d\vec{x}^2$$

$$= A(t, r)dt^2 - B(t, r)r dt dr - C(t, r)r^2 dr^2 - D(t, r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)$$

$$= A(t, r)dt^2 - \tilde{B}(t, \tilde{r})dt d\tilde{r} - \tilde{C}(t, \tilde{r})d\tilde{r}^2 - \tilde{D}(t, \tilde{r}) (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$= A'(t, \tilde{r})dt^2 - B'(t, \tilde{r})dt d\tilde{r} - C'(t, \tilde{r})d\tilde{r}^2 - \tilde{r}^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(22) 

where we have set $\tilde{r}^2 = D(t, r)$ and we redefined the $A$, $\tilde{B}$ and $\tilde{C}$. The next step will be to introduce a new timelike coordinate $\tilde{t}$ as

$$d\tilde{t} = \Phi(t, \tilde{r}) \left[ A'(t, \tilde{r})dt - \frac{1}{2} B'(t, \tilde{r})d\tilde{r} \right]$$

where $\Phi(t, \tilde{r})$ is an integrating factor that makes the right-hand side an exact differential.
By squaring we obtain
\[d\tilde{t}^2 = \Phi^2 \left( A'^2 dt^2 - A'B' dt d\tilde{r} + \frac{1}{4} B'^2 d\tilde{r}^2 \right)\]
from which we find
\[A' dt^2 - B' dt d\tilde{r} = \frac{1}{A' \Phi^2} d\tilde{t}^2 - \frac{B'}{4A'} d\tilde{r}^2\]

Thus by defining the new functions \( \hat{A} = 1/(A' \Phi)^2 \) and \( \hat{B} = C + B'/(4A') \) the metric (23) becomes diagonal
\[ds^2 = \hat{A}(\tilde{t}, \tilde{r}) d\tilde{t}^2 - \hat{B}(\tilde{t}, \tilde{r}) d\tilde{r}^2 - \tilde{r}^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{24}\]
or by dropping the ‘hats’ and ‘tildes’
\[ds^2 = A(t, r) dt^2 - B(t, r) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{25}\]

- Thus the **general isotropic metric** is specified by two functions of \( t \) and \( r \), namely \( A(t, r) \) and \( B(t, r) \).
- Also, surfaces for \( t \) and \( r \) constant are 2-spheres (isotropy of the metric).
- Since \( B(t, r) \) is not unity we cannot assume that \( r \) is the radial distance.
Schwarzschild Solution

A typical solution of Einstein’s equations describing spherically symmetric spacetimes has the form:

\[ ds^2 = e^{\nu(t,r)} dt^2 - e^{\lambda(t,r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  

(26)

We next calculate the components of the Ricci tensor

\[ R_{\mu\nu} = 0. \]  

(27)

Then the Christoffel symbols for the metric (26) are:

\[
\begin{align*}
\Gamma^0_{00} & = \frac{\dot{\nu}}{2}, & \Gamma^1_{00} & = \frac{\nu'}{2} e^{\nu-\lambda}, & \Gamma^0_{01} & = \frac{\nu'}{2} \\
\Gamma^0_{11} & = \frac{\dot{\lambda}}{2}, & \Gamma^0_{11} & = \frac{\lambda'}{2} e^{\lambda-\nu}, & \Gamma^0_{11} & = \frac{\lambda'}{2} \\
\Gamma^1_{12} & = \frac{1}{r}, & \Gamma^3_{13} & = \frac{1}{r}, & \Gamma^1_{22} & = -re^{-\lambda} \\
\Gamma^3_{23} & = \cot \theta, & \Gamma^3_{33} & = -re^{-\lambda} \sin^2 \theta, & \Gamma^2_{33} & = \sin \theta \cos \theta
\end{align*}
\]  

(28-31)

where \( \dot{\cdot} = \partial / \partial t \) and \( \cdot' = \partial / \partial r \)
Then the components of the Ricci tensor will be:

\[
R_{11} = -\frac{\nu''}{2} - \frac{\nu'^2}{4} + \frac{\nu' \lambda'}{4} + \frac{\lambda'}{r} + e^{\lambda-\nu} \left[ \frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\nu} \dot{\lambda}}{4} \right] \quad (32)
\]

\[
R_{00} = e^{\nu-\lambda} \left[ \frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu' \lambda'}{4} + \frac{\nu'}{r} \right] - \frac{\ddot{\lambda}}{2} - \frac{\dot{\lambda}^2}{4} + \frac{\dot{\nu} \dot{\lambda}}{4} \quad (33)
\]

\[
R_{10} = \frac{\dot{\lambda}}{r} \quad (34)
\]

\[
R_{22} = -e^{-\lambda} \left[ 1 + r \left( \nu' - \lambda' \right) \right] + 1 \quad (35)
\]

\[
R_{33} = \sin^2 \theta R_{22} \quad (36)
\]

in the absence of matter (outside the source) \( R_{\mu\nu} = 0 \) we can prove that \( \lambda(r) = -\nu(r) \), i.e. the solution is independent of time (how and why?).
We will use the above relations to find the functional form of \( \nu(t, r) \) and \( \lambda(t, r) \) in an empty space i.e. when \( R_{\mu\nu} = 0 \).

Then from the sum: \( R_{11} + e^{\lambda - \nu} R_{00} = 0 \) we get:

\[
\nu' + \lambda' = 0 \tag{37}
\]

while from

\[
R_{10} = \frac{\dot{\lambda}}{r} = 0 \tag{38}
\]

we conclude that \( \lambda = \lambda(r) \).

From \( R_{22} = 0 \) (eq. 35) we get:

\[
\nu' - \lambda' = \frac{2}{r}(e^\lambda - 1) \tag{39}
\]

Thus

\[
\nu' = \frac{1}{r}(e^\lambda - 1) \tag{40}
\]

\[
\lambda' = -\frac{1}{r}(e^\lambda - 1) \tag{41}
\]
Since the right side of the above differential equation for $\nu$ is function only of $r$ the function $\nu = \nu(t, r)$ can be written as:

$$\nu(t, r) = \alpha(t) + \tilde{\nu}(r) \quad (42)$$

which leads to a redefinition of the time coordinate $\tilde{t}$,

$$d\tilde{t} = e^{\alpha/2}dt \quad (43)$$

i.e the time dependence of $\nu(t, r)$ is ”absorbed” by the change of variable from $t$ to $\tilde{t}$.
Thus from (37) we lead to the relation:

$$\lambda(r) = -\tilde{\nu}(r) \quad (44)$$

i.e. both unknown components of the metric tensor are independent of time and for simplicity we will write $\nu(r)$ instead of $\tilde{\nu}(r)$.

**BIRKOFF’s THEOREM**
*If the geometry of a spacetime is spherically symmetric and is solution of the Einstein’s equations, then it is described by the Schwarzschild solution.*
The solution of equation (41) provides the function $\lambda(r)$. We can get it by substituting $f = e^{-\lambda}$ then eq. (41) will be written as:

$$rf' + f = 1 \quad (45)$$

with obvious solution:

$$f = e^{-\lambda} = e^\nu = 1 - \frac{k}{r} \quad (46)$$

where $k$ and will be determined by the boundary conditions. In the present case for $r \to \infty$ our solutions should lead to Newton's solution that is

$$g_{00} = 1 + \frac{2}{c^2} U(r) \quad (47)$$

where $U(r)$ is the Newtonian potential and for the case of spherical symmetry is

$$U(r) = -\frac{GM}{r} \quad (48)$$

which leads to

$$k = \frac{2GM}{c^2} \quad (49)$$

and Schwarzschild solutions gets the form:
\[ ds^2 = \left( 1 - \frac{2GM}{rc^2} \right) c^2 dt^2 - \left( 1 - \frac{2GM}{rc^2} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

- **Sun**: \( M_\odot \approx 2 \times 10^{33} \text{gr} \) and \( R_\odot = 696.000 \text{km} \)

\[
\frac{2GM}{rc^2} \approx 4 \times 10^{-6}
\]

- **Neutron star**: \( M \approx 1.4M_\odot \) and \( R_\odot \approx 10 - 15 \text{km} \)

\[
\frac{2GM}{rc^2} \approx 0.3 - 0.5
\]

- **Neutonian limit**:

\[
g_{00} \approx \eta_{00} + h_{00} = 1 + \frac{2U}{c^2} \quad \Rightarrow \quad U = \frac{GM}{r}
\]
Schwarzschild Solution: Geodesics

\[ \ddot{r} - \frac{1}{2} \nu' \dot{r}^2 - re^{\nu} \dot{\theta}^2 - re^{\nu} \sin^2 \theta \dot{\phi}^2 + \frac{1}{2} e^{2\nu} \nu' \dot{t}^2 = 0 \]  \hspace{1cm} (50)

\[ \ddot{\theta} + \frac{2}{r} \dot{r} \dot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 = 0 \]  \hspace{1cm} (51)

\[ \ddot{\phi} + \frac{2}{r} \dot{r} \dot{\phi} + 2 \frac{\cos \theta}{\sin \theta} \theta \dot{\phi} \dot{\theta} = 0 \]  \hspace{1cm} (52)

\[ \ddot{t} + \nu' \dot{r} \dot{t} = 0 \]  \hspace{1cm} (53)

and from \( g_{\lambda \mu} \dot{x}^\lambda \dot{x}^\mu = 1 \) (here we assume a massive particle) we get:

\[-e^{\nu} \dot{t}^2 + e^{\lambda} \dot{r}^2 + r^2 \dot{\theta} + r^2 \sin^2 \theta \dot{\phi}^2 = 1. \]  \hspace{1cm} (54)

If a geodesic is passing through a point \( P \) in the equatorial plane \( (\theta = \pi/2) \) and has a tangent at \( P \) situated also in this plane \( (\dot{\theta} = 0 \) at \( P \)\) then from (51) we get \( \ddot{\theta} = 0 \) at \( P \) and all higher derivatives are also vanishing at \( P \). That is, the geodesic lies entirely in the plane defined by \( P \), the tangent at \( P \) and the center of symmetry of the space. Since the symmetry planes are equivalent to each other, it will be sufficient to discuss the geodesics lying on one of these planes e.g. the equatorial plane \( \theta = \pi/2 \).
The geodesics on the equatorial plane are:

\[ \ddot{r} - \frac{1}{2} \nu' r^2 - r e^\nu \dot{\phi}^2 + \frac{1}{2} e^{2\nu} \nu' \dot{t}^2 = 0 \quad (55) \]
\[ \ddot{\phi} + \frac{2}{r} \dot{r} \dot{\phi} = 0 \quad (56) \]
\[ \ddot{t} + \nu' \dot{r} \dot{t} = 0 \quad (57) \]
\[ -e^\nu \dot{t}^2 + e^\lambda r^2 + r^2 \dot{\phi}^2 = 1 \quad (58) \]

Then from equations (56) and (57) we can easily prove (how?) that:

\[ \frac{d}{d\tau} \left( r^2 \dot{\phi} \right) = 0 \quad \Rightarrow \quad r^2 \dot{\phi} = L = \text{const} : \text{(Angular Momentum)} (59) \]
\[ \frac{d}{d\tau} \left( e^\nu \dot{t} \right) = 0 \quad \Rightarrow \quad e^\nu \dot{t} = E = \text{const} : \text{(Energy)} (60) \]

- For a massive particle with unit rest mass, by assuming that \( \tau \) is the affine parameter for its motion we get \( p^\mu = \dot{x}^\mu \). Thus

\[ p_0 = g_{00} \dot{t} = e^\nu \dot{t} = E \quad \text{and} \quad p_\phi = g_{\phi\phi} \dot{\phi} = -r^2 \dot{\phi} = -L \quad (61) \]
• An observer with 4-velocity $U^\mu$ will find that the energy of a particle with 4-momentum $p^\mu$ is:

$$\mathcal{E} = p_\mu U^\mu$$

An observer at infinity, $U^\mu = (1, 0, 0, 0)$ will find $\mathcal{E} = p_0 = E c^2$. Actually, for a particle with rest mass $m_0$ we get $E = \mathcal{E} / (m_0 c^2)$ i.e. it is the energy per unit rest-mass.

For a particle at infinity we assume $\mathcal{E} = m_0 c^2$.

Finally, by substituting eqns (60) and (59) into (58) we get the “energy equation” for the $r$-coordinate:

$$\dot{r}^2 + \frac{e^\nu}{r^2} L^2 + e^\nu = E^2$$

(62)

which suggests that at $r \to \infty$ we get that $E = 1$. 
By combining the 2 integrals of motion and eqn (58) we can eliminate the proper time $\tau$ to derive an equation for a 3D path of the particle (how?)

$$\left(\frac{d}{d\phi}u\right)^2 + u^2 = \frac{E^2 - 1}{L^2} + \frac{2Mu}{L^2} + 2Mu^3$$  \hspace{1cm} (63)

where we have used $u = 1/r$ and

$$\dot{r} = \frac{dr}{d\tau} = \frac{dr}{d\phi} \frac{d\phi}{d\tau} = \frac{L}{r^2} \frac{dr}{d\phi}$$

The previous equation can be written in a form similar to Kepler’s equation of Newtonian mechanics (how?) i.e.

$$\frac{d^2}{d\phi^2}u + u = \frac{M}{L^2} + 3Mu^2$$  \hspace{1cm} (64)

The term $3Mu^2$ is the relativistic correction to the Newtonian equation. This term for the trajectories of planets in the solar system, where $M = M_\odot = 1.47664 \text{ km}$, and for orbital radius $r \approx 4.6 \times 10^7 \text{ km}$ (Mercury) and $r \approx 1.5 \times 10^8 \text{ km}$ (Earth) gets very small values $\approx 3 \times 10^{-8}$ and $\approx 10^{-8}$ correspondingly.
Radial motion of massive particles

For the radial motion $\phi$ is constant, which implies that $L = 0$ and eqn (62) reduces to

$$\dot{r}^2 = E^2 - e^\nu$$

and by differentiation we get an equation which reminds the equivalent one of Newtonian gravity i.e.

$$\ddot{r} = \frac{M}{r^2}.$$ (66)

• If a particle is dropped from the rest at $r = R$ ($\dot{r} = 0$) we get that $E^2 = e^{\nu(R)} = 1 - 2M/R$ and (65) will be written

$$\dot{r}^2 = 2M \left( \frac{1}{r} - \frac{1}{R} \right)$$

which is again similar to the Newtonian formula for the gain of kinetic energy due to the loss in gravitational potential energy for a particle (of unit mass) falling from rest at $r = R$. 

Einstein’s Theory of Gravity
For a particle dropped from the rest at infinity $E = 1$ the geodesic equations are simplified (how?):

\[
\frac{dt}{d\tau} = e^{-\nu} \quad \text{and} \quad \frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}}
\] (68)

The component of the 4-velocity will be:

\[
u^\mu = \frac{dx^\mu}{d\tau} = \left( e^{-\nu}, -\sqrt{\frac{2M}{r}}, 0, 0 \right)
\] (69)

Then by integrating the second of (68) and by assuming that at $\tau = \tau_0 = 0$ that $r = r_0$ we get

\[
\tau = \frac{2}{3} \sqrt{\frac{r_0^3}{2M}} - \frac{2}{3} \sqrt{\frac{r^3}{2M}}
\] (70)

which suggests that for $r = 0$ we get $\tau \to \frac{2}{3} \sqrt{\frac{r_0^3}{2M}}$ i.e. the particle takes finite proper time to reach $r = 0$. 

Einstein’s Theory of Gravity
If we want to map the trajectory of the particle in the \((r, t)\) coordinates we need to solve the equation

\[
\frac{dr}{dt} = \frac{d}{d\tau} \frac{d\tau}{dt} = -e^{-\nu} \sqrt{\frac{2M}{r}} \tag{71}
\]

The integration leads to the relation

\[
t = \frac{2}{3} \left( \sqrt{\frac{r_0^3}{2M}} - \sqrt{\frac{r^3}{2M}} \right) + 4M \left( \sqrt{\frac{r_0}{2M}} - \sqrt{\frac{r}{2M}} \right) \\
+ 2M \ln \left| \frac{\sqrt{r/2M - 1}}{\sqrt{r/2M - 1}} \right| \tag{72}
\]

where we have chosen that for \(t = 0\) to set \(r = r_0\).
Notice that when \(r \to 2M\) then \(t \to \infty\). In other words, for an observer at infinity, it takes \textit{infinite time} for a particle to reach \(r = 2M\).

\begin{center}
\textbf{QUESTION} : Can you find what will be the velocity of the radially falling particle for a stationary observer at coordinate radius \(\tilde{r}\)?
\end{center}
Figure: Radial fall from rest towards a Schwarzschild BH as described by a comoving observer (proper time $\tau$) and by a distant observer (Schwarzschild coordinate time $t$)

$$\tau = \frac{2}{3} \sqrt{\frac{r_0^3}{2M}} - \frac{2}{3} \sqrt{\frac{r^3}{2M}}$$

$$t = \frac{2}{3} \left( \sqrt{\frac{r_0^3}{2M}} - \sqrt{\frac{r^3}{2M}} \right) + 4M \left( \sqrt{\frac{r_0}{2M}} - \sqrt{\frac{r}{2M}} \right) + 2M \ln \left| \left( \frac{\sqrt{r/2M} + 1}{\\sqrt{r/2M} - 1} \right) \left( \frac{\sqrt{r_0/2M} - 1}{\\sqrt{r_0/2M} + 1} \right) \right|$$
The motion of massive particles in the equatorial plane is described by eqn (64)

\[
\frac{d^2}{d\phi^2} u + u = \frac{M}{L^2} + 3Mu^2
\]  

(73)

For circular motions \( r = \frac{1}{u} = \text{const} \) and \( \dot{r} = \ddot{r} = 0 \). Thus we get

\[
L^2 = \frac{r^2M}{r - 3M}
\]  

(74)

if we also put \( \dot{r} = 0 \) in eqn (62) we get:

\[
E = \frac{1 - 2M/r}{\sqrt{1 - 3M/r}}
\]  

(75)

The energy of a particle with rest-mass \( m_0 \) in a circular radius \( r \) is then given by \( \mathcal{E} = E \cdot (m_0c^2) \).

For the circular orbits to be bound we require \( \mathcal{E} < m_0c^2 \), so the limit on \( r \) for an orbit to be bound is given by \( E = 1 \) which leads to

\[
(1 - 2M/r)^2 = 1 - 3M/r \quad \text{true when} \quad r = 4M \text{ or } r = \infty
\]  

(76)

Thus over the range \( 4M < r < \infty \) circular orbits are bound.
Figure: The variation of $E = \mathcal{E}/(m_0 c^2)$ as a function of $r/M$ for a circular orbit of a massive particle in the Schwarzschild geometry.
Figure: The shape of a bound orbit outside a spherical star or a black-hole
From the integral of motion \( r^2 \dot{\phi} = L \) and eqn (74) we get

\[
\left( \frac{d\phi}{d\tau} \right)^2 = \frac{M}{r^2(r - 3M)} \tag{77}
\]

**NOTICE:** This equation cannot be satisfied for circular orbits with \( r < 3M \). Such orbits cannot be geodesics and cannot be followed by freely falling particles.

We can also calculate an expression for \( \Omega = \frac{d\phi}{dt} \)

\[
\Omega^2 = \left( \frac{d\phi}{dt} \right)^2 = \left( \frac{d\phi}{d\tau} \frac{d\tau}{dt} \right)^2 = \frac{e^{2\nu}}{E^2} \left( \frac{d\phi}{d\tau} \right)^2 = \frac{M}{r^3} \tag{78}
\]

which is equivalent to Kepler’s law in Newtonian gravity.
Stability of massive particle orbits

According to the previous discussion the closest bound orbit around a massive body is at \( r = 4M \), however we cannot yet determine whether this orbit is stable.

In Newtonian theory the particle motion in a central potential is described by:

\[
\frac{1}{2} \left( \frac{dr}{dt} \right)^2 + V_{\text{eff}}(r) = E^2
\]  

where

\[
V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{L^2}{2r^2}
\]  

The bound orbits have two turning points while the circular orbit corresponds to the special case where the particle sits in the minimum of the effective potential.
In GR the ‘energy’ equation (62) is

\[ r^2 + \frac{e^\nu}{r^2} L^2 + e^\nu = E^2 \]  \hspace{1cm} (81)

which leads to an effective potential of the form

\[ V_{\text{eff}}(r) = \frac{e^\nu}{r^2} L^2 + e^\nu = -\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} \]  \hspace{1cm} (82)

Circular orbits occur where \( dV_{\text{eff}}/dr = 0 \) that is:

\[ \frac{dV_{\text{eff}}}{dr} = \frac{M}{r^2} - \frac{L^2}{r^3} + \frac{3ML^2}{r^4} \]  \hspace{1cm} (83)

so the extrema are located at the solutions of the eqn \( Mr^2 - Lr + 3ML^2 = 0 \) which occur at

\[ r = \frac{L}{2M} \left( L \pm \sqrt{L^2 - 12M^2} \right) \]
Note that if $L = \sqrt{12}M = 2\sqrt{3}M$ then there is only one extremum and no turning points in the orbit for lower values of $L$. Thus the **innermost stable circular orbit** (ISCO) has

$$r_{\text{ISCO}} = 6M \quad \text{and} \quad L = 2\sqrt{3}M \approx 3.46M$$

and it is unique in satisfying both $dV_{\text{eff}}/dr = 0$ and $d^2V_{\text{eff}}/dr^2 = 0$, the latter is the condition for **marginal stability** of the orbit.

**Figure:** The dots indicate the locations of **stable circular orbits** which occur at the local minimum of the potential. The local maxima in the potential are the locations of the **unstable circular orbits**.
Figure: Orbits for $L = 4.3$ and different values of $E$.

- The **upper** shows *circular orbits*. A stable (outer) and an unstable (inner) one.
- The **lower** shows *bound orbits*, the particle moves between two turning points marked by dotted circles.
Figure: Orbits for $L = 4.3$ and different values of $E$.

- The upper shows a **scattering orbit** the particle comes in from infinity passes around the center of attraction and moves out to infinity again.
- The lower shows **plunge orbit**, in which the particle comes in from infinity.
Trajectories of photons

Photons, as any zero rest mass particle, move on null geodesics. In this case we cannot use the proper time $\tau$ as the parameter to characterize the motion and thus we will use some affine parameter $\sigma$. We will study photon orbits on the equatorial plane and the equation of motion will be in this case

$$e^\nu \dot{t} = E$$  \hspace{1cm} (84)

$$e^\nu \dot{t}^2 - e^{-\nu} \dot{r}^2 - r^2 \dot{\phi}^2 = 0$$  \hspace{1cm} (85)

$$r^2 \dot{\phi} = L$$  \hspace{1cm} (86)

The equivalent to eqn (62) for photons is

$$\dot{r}^2 + \frac{e^\nu}{r^2} L^2 = E^2$$  \hspace{1cm} (87)

while the equivalent of eqn (64) is

$$\frac{d^2}{d\phi^2} u + u = 3Mu^2$$  \hspace{1cm} (88)
Radial motion of photons

For radial motion $\dot{\phi} = 0$ and we get

$$e^{\nu} \dot{t}^2 - e^{-\nu} \dot{r}^2 = 0$$

from which we obtain

$$\frac{dr}{dt} = \pm \left(1 - \frac{2M}{r}\right)$$

(89)

Integration leads to:

**Outgoing Photon**

$$t = r + 2M \ln \left|\frac{r}{2M} - 1\right| + \text{const}$$

**Incoming Photon**

$$t = -r - 2M \ln \left|\frac{r}{2M} - 1\right| + \text{const}$$

Figure: Radially infalling particle emitting a radially outgoing photon.
Circular motion of photons

For circular orbits we have \( r = \text{constant} \) and thus from eqn (88) we see that the only possible radius for a circular photon orbit is:

\[
r = 3M \quad \left( \text{or} \quad r = \frac{3GM}{c^2} \right)
\]  

There are no such orbits around typical stars because their radius is much larger than \( 3M \) (in geometrical units). But outside the black hole there can be such an orbit.
Stability of photon orbits

We can rewrite the “energy” equation (87) as

\[
\frac{\dot{r}^2}{L^2} + \frac{e^\nu}{r^2} = \frac{1}{D^2}
\]

where \( D = L/E \) and \( V_{\text{eff}} = e^\nu/r \)

**Figure:** The effective potential for photon orbits.

We can see that \( V_{\text{eff}} \) has a single maximum at \( r = 3M \) where the value of the potential is \( 1/(27M^2) \). Thus the circular orbit \( r = 3M \) is unstable.

There are no stable circular photon orbits in the Schwarzschild geometry.
The slow-rotation limit: Dragging of inertial frames

\[ ds^2 = ds^2_{\text{Schw}} + \frac{4Ma}{r} \sin^2 \theta dt d\phi \]  

(92)

where \( J = Ma \) is the angular momentum.

The contravariant components of the particles 4-momentum will be

\[ p^\phi = g^{\phi \mu} p_\mu = g^{\phi t} p_t + g^{\phi \phi} p_\phi \quad \text{and} \quad p^t = g^{t \mu} p_\mu = g^{tt} p_t + g^{t \phi} p_\phi \]

If we assume a particle with zero angular momentum, i.e. \( p_\phi = 0 \) along the geodesic then the particle’s trajectory is such that

\[ \frac{d \phi}{dt} = \frac{p^\phi}{p^t} = \frac{g^{t \phi}}{g^{tt}} = \frac{2Ma}{r^3} = \omega(r) \]

which is the coordinate angular velocity of a zero-angular-momentum particle.

A particle dropped “straight in” from infinity (\( p_\phi = 0 \)) is dragged just by the influence of gravity so that acquires an angular velocity in the same sense as that of the source of the metric. The effect weakens with the distance and makes the angular momentum of the source measurable in practice.
## Useful Constants in Geometrical Units

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Speed of light</strong></td>
<td>(c = 299,792.458 \text{ km/s} = 1)</td>
</tr>
<tr>
<td><strong>Planck's constant</strong></td>
<td>(\hbar = 1.05 \times 10^{-27} \text{ erg} \cdot \text{s} = 2.612 \times 10^{-66} \text{ cm}^2)</td>
</tr>
<tr>
<td><strong>Gravitation constant</strong></td>
<td>(G = 6.67 \times 10^{-8} \text{ cm}^3/\text{g} \cdot \text{s}^2 = 1)</td>
</tr>
<tr>
<td><strong>Energy</strong></td>
<td>(\text{eV} = 1.602 \times 10^{-12} \text{erg} = 1.16 \times 10^4 \text{K})</td>
</tr>
<tr>
<td><strong>Distance</strong></td>
<td>(1 \text{ pc} = 3.09 \times 10^{13} \text{km} = 3.26 \text{ ly})</td>
</tr>
<tr>
<td><strong>Time</strong></td>
<td>(1 \text{ yr} = 3.156 \times 10^7 \text{ sec})</td>
</tr>
<tr>
<td><strong>Light year</strong></td>
<td>(1 \text{ ly} = 9.46 \times 10^{12} \text{km})</td>
</tr>
<tr>
<td><strong>Astronomical unit (AU)</strong></td>
<td>(1 \text{AU} = 1.5 \times 10^8 \text{ km})</td>
</tr>
<tr>
<td><strong>Earth's mass</strong></td>
<td>(M_\oplus = 5.97 \times 10^{27} \text{ g})</td>
</tr>
<tr>
<td><strong>Earth's radius (equator)</strong></td>
<td>(R_\oplus = 6378 \text{ km})</td>
</tr>
<tr>
<td><strong>Solar Mass</strong></td>
<td>(M_\odot = 1.99 \times 10^{33} \text{ g} = 1.47664 \text{ km})</td>
</tr>
<tr>
<td><strong>Solar Radius</strong></td>
<td>(R_\odot = 6.96 \times 10^5 \text{ km})</td>
</tr>
</tbody>
</table>
For a given value of angular momentum Kepler’s equation (64)

\[
\frac{d^2}{d\phi^2} u + u = \frac{M}{L^2} + 3Mu^2
\]  

(without the GR term \(3Mu^2\)) admits a solution given by

\[
u = \frac{M}{L^2} [1 + e \cos(\phi + \phi_0)]
\]  

(93)

where \(e\) and \(\phi_0\) are integration constants.

- We can set \(\phi_0 = 0\) by rotating the coordinate system by \(\phi_0\).
- The other constant \(e\) is the **eccentricity** of the orbit which is an ellipse if \(e < 1\).

Now since the term \(3Mu^2\) is small we can use perturbation theory to get a solution of equation (64).

\[
\frac{d^2}{d\phi^2} u + u \approx \frac{M}{L^2} + \frac{3M^3}{L^4} \left[1 + e \cos(\phi)\right]^2 \approx \frac{M}{L^2} + \frac{3M^3}{L^4} + \frac{6eM^3}{L^4} \cos(\phi)
\]

Because, \(3M^3/L^4 \ll M/L^2\) and can be omitted and its corrections will be small periodic elongations of the semiaxis of the ellipse.
The term $6eM^3/L^4 \cos(\phi)$ is also small but has an accumulative effect which can be measured. Thus the solution of the relativistic form of Kepler's equation (94) becomes ($k = 3M^2/L^2$)

$$u = \frac{M}{L^2} \left[ 1 + e \cos \phi + \frac{3eM^2}{L^2} \phi \sin \phi \right] \approx \frac{M}{L^2} \left\{ 1 + e \cos[\phi(1 - k)] \right\}$$ \hspace{1cm} (94)

The perihelion of the orbit can be found by maximizing $u$ that is when $\cos[\phi(1 - k)] = 1$ or better if $\phi(1 - k) = 2n\pi$. This means that after each rotation around the Sun the angle of the perihelion will increase by

$$\phi_n = \frac{2n\pi}{1 - k} \quad \Rightarrow \quad \phi_{n+1} - \phi_n = \frac{2\pi}{1 - k} \approx 2\pi(1 + k) = 2\pi + \frac{6\pi M^2}{L^2}$$ \hspace{1cm} (95)
i.e. in the relativistic orbit the perihelion is no longer a fixed point, as it was in the Newtonian elliptic orbit but it moves in the direction of the motion of the planet, advancing by the angle

\[ \delta \phi \approx \frac{6\pi M^2}{L^2} \]  

By using the well know relation from Newtonian Celestial Mechanics \( L^2 = Mr_0(1 - e^2) \) connecting the perihelion distance \( r_0 \) with \( L \) we derive a more convenient form for the perihelion advance

\[ \delta \phi \approx \frac{6\pi M}{r_0(1 - e^2)} \]  

Solar system measurements

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>(43.11 ± 0.45)''</td>
<td>43.03''</td>
<td></td>
</tr>
<tr>
<td>Venus</td>
<td>(8.4 ± 4.8)''</td>
<td>8.6''</td>
<td></td>
</tr>
<tr>
<td>Earth</td>
<td>(5.0 ± 1.2)''</td>
<td>3.8''</td>
<td></td>
</tr>
</tbody>
</table>

In binary pulsars separated by \( 10^6 \text{km} \) the perihelion advance is extremely important and it can be up to \( \sim 2^\circ/\text{year} \) (about 1000 orbital rotations).
## Sources of the precession of perihelion for Mercury

<table>
<thead>
<tr>
<th>Amount (arcsec/century)</th>
<th>Gause</th>
</tr>
</thead>
<tbody>
<tr>
<td>5025.6</td>
<td>Coordinate (precession of the equinoxes)&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td>531.4</td>
<td>Gravitational tugs of the other planets</td>
</tr>
<tr>
<td>0.0254</td>
<td>Oblateness of the Sun</td>
</tr>
<tr>
<td>42 ± 0.04</td>
<td>General relativity</td>
</tr>
<tr>
<td>5600.0</td>
<td>Total</td>
</tr>
<tr>
<td>5599.7</td>
<td>Observed</td>
</tr>
</tbody>
</table>

---

<sup>a</sup>Precession refers to a change in the direction of the axis of a rotating object. There are two types of precession, torque-free and torque-induced.

![Einstein's Theory of Gravity](image)
Deflection of light rays due to presence of a gravitational field is a prediction of Einstein dated even before the GR and has been verified in 1919.

Photons follow null geodesics, this means that $ds = 0$ and the integrals of motion ($E$ and $L$) are divergent but not their ratio $L/E$. Thus the photon’s equation of motion on the equatorial plane of Schwarzschild spacetime will be:

$$\frac{d^2 u}{d\phi^2} + u = 3Mu^2.$$  (98)

with an approximate solution (we omit at the moment the term $3Mu^2$)

$$u = \frac{1}{b} \cos(\phi + \phi_0)$$  (99)

which describes a straight line, where $b$ and $\phi_0$ are integration constants. Actually, with an appropriate rotation $\phi_0 = 0$ and the length $b$ is the distance of the line from the origin.
Since the term $3M u^2$ is very small we can substitute $u$ with the Newtonian solution (99) and we need to solve the non-homogeneous ODE

$$\frac{d^2 u}{d\phi^2} + u = \frac{3M}{b^2} \cos^2 \phi.$$  

(100)

admitting a solution of the form

$$u = \frac{\cos \phi}{b} + \frac{M}{b^2} (1 + \sin^2 \phi)$$

(101)

which for distant observers ($r \to \infty$) i.e. $u \to 0$, and we get a relation between $\phi$, $M$ and the parameter $b$

$$\frac{\cos \phi}{b} + \frac{M}{b^2} (1 + \sin^2 \phi) = 0$$

(102)

and since $r \to \infty$, this means that $\phi \to \pi/2 + \epsilon$ and $\cos \phi \to 0 + \epsilon$ and $\sin \phi \to 1 - \epsilon$ we get:

$$\epsilon \approx -\frac{2M}{b}$$

(103)

Since, $\phi \to \pi/2 + 2M/b$ for $r \to \infty$ on the one side & $\phi \to 3\pi/2 - 2M/b$ on the other side the total deviation will be the sum of the two i.e.

$$\delta \phi = \frac{4M}{b}.$$

(104)
For a light ray tracing the surface of the Sun gives a deflection of \( \sim 1.75'' \).

The deflection of light rays is a quite common phenomenon in Astronomy and has many applications. We typically observe “crosses” or “rings”

**Figure:** *Einstein Cross* (G2237+030) is the most characteristic case of gravitational lens where a galaxy at a distance \( 5 \times 10^8 \) lys focuses the light from a quasar who is behind it in a distance of \( 8 \times 10^9 \) lys. The focusing creates 4 symmetric images of the same quasar. The system has been discovered by John Huchra.

**Figure:** “Einstein rings” are observed when the source, the focusing body and Earth are on the same line of sight. This ring has been discovered by Hubble space telescope.
The Classical Tests: Gravitational Redshift

Figure: Let’s assume 3 static observers on a Schwarzschild spacetime, one very close to the source of the field, the other in a medium distance from the source and the third at infinity.

The clocks of the 3 observers ticking with different rates. The clocks of the two closer to the source are ticking slower than the clock of the observer at infinity who measures the so called ‘coordinate time’ i.e.

\[ d\tau_1 = \left(1 - \frac{2M}{r_1}\right)^{1/2} \, dt \quad \text{and} \quad d\tau_2 = \left(1 - \frac{2M}{r_2}\right)^{1/2} \, dt \]

this means that

\[
\frac{d\tau_2}{d\tau_1} = \frac{\left(1 - \frac{2M}{r_2}\right)^{1/2}}{\left(1 - \frac{2M}{r_1}\right)^{1/2}} \approx 1 + \frac{M}{r_1} - \frac{M}{r_2}.\quad (105)
\]
If the 1st observer sends light signals on a specific wavelength $\lambda_1$ from $c = \lambda/\tau$ we get a relation between the wavelength of the emitted and received signals

$$\frac{\lambda_2}{\lambda_1} \approx 1 + \frac{M}{r_1} - \frac{M}{r_2} \Rightarrow \frac{\lambda_2 - \lambda_1}{\lambda_1} = \frac{\Delta \lambda}{\lambda_1} = \frac{M}{r_1} - \frac{M}{r_2}. \quad (106)$$

A similar relation can be found for the frequency of the emitted signal:

$$\frac{\nu_1}{\nu_2} = \frac{\left(1 - \frac{2M}{r_2}\right)^{1/2}}{\left(1 - \frac{2M}{r_1}\right)^{1/2}}. \quad (107)$$

While the photon redshift $z$ is defined by

$$1 + z = \frac{\nu_1}{\nu_2} \quad (108)$$

**QUESTION**: What will be the redshift for signals emitted from the surface of the Sun, a neutron star and a black hole?
The Classical Tests : Radar Delay

A more recent test (late 60s) where the delay of the radar signals caused by the gravitational field of Sun was measured. This experiment suggested and performed by I.I. Shapiro and his collaborators.

The line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ for the light rays i.e. for $ds = 0$ on the equatorial plane has the form

$$0 = \left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{dr}{dt}\right)^2 - r^2 \left(\frac{d\phi}{dt}\right)^2$$  \hspace{1cm} (109)

For the study of the radial motion we should substitute the term $d\phi/dt$ from the integrals of motion (59) and (60) i.e. we can create the quantity:

$$D = \frac{L}{E} = r^2 \left(1 - \frac{2M}{r}\right)^{-1} \frac{d\phi}{dt}$$  \hspace{1cm} (110)
Then eqn (109) becomes

\[
\left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{dr}{dt}\right)^2 - \frac{D^2}{r^2} \left(1 - \frac{2M}{r}\right)^2 = 0. \tag{111}
\]

At the point of the closest approach to the Sun, \( r_0 \), there should be \( \frac{dr}{dt} = 0 \) and thus we get the value of \( D^2 = \frac{r_0^2}{1-2M/r_0} \). Leading to an equation for the radial motion:

\[
\frac{dr}{dt} = \left(1 - \frac{2M}{r}\right) \left[1 - \left(\frac{r_0}{r}\right)^2 \frac{1 - 2M/r}{1 - 2M/r_0}\right]^{1/2} \tag{112}
\]

leading to

\[
t_1 &= \int_{r_0}^{r_1} \frac{dr}{(1 - \frac{2M}{r}) \sqrt{1 - \left(\frac{r_0}{r}\right)^2 \frac{1 - 2M/r}{1 - 2M/r_0}}} \\
&= \sqrt{r_1^2 - r_0^2} + 2M \ln \left[ \frac{r_1 \sqrt{r_1^2 - r_0^2}}{r_0} \right] + M \sqrt{\frac{r_1 - r_0}{r_1 + r_0}} \\
&\approx r_1 + 2M \ln \left(\frac{2r_1}{r_0}\right) + M
\]
For flat space we have $t_1 = r_1$ i.e. the term $\tilde{t}_1 = 2M \ln(2r_1/r_0) + M$ is the relativistic correction for the first part of the orbit in the same way we get a similar contribution as the signal returns to Earth. Thus the total "extra time" is:

$$\Delta T = 2(\tilde{t}_1 + \tilde{t}_2) = 4M \left[ 1 + \ln \left( \frac{4r_1 r_2}{r_0^2} \right) \right]. \quad (113)$$
Figure: Comparison of the experimental results with the prediction of the theory. The results are from I.I. Shapiro’s experiment (1970) using Venus as reflector.
Figure: As a pulsar passes behind its heavy companion star, its pulses are delayed by the mass of the companion. (Credit B. Saxton/NRAO/AUI)

- If a pulsar is in orbit around a companion WD, its pulses of light will follow the space curve caused by that star. When the companion WD star is in front of the pulsar, the pulses take a little longer to reach us than when the WD is clear of the pulsar. The amount of delay tells you the amount of mass of the star causing the delay. ¹

- Astronomers using the NSF’s Green Bank Telescope (GBT) have discovered the most massive neutron star ($2M_\odot$) yet found, a discovery with strong and wide-ranging impacts across several fields of physics and astrophysics. ¹

¹NRAO: http://www.nrao.edu/index.php/learn/science/weighing-pulsars