1st Set of Problems

1. If $A_\mu = (xy, 3y, yz)$ is a covariant vector, find its components in spherical coordinates (r, θ, ϕ).

2. Show that:
$$T_{(\alpha\beta)} = \frac{1}{2}\left(T_{\alpha\beta} + T_{\beta\alpha}\right)\,, \quad T_{[\alpha\beta]} = \frac{1}{2}\left(T_{\alpha\beta} - T_{\beta\alpha}\right)$$

3. Prove two of the following relations for the covariant differentiation
$$a_{\lambda;\mu} = a_{\lambda,\mu} - \Gamma^{\rho}_{\mu\lambda} a_\rho \quad (1)$$
$$T^{\lambda\mu}{}_{;\nu} = T^{\lambda\mu}{}_{,\nu} + \Gamma^{\lambda}_{\alpha\nu} T^{\alpha\mu} + \Gamma^{\mu}_{\alpha\nu} T^{\lambda\alpha} \quad (2)$$
$$T^{\lambda}{}_{\mu;\nu} = T^{\lambda}{}_{\mu,\nu} + \Gamma^{\lambda}_{\alpha\nu} T^{\alpha}{}_{\mu} - \Gamma^{\alpha}_{\mu\nu} T^{\lambda}{}_{\alpha} \quad (3)$$
$$T_{\lambda\mu;\nu} = T_{\lambda\mu,\nu} - \Gamma^{\alpha}_{\lambda\nu} T_{\mu\alpha} - \Gamma^{\alpha}_{\mu\nu} T_{\lambda\alpha} \quad (4)$$

4. (*) Prove the following relation for the parallel transport of a vector around a close path
$$\delta a^\lambda = -\frac{1}{2} a^\beta R^{\lambda}_{\beta\nu\sigma}\left(dx^\sigma \delta x^\nu - dx^\nu \delta x^\sigma\right)$$

5. Transform the Euclidean metric tensor from Cartesian coordinates to cylindrical ones.

6. Find the Christoffel symbols and the geodesic equations for one of the metrics:
$$ds^2 = dv^2 + [u^2 - v^2]du^2$$
$$ds^2 = dv^2 - v^2 du^2$$

7. For the metric $ds^2 = dr^2 + r^2 d\theta^2$, find the geodesic equations and show the following relations:
$$r^2 \frac{d\theta}{ds} = \text{constant}$$
$$\left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\theta}{ds}\right)^2 = 1$$

8. (*) Prove that :
$$a_{\lambda;\mu;\nu} - a_{\lambda;\nu;\mu} = R^{\kappa}_{\lambda\mu\nu} a_\kappa$$