Solutions for the 1st set of problems

1. The transformation from spherical to Cartesian coordinates is

$$x = r\cos\theta\cos\phi,$$
$$y = r\cos\theta\sin\phi,$$
$$z = r\sin\theta.$$

From the transformation rule of covariant vectors

$$\tilde{A}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} A_\mu$$

we thus obtain

$$\begin{aligned}
\tilde{A}_r &= \frac{\partial x}{\partial r} A_x + \frac{\partial y}{\partial r} A_y + \frac{\partial z}{\partial r} A_z \\
&= r\cos\theta\sin\phi\left(r\cos^2\theta\cos^2\phi + 3\cos\theta\sin\phi + r\sin^2\theta\right), \\
\tilde{A}_\theta &= r\cos\theta\sin\phi\left(-r^2\cos\theta\sin\theta\cos^2\phi - 3r\sin\theta\sin\phi + r^2\sin\theta\cos\theta\right), \\
&= r\cos\theta\sin\phi\left(r^2\cos\theta\sin\theta\sin^2\phi - 3r\sin\theta\sin\phi\right), \\
\tilde{A}_\phi &= r\cos\theta\sin\phi\left(-r^2\cos^2\theta\sin\phi\cos\phi + 3r\cos\theta\cos\phi\right).
\end{aligned}$$

2. We show that $T_{[\alpha\beta]}$ is antisymmetric:

$$T_{[\alpha\beta]} = \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}) = \frac{1}{2}(-T_{\beta\alpha} + T_{\alpha\beta}) = -\frac{1}{2}(T_{\beta\alpha} - T_{\alpha\beta}) = T_{[\beta\alpha]}.$$

By replacing the minus by a plus one also proves the symmetric case.

3. As in the lecture we start by calculating the partial derivative of a_λ with respect to x^μ:

$$\begin{aligned}
a_{\lambda,\mu} &= \frac{\partial}{\partial x^\mu}\left(\frac{\partial \tilde{x}^\rho}{\partial x^\lambda}\tilde{a}_\rho\right) = \frac{\partial^2 \tilde{x}^\rho}{\partial x^\lambda \partial x^\mu}\tilde{a}_\rho + \frac{\partial \tilde{x}^\rho}{\partial x^\lambda}\frac{\partial}{\partial x^\mu}\tilde{a}_\rho \\
&= \frac{\partial^2 \tilde{x}^\rho}{\partial x^\lambda \partial x^\mu}\frac{\partial \tilde{x}^\sigma}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial \tilde{x}^\rho}\tilde{a}_\sigma + \frac{\partial \tilde{x}^\rho}{\partial x^\lambda}\frac{\partial \tilde{x}^\sigma}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\sigma}\tilde{a}_\rho.
\end{aligned}$$

Now we consider the following calculation:

$$\begin{aligned}
0 &= \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \tilde{x}^\beta}{\partial x^\gamma}\frac{\partial x^\delta}{\partial \tilde{x}^\beta}\right) \\
&= \frac{\partial^2 \tilde{x}^\beta}{\partial x^\gamma \partial x^\alpha}\frac{\partial x^\delta}{\partial \tilde{x}^\beta} + \frac{\partial \tilde{x}^\beta}{\partial x^\gamma}\frac{\partial}{\partial x^\alpha}\frac{\partial x^\delta}{\partial \tilde{x}^\beta} \\
&= \frac{\partial^2 \tilde{x}^\beta}{\partial x^\gamma \partial x^\alpha}\frac{\partial x^\delta}{\partial \tilde{x}^\beta} + \frac{\partial \tilde{x}^\beta}{\partial x^\gamma}\frac{\partial \tilde{x}^\sigma}{\partial x^\alpha}\frac{\partial^2 x^\delta}{\partial \tilde{x}^\beta \partial \tilde{x}^\sigma}.
\end{aligned}$$

Hence,

$$\frac{\partial^2 \tilde{x}^\rho}{\partial x^\lambda \partial x^\mu}\frac{\partial x^\alpha}{\partial \tilde{x}^\rho} = -\frac{\partial \tilde{x}^\rho}{\partial x^\lambda}\frac{\partial \tilde{x}^\sigma}{\partial x^\mu}\frac{\partial^2 x^\alpha}{\partial \tilde{x}^\rho \partial \tilde{x}^\sigma}$$

and

$$\begin{aligned}
a_{\lambda,\mu} &= \frac{\partial \tilde{x}^\rho}{\partial x^\lambda}\frac{\partial \tilde{x}^\sigma}{\partial x^\mu}\tilde{a}_{\rho,\sigma} - \frac{\partial \tilde{x}^\rho}{\partial x^\lambda}\frac{\partial \tilde{x}^\sigma}{\partial x^\mu}\frac{\partial^2 x^\alpha}{\partial \tilde{x}^\rho \partial \tilde{x}^\sigma}\frac{\partial \tilde{x}^\beta}{\partial x^\alpha}\tilde{a}_\beta \\
&= \frac{\partial \tilde{x}^\rho}{\partial x^\lambda}\frac{\partial \tilde{x}^\sigma}{\partial x^\mu}\left(\tilde{a}_{\rho,\sigma} - \frac{\partial^2 x^\alpha}{\partial \tilde{x}^\rho \partial \tilde{x}^\sigma}\frac{\partial \tilde{x}^\beta}{\partial x^\alpha}\tilde{a}_\beta\right) \\
&= \frac{\partial \tilde{x}^\rho}{\partial x^\lambda}\frac{\partial \tilde{x}^\sigma}{\partial x^\mu}\left(\tilde{a}_{\rho,\sigma} - \tilde{\Gamma}^\beta_{\rho\sigma}\tilde{a}_\beta\right)
\end{aligned}$$

In analogy to the discussion in the lecture we conclude that

$$a_{\lambda;\mu} = \left(a_{\rho,\sigma} - \Gamma^{\beta}_{\rho\sigma} a_\beta\right)$$

transforms as a tensor of rank two.

Analogously we calculate for an $r+s$-Tensor:

$$\begin{aligned}
T^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s,\mu} &= \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\mu_1}}{\partial\tilde{x}^{\lambda_1}}\cdots\frac{\partial x^{\mu_r}}{\partial\tilde{x}^{\lambda_r}}\frac{\partial\tilde{x}^{\sigma_1}}{\partial x^{\nu_1}}\cdots\frac{\partial\tilde{x}^{\sigma_s}}{\partial x^{\nu_s}}\tilde{T}^{\lambda_1\ldots\lambda_r}{}_{\sigma_1\ldots\sigma_s}\right)\\
&= \frac{\partial^2 x^{\mu_1}}{\partial\tilde{x}^{\lambda_1}\partial\tilde{x}^{\rho}}\frac{\partial\tilde{x}^{\rho}}{\partial x^\mu}\frac{\partial x^{\mu_2}}{\partial\tilde{x}^{\lambda_2}}\cdots\frac{\partial x^{\mu_r}}{\partial\tilde{x}^{\lambda_r}}\frac{\partial\tilde{x}^{\sigma_1}}{\partial x^{\nu_1}}\cdots\frac{\partial\tilde{x}^{\sigma_s}}{\partial x^{\nu_s}}\tilde{T}^{\lambda_1\ldots\lambda_r}{}_{\sigma_1\ldots\sigma_s}\\
&\quad+\ldots+\frac{\partial x^{\mu_1}}{\partial\tilde{x}^{\lambda_1}}\cdots\frac{\partial x^{\mu_{r-1}}}{\partial\tilde{x}^{\lambda_{r-1}}}\frac{\partial^2 x^{\mu_r}}{\partial\tilde{x}^{\lambda_r}\partial\tilde{x}^{\rho}}\frac{\partial\tilde{x}^{\rho}}{\partial x^\mu}\frac{\partial\tilde{x}^{\sigma_1}}{\partial x^{\nu_1}}\cdots\frac{\partial\tilde{x}^{\sigma_s}}{\partial x^{\nu_s}}\tilde{T}^{\lambda_1\ldots\lambda_r}{}_{\sigma_1\ldots\sigma_s}\\
&\quad+\frac{\partial x^{\mu_1}}{\partial\tilde{x}^{\lambda_1}}\cdots\frac{\partial x^{\mu_r}}{\partial\tilde{x}^{\lambda_r}}\frac{\partial^2\tilde{x}^{\sigma_1}}{\partial x^{\nu_1}\partial x^{\mu}}\frac{\partial\tilde{x}^{\sigma_2}}{\partial x^{\nu_2}}\cdots\frac{\partial\tilde{x}^{\sigma_s}}{\partial x^{\nu_s}}\tilde{T}^{\lambda_1\ldots\lambda_r}{}_{\sigma_1\ldots\sigma_s}\\
&\quad+\ldots+\frac{\partial x^{\mu_1}}{\partial\tilde{x}^{\lambda_1}}\cdots\frac{\partial x^{\mu_r}}{\partial\tilde{x}^{\lambda_r}}\frac{\partial\tilde{x}^{\sigma_1}}{\partial x^{\nu_1}}\cdots\frac{\partial\tilde{x}^{\sigma_{s-1}}}{\partial x^{\nu_{s-1}}}\frac{\partial^2\tilde{x}^{\sigma_s}}{\partial x^{\nu_s}\partial x^{\mu}}\tilde{T}^{\lambda_1\ldots\lambda_r}{}_{\sigma_1\ldots\sigma_s}\\
&\quad+\frac{\partial x^{\mu_1}}{\partial\tilde{x}^{\lambda_1}}\cdots\frac{\partial x^{\mu_r}}{\partial\tilde{x}^{\lambda_r}}\frac{\partial\tilde{x}^{\sigma_1}}{\partial x^{\nu_1}}\cdots\frac{\partial\tilde{x}^{\sigma_s}}{\partial x^{\nu_s}}\frac{\partial\tilde{x}^{\nu}}{\partial x^{\mu}}\tilde{T}^{\lambda_1\ldots\lambda_r}{}_{\sigma_1\ldots\sigma_s,\nu}.
\end{aligned}$$

As in the case of the (co-)vector we replace the second derivatives as follows:

$$\begin{aligned}
\frac{\partial^2 x^{\mu_i}}{\partial\tilde{x}^{\lambda_i}\partial\tilde{x}^{\rho}} &= \frac{\partial^2 x^{\alpha_i}}{\partial\tilde{x}^{\lambda_i}\partial\tilde{x}^{\rho}}\frac{\partial x^{\mu_i}}{\partial\tilde{x}^{\rho_i}}\frac{\partial\tilde{x}^{\rho_i}}{\partial x^{\alpha_i}}\\
&= \tilde{\Gamma}^{\rho_i}_{\lambda_i\rho}\frac{\partial x^{\mu_i}}{\partial\tilde{x}^{\rho_i}},\\
\frac{\partial^2\tilde{x}^{\sigma_i}}{\partial x^{\nu_i}\partial x^{\mu}} &= \frac{\partial^2\tilde{x}^{\rho_i}}{\partial x^{\nu_i}\partial x^{\mu}}\frac{\partial x^{\alpha_i}}{\partial\tilde{x}^{\rho_i}}\frac{\partial\tilde{x}^{\sigma_i}}{\partial x^{\alpha_i}} = -\frac{\partial^2 x^{\alpha_i}}{\partial\tilde{x}^{\beta_i}\partial\tilde{x}^{\gamma_i}}\frac{\partial\tilde{x}^{\beta_i}}{\partial\tilde{x}^{\nu_i}}\frac{\partial\tilde{x}^{\gamma_i}}{\partial x^{\mu}}\frac{\partial\tilde{x}^{\sigma_i}}{\partial x^{\alpha_i}}\\
&= -\tilde{\Gamma}^{\sigma_i}_{\beta_i\gamma_i}\frac{\partial\tilde{x}^{\beta_i}}{\partial\tilde{x}^{\nu_i}}\frac{\partial\tilde{x}^{\gamma_i}}{\partial x^{\mu}}.
\end{aligned}$$

The partial derivative hence becomes

$$\begin{aligned}
T^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s,\mu} = \frac{\partial\tilde{x}^{\rho}}{\partial x^\mu}\frac{\partial x^{\mu_1}}{\partial\tilde{x}^{\lambda_1}}\cdots\frac{\partial x^{\mu_r}}{\partial\tilde{x}^{\lambda_r}}\frac{\partial\tilde{x}^{\sigma_1}}{\partial x^{\nu_1}}\cdots\frac{\partial\tilde{x}^{\sigma_s}}{\partial x^{\nu_s}}\Big[&\\
\tilde{T}^{\lambda_1\ldots\lambda_r}{}_{\sigma_1\ldots\sigma_s,\rho}&\\
+\tilde{\Gamma}^{\lambda_1}_{\alpha\rho}\tilde{T}^{\alpha\lambda_2\ldots\lambda_r}{}_{\sigma_1\ldots\sigma_s}+\ldots+\tilde{\Gamma}^{\lambda_r}_{\alpha\rho}\tilde{T}^{\lambda_1\ldots\lambda_{r-1}\alpha}{}_{\sigma_1\ldots\sigma_s}&\\
-\tilde{\Gamma}^{\alpha}_{\sigma_1\rho}\tilde{T}^{\lambda_1\ldots\lambda_r}{}_{\alpha\sigma_2\ldots\sigma_s}-\ldots-\tilde{\Gamma}^{\alpha}_{\sigma_s\rho}\tilde{T}^{\lambda_1\ldots\lambda_r}{}_{\sigma_1\ldots\sigma_{s-1}\alpha}\Big]&
\end{aligned}$$

and we conclude that

$$\begin{aligned}
&T^{\lambda_1\ldots\lambda_r}{}_{\sigma_1\ldots\sigma_s,\rho}\\
&+\Gamma^{\lambda_1}_{\alpha\rho}T^{\alpha\lambda_2\ldots\lambda_r}{}_{\sigma_1\ldots\sigma_s}+\ldots+\Gamma^{\lambda_r}_{\alpha\rho}T^{\lambda_1\ldots\lambda_{r-1}\alpha}{}_{\sigma_1\ldots\sigma_s}\\
&-\Gamma^{\alpha}_{\sigma_1\rho}T^{\lambda_1\ldots\lambda_r}{}_{\alpha\sigma_2\ldots\sigma_s}-\ldots-\Gamma^{\alpha}_{\sigma_s\rho}T^{\lambda_1\ldots\lambda_r}{}_{\sigma_1\ldots\sigma_{s-1}\alpha}
\end{aligned}$$

transforms as a $(r+s+1)$-Tensor.

4. In an infinitesimal neighborhood of a point we consider parallel transports of the vector a^λ from P over a point A to the point Q and from P over another point B to Q.

   We assume that dx^μ connects P with A and B with Q, and that δx^μ connects A with Q and P with B. The transported vector at A is then

$$a^\lambda + a^\lambda{}_{,\nu}dx^\nu + \Gamma^\lambda_{\mu\nu}a^\mu dx^\nu.$$

Transporting this vector from A to Q results in

$$
\begin{aligned}
& a^\lambda + a^\lambda{}_{,\rho}\delta x^\rho + \Gamma^\lambda_{\mu\rho} a^\mu \delta x^\rho \\
& + a^\lambda{}_{,\nu} dx^\nu + a^\lambda{}_{,\nu,\rho} dx^\nu \delta x^\rho + \Gamma^\lambda_{\sigma\rho} a^\sigma{}_{,\nu} dx^\nu \delta x^\rho \\
& + \Gamma^\lambda_{\mu\nu} a^\mu dx^\nu + \Gamma^\lambda_{\mu\nu,\rho} a^\mu dx^\nu \delta x^\rho + \Gamma^\lambda_{\mu\nu} a^\mu{}_{,\rho} dx^\nu \delta x^\rho + \Gamma^\lambda_{\sigma\rho}\Gamma^\sigma_{\mu\nu} a^\mu dx^\nu \delta x^\rho
\end{aligned}
$$

The same procedure can be applied for the path over B. In this case, however we first have to transport with respect to δx^μ and afterwards with respect to dx^μ This means the two vectors must be exchanged and we obtain

$$
\begin{aligned}
& a^\lambda + a^\lambda{}_{,\rho} dx^\rho + \Gamma^\lambda_{\mu\rho} a^\mu dx^\rho \\
& + a^\lambda{}_{,\nu} \delta x^\nu + a^\lambda{}_{,\nu,\rho} \delta x^\nu dx^\rho + \Gamma^\lambda_{\sigma\rho} a^\sigma{}_{,\nu} \delta x^\nu dx^\rho \\
& + \Gamma^\lambda_{\mu\nu} a^\mu \delta x^\nu + \Gamma^\lambda_{\mu\nu,\rho} a^\mu \delta x^\nu dx^\rho + \Gamma^\lambda_{\mu\nu} a^\mu{}_{,\rho} \delta x^\nu dx^\rho + \Gamma^\lambda_{\sigma\rho}\Gamma^\sigma_{\mu\nu} a^\mu \delta x^\nu dx^\rho
\end{aligned}
$$

If $\delta x^\nu dx^\rho = dx^\rho \delta x^\nu$ then the difference of these expressions is (since the Christoffel symbols are symmetric in the lower indices)

$$
\begin{aligned}
& \Gamma^\lambda_{\mu\nu,\rho} a^\mu dx^\nu \delta x^\rho + \Gamma^\lambda_{\sigma\rho}\Gamma^\sigma_{\mu\nu} a^\mu dx^\nu \delta x^\rho - \Gamma^\lambda_{\mu\nu,\rho} a^\mu dx^\rho \delta x^\nu - \Gamma^\lambda_{\sigma\rho}\Gamma^\sigma_{\mu\nu} a^\mu dx^\rho \delta x^\nu \\
&= \frac{a^\mu}{2}\Big(\Gamma^\lambda_{\mu\nu,\rho} dx^\nu \delta x^\rho + \Gamma^\lambda_{\mu\rho,\nu} dx^\rho \delta x^\nu + \Gamma^\lambda_{\sigma\rho}\Gamma^\sigma_{\mu\nu} dx^\nu \delta x^\rho + \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\mu\rho} dx^\rho \delta x^\nu \\
&\qquad - \Gamma^\lambda_{\mu\nu,\rho} dx^\rho \delta x^\nu - \Gamma^\lambda_{\mu\rho,\nu} dx^\nu \delta x^\rho - \Gamma^\lambda_{\sigma\rho}\Gamma^\sigma_{\mu\nu} dx^\rho \delta x^\nu - \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\mu\rho} dx^\nu \delta x^\rho\Big) \\
&= \frac{a^\mu}{2}\left(\Gamma^\lambda_{\mu\nu,\rho} - \Gamma^\lambda_{\mu\rho,\nu} + \Gamma^\lambda_{\sigma\rho}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\mu\rho}\right)\left(dx^\nu \delta x^\rho - dx^\rho \delta x^\nu\right) \\
&= \frac{1}{2} R^\lambda{}_{\mu\rho\nu} a^\mu \left(dx^\nu \delta x^\rho - dx^\rho \delta x^\nu\right).
\end{aligned}
$$

5. The Euclidean metric tensor in Cartesian coordinates is

$$g_{\mu\nu} = \delta_{\mu\nu}.$$

The transformation from cylindrical to Cartesian coordinates is

$$
\begin{aligned}
x &= r\cos\phi, \\
y &= r\sin\phi, \\
z &= \tilde{z}.
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
\tilde{g}_{rr} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial r} g_{xx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r} g_{yy} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial r} g_{zz} \\
&\qquad + 2\frac{\partial x}{\partial r}\frac{\partial y}{\partial r} g_{xy} + 2\frac{\partial x}{\partial r}\frac{\partial z}{\partial r} g_{xz} + 2\frac{\partial y}{\partial r}\frac{\partial z}{\partial r} g_{yz} \\
&= \cos^2\phi + \sin^2\phi = 1, \\
\tilde{g}_{\phi\phi} &= r^2\sin^2\phi + r^2\cos^2\phi = r^2, \\
\tilde{g}_{\tilde{z}\tilde{z}} &= 1, \\
\tilde{g}_{r\phi} &= -r\sin\phi\cos\phi + r\sin\phi\cos\phi = 0, \\
\tilde{g}_{r\tilde{z}} &= 0, \\
\tilde{g}_{\phi\tilde{z}} &= 0.
\end{aligned}
$$

6. The Christoffel symbols of $ds^2 = dv^2 + (u^2 - v^2)du^2$ are

$$\Gamma^u_{uu} = \frac{u}{u^2 - v^2}, \qquad \Gamma^v_{uu} = v, \qquad \Gamma^u_{vu} = \Gamma^u_{uv} = -\frac{v}{u^2 - v^2}.$$

the other Christoffel symbols vanish.

The geodesic equations are thus

$$\ddot{v} + v(\dot{u})^2 = 0, \qquad \ddot{u} + \frac{u(\dot{u})^2}{u^2 - v^2} - 2\frac{v\dot{u}\dot{v}}{u^2 - v^2} = 0.$$

The Christoffel symbols of $ds^2 = dv^2 - v^2du^2$ are

$$\Gamma^v_{uu} = v, \qquad \Gamma^u_{vu} = \Gamma^u_{uv} = \frac{1}{v},$$

the other Christoffel symbols vanish.

The geodesic equations are thus

$$\ddot{v} + v(\dot{u})^2 = 0, \qquad \ddot{u} + 2\frac{\dot{u}\dot{v}}{v} = 0.$$

7. The Christoffel symbols of $ds^2 = dr^2 + r^2d\theta^2$ are

$$\Gamma^r_{\theta\theta} = -r, \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r},$$

the other Christoffel symbols vanish.

The geodesic equations are thus

$$\ddot{r} - r(\dot{\theta})^2 = 0, \qquad \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0.$$

Multiplying the second equation by r^2 we obtain

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0,$$

which is equivalent to

$$\frac{d}{d\lambda}\left(r^2\dot{\theta}\right) = 0.$$

For later convenience we calculate

$$\frac{d}{d\lambda}\left(r^2\dot{\theta}^2\right) = \frac{d}{d\lambda}\left(r^2\dot{\theta}\right)\dot{\theta} + \left(r^2\dot{\theta}\right)\ddot{\theta} = r^2\dot{\theta}\ddot{\theta}.$$

Multiplying the first equation by $2\dot{r}$ we obtain

$$\begin{aligned} 2\dot{r}\ddot{r} - 2r\dot{r}(\dot{\theta})^2 = 0 &\Rightarrow \frac{d}{d\lambda}\left(\dot{r}^2\right) - (\dot{\theta})^2\frac{d}{d\lambda}\left(r^2\right) = 0 \\ &\Rightarrow \frac{d}{d\lambda}\left(\dot{r}^2\right) - \dot{\theta}\frac{d}{d\lambda}\left(r^2\dot{\theta}\right) + r^2\dot{\theta}\ddot{\theta} = 0. \end{aligned}$$

Using that $\frac{d}{d\lambda}\left(r^2\dot{\theta}\right) = 0$ we get

$$\frac{d}{d\lambda}\left(\dot{r}^2\right) + r^2\dot{\theta}\ddot{\theta} = 0 \Rightarrow \frac{d}{d\lambda}\left(\dot{r}^2\right) + \frac{d}{d\lambda}\left(r^2\dot{\theta}^2\right) = 0,$$

and thus

$$\dot{r}^2 + r^2\dot{\theta}^2 = \text{const.}$$

On the other hand the left hand side of this equation is the square of the rate of change of the arclength ($\dot{s}^2$). Thus, if we use the arclength as the curve parameter then the right hand side is one ($(ds/ds)^2 = 1$).

8. We write the covariant derivative in terms of partial derivatives and Christoffel symbols:

$$\begin{aligned}
a_{\lambda;\mu;\nu} &= a_{\lambda;\mu,\nu} - \Gamma^{\sigma}_{\lambda\nu} a_{\sigma;\mu} - \Gamma^{\sigma}_{\mu\nu} a_{\lambda;\sigma} \\
&= a_{\lambda,\mu,\nu} - (\Gamma^{\sigma}_{\lambda\mu} a_{\sigma})_{,\nu} - \Gamma^{\sigma}_{\lambda\nu}(a_{\sigma,\mu} - \Gamma^{\rho}_{\sigma\mu} a_{\rho}) - \Gamma^{\sigma}_{\mu\nu}(a_{\lambda,\sigma} - \Gamma^{\rho}_{\lambda\sigma} a_{\rho}) \\
&= -\Gamma^{\sigma}_{\lambda\mu,\nu} a_{\sigma} + \Gamma^{\sigma}_{\lambda\nu}\Gamma^{\rho}_{\sigma\mu} a_{\rho} \\
&\quad + a_{\lambda,\mu,\nu} - \Gamma^{\sigma}_{\lambda\mu} a_{\sigma,\nu} - \Gamma^{\sigma}_{\lambda\nu} a_{\sigma,\mu} - \Gamma^{\sigma}_{\mu\nu} a_{\lambda,\sigma} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\lambda\sigma} a_{\rho}.
\end{aligned}$$

The terms in the last line are symmetric under exchange of μ and ν. When we consider $a_{\lambda;\mu;\nu} - a_{\lambda;\nu;\mu}$ these terms thus vanish. Hence,

$$\begin{aligned}
a_{\lambda;\mu;\nu} - a_{\lambda;\nu;\mu} &= \Gamma^{\sigma}_{\lambda\nu,\mu} a_{\sigma} - \Gamma^{\sigma}_{\lambda\mu,\nu} a_{\sigma} + \Gamma^{\sigma}_{\lambda\nu}\Gamma^{\rho}_{\sigma\mu} a_{\rho} - \Gamma^{\sigma}_{\lambda\mu}\Gamma^{\rho}_{\sigma\nu} a_{\rho} \\
&= R^{\sigma}_{\lambda\mu\nu} a_{\sigma}.
\end{aligned}$$