2nd Set of Problems

1. Prove the the following so-called quotient theorem:
   Consider the object $T_{\lambda\mu\nu}$. If for every 4-vector A^λ the quantity $A^\lambda T_{\lambda\mu\nu}$ is a tensor then also $T_{\lambda\mu\nu}$ is a tensor.
   Conversely, if $T_{\lambda\mu\nu}$ is a tensor and A^λ is a 4-vector then $A^\lambda T_{\lambda\mu\nu}$ is a tensor.

2. Let $\xi^0 = t$, ξ^1, ξ^2, ξ^3 be the coordinates of an inertial frame, i.e. $ds^2 = dt^2 - (d\xi^1)^2 - (d\xi^2)^2 - (d\xi^3)^2$. Let further x^0, x^1, x^2, x^3 be the coordinates of a rotating frame that are subject to the following transformation

$$\begin{aligned}
\xi^0 &= x^0, \\
\xi^1 &= x^1 \cos(\omega t) - x^2 \sin(\omega t), \\
\xi^2 &= x^1 \sin(\omega t) + x^2 \cos(\omega t), \\
\xi^3 &= x^3,
\end{aligned}$$

   where ω is constant and $\omega^2((x^1)^2 + (x^2)^2) \ll 1$.

   (a) Calculate the Christoffel symbols through

$$\Gamma^\mu_{\alpha\beta} = \frac{\partial^2 \xi^\rho}{\partial x^\alpha \partial x^\beta} \frac{\partial x^\mu}{\partial \xi^\rho}.$$

   (b) Set up the equations of motion in the rotating coordinate system as

$$\frac{d^2 x^k}{dt^2} + \Gamma^k_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0$$

   and interpret the occurring terms.

   (c) Calculate the components $g_{\mu\nu}(x)$, $g^{\mu\nu}(x)$ of the metric tensor in the rotating system.

3. (a) Calculate the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$ for polar coordinates

$$x^1 = r, \qquad x^2 = \theta, \qquad x^3 = \phi$$

   from the transformation equations with Cartesian coordinates ξ^1, ξ^2, ξ^3 through

$$\Gamma^\mu_{\alpha\beta} = \frac{\partial^2 \xi^\rho}{\partial x^\alpha \partial x^\beta} \frac{\partial x^\mu}{\partial \xi^\rho}.$$

   (b) Using these Christoffel symbols set up the equations of motion (free from forces) in the plane defined through $\phi = \pi/2$, i.e. in planar polar coordinates.

   (c) Determine the Euler-Lagrange equations from the Lagrangian function in planar polar coordinates for a particle that moves free from forces in this plane. Compare the results of part (3b) with the equations of motion from the Euler-Lagrange approach.

4. (a) From the 2-dimensional line element ds^2 calculate the components g_{ik} of the metric tensor for the sphere, the cylinder barrel and a rotational paraboloid.

   (b) For the sphere and the cylinder barrel calculate the Γ^i_{kl} from the metric.

5. Consider the 2-dimensional Riemannian spaces below. Let C_1, C_2 be two curves given parametrically (parameter $t \geq 0$).
   Calculate the arc lengths s_1, s_2 of C_1, C_2 respectively and specify an invariant parametrisation of these curves. Furthermore determine the tangent vectors of C_1, C_2.

   (a) $(x^1, x^2) = (\theta, \phi)$ with the metric

$$
\begin{aligned}
& ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2, \quad (a = \text{const.}), \\
& C^1 : t \to (\theta(t), \phi_0), \quad \theta(0) = 0, \\
& C^2 : t \to (\theta_0, \phi(t), \quad \phi(0) = 0.
\end{aligned}
$$

   (b) $(x^1, x^2) = (r, \phi)$ with the metric

$$
\begin{aligned}
& ds^2 = \frac{dr^2}{1 \pm r^2} + r^2 d\phi^2 \quad \text{(for both signs)}, \\
& C^1 : t \to (r(t), \phi_0), \quad r(0) = 0, \\
& C^2 : t \to (r_0, \phi(t)), \quad \phi(0) = 0.
\end{aligned}
$$

   Assume that r_0, θ_0, ϕ_0 are constant and that $\dot{r}, \dot{\theta}, \dot{\phi} > 0$.

6. Prove that if vector a^λ is tangent in a point P of a geodesic curve it will be tangent to this curved if parallel transported along the curve.

7. Prove that the covariant divergence of the Einstein tensor vanishes i.e. that

$$
G^\mu{}_{\nu;\mu} = \left(R^\mu{}_\nu - \frac{1}{2} \delta^\mu{}_\nu R \right)_{;\mu} = 0 \,. \tag{1}
$$

8. Prove that

$$
S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \quad \Rightarrow \quad S = S^\mu{}_\mu = g^{\mu\nu} S_{\mu\nu} = 0 \,. \tag{2}
$$

$$
g^{\lambda\rho} C_{\lambda\mu\nu\rho} = 0 \tag{3}
$$

9. Can you explain why the $g^{\lambda\rho} C_{\lambda\mu\nu\rho} = 0$ condition means 10 less independent components for the Weyl tensor?