

## Solutions for the 2nd set of problems

1. • It is known that  $A^\lambda T_{\lambda\mu\nu}$  is for every 4-vector  $A^\lambda$  a tensor. It follows that this tensor transforms as follows

$$A^\lambda \widetilde{T}_{\lambda\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} A^\lambda T_{\lambda\alpha\beta}.$$

Moreover  $A^\lambda$  can be written in terms of  $\tilde{A}^\lambda$  what implies

$$A^\lambda T_{\lambda\alpha\beta} = \frac{\partial x^\lambda}{\partial \tilde{x}^\gamma} \tilde{A}^\gamma T_{\lambda\alpha\beta}.$$

On the other hand we have in the new coordinate system

$$\begin{aligned} A^\lambda \widetilde{T}_{\lambda\mu\nu} &= \tilde{A}^\lambda \tilde{T}_{\lambda\mu\nu} \\ &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} \frac{\partial x^\lambda}{\partial \tilde{x}^\gamma} \tilde{A}^\gamma T_{\lambda\alpha\beta}, \end{aligned}$$

and since this is valid for every vector  $A^\lambda$  we get

$$\tilde{T}_{\lambda\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} \frac{\partial x^\gamma}{\partial \tilde{x}^\lambda} T_{\lambda\alpha\beta}$$

- Conversely, if  $T_{\lambda\mu\nu}$  is a tensor and  $A^\lambda$  is a vector then

$$\begin{aligned} A^\lambda T_{\lambda\mu\nu} &= \frac{\partial x^\lambda}{\partial \tilde{x}^\gamma} \frac{\partial \tilde{x}^\delta}{\partial x^\lambda} \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \tilde{A}^\gamma \tilde{T}_{\delta\alpha\beta} \\ &= \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \tilde{A}^\gamma \tilde{T}_{\gamma\alpha\beta}. \end{aligned}$$

2. It is clear that in the inertial frame the Christoffel symbols vanish. We can thus use the given formula. The transformation from the inertial into the rotating frame is

$$\begin{aligned} x^0 &= \xi^0, \\ x^1 &= \xi^1 \cos(\omega t) + \xi^2 \sin(\omega t), \\ x^2 &= \xi^2 \cos(\omega t) - \xi^1 \sin(\omega t), \\ x^3 &= \xi^3. \end{aligned}$$

- (a) Obviously the second derivatives of  $\xi^0$  and  $\xi^3$  vanish. Moreover any derivative of  $\xi^1$  and  $\xi^2$  with respect to  $x^3$  vanishes and the derivative of  $x^0$  and  $x^1$  with respect to  $\xi^1$  and  $\xi^2$  also vanish. Finally the second derivatives of  $\xi^\rho$  w.r.t.  $x^1, x^2, x^3$  are zero. Therefore the following identities are satisfied:

$$0 = \Gamma_{\alpha\beta}^0 = \Gamma_{\alpha\beta}^3 = \Gamma_{3\beta}^\alpha = \Gamma_{\beta 3}^\alpha = \Gamma_{11}^\alpha = \Gamma_{12}^\alpha = \Gamma_{22}^\alpha$$

What remains are the Christoffel symbols  $\Gamma_{00}^1, \Gamma_{01}^1, \Gamma_{02}^1, \Gamma_{00}^2, \Gamma_{01}^2, \Gamma_{02}^2$ . For them we get

$$\begin{aligned}\Gamma_{00}^1 &= -\omega^2 x^1, \\ \Gamma_{01}^1 &= 0, \\ \Gamma_{02}^1 &= -\omega, \\ \Gamma_{00}^2 &= -\omega^2 x^2, \\ \Gamma_{01}^2 &= \omega, \\ \Gamma_{02}^2 &= 0.\end{aligned}$$

(b) The equations of motion in the rotating coordinate system are thus

$$\begin{aligned}\ddot{x}^0 &= 0, \\ \ddot{x}^1 - \omega^2 x^1 - \omega \dot{x}^2 &= 0, \\ \ddot{x}^2 - \omega^2 x^2 + \omega \dot{x}^1 &= 0, \\ \ddot{x}^3 &= 0.\end{aligned}$$

The terms  $-\omega^2 x^1$  and  $-\omega^2 x^2$  describe a force in the direction of the position vector. These are the centrifugal forces. The terms  $-\omega \dot{x}^2$  and  $\omega \dot{x}^1$  describe a force orthogonal to the velocity of the particle, the Coriolis force.

(c)

$$g_{\mu\nu}(x) = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}.$$

One obtains

$$\begin{aligned}g_{00}(x) &= 1 - \omega^2((x^1)^2 + (x^2)^2), \\ g_{01}(x) &= \omega x^2, & g_{02}(x) &= -\omega x^1, & g_{03}(x) &= 0, \\ g_{11}(x) &= -1, & g_{12}(x) &= 0, & g_{13}(x) &= 0, \\ & & g_{22}(x) &= -1, & g_{23}(x) &= 0, \\ & & & & g_{33}(x) &= -1.\end{aligned}$$

Analogously we get

$$\begin{aligned}g^{00}(x) &= 1, \\ g^{01}(x) &= \omega x^2, & g^{02}(x) &= -\omega x^1, & g^{03}(x) &= 0, \\ g^{11}(x) &= \omega^2(x^2)^2 - 1, & g^{12}(x) &= -\omega^2 x^1 x^2, & g^{13}(x) &= 0, \\ & & g^{22}(x) &= \omega^2(x^1)^2 - 1, & g^{23}(x) &= 0, \\ & & & & g^{33}(x) &= -1.\end{aligned}$$

3. The transformation from Cartesian to polar coordinates is defined as follows

$$\begin{aligned}x &= r \cos \theta \cos \phi, \\y &= r \cos \theta \sin \phi, \\z &= r \sin \theta,\end{aligned}$$

and the back transformation is

$$\begin{aligned}r &= \sqrt{x^2 + y^2 + z^2}, \\ \theta &= \arctan \frac{z}{\sqrt{x^2 + y^2}}, \\ \phi &= \arctan \frac{y}{x}.\end{aligned}$$

(a) For the non vanishing Christoffel symbols we get

$$\begin{aligned}\Gamma_{12}^2 &= \frac{1}{r}, \\ \Gamma_{13}^3 &= \frac{1}{r}, \\ \Gamma_{22}^1 &= -r, \\ \Gamma_{23}^3 &= -\frac{\sin \theta}{\cos \theta}, \\ \Gamma_{33}^1 &= -r \cos^2 \theta, \\ \Gamma_{33}^2 &= \sin \theta \cos \theta.\end{aligned}$$

(b) The equations of motion are thus

$$\begin{aligned}\ddot{r} - r\dot{\theta}^2 - r \cos^2 \theta \dot{\phi}^2 &= 0, \\ \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} + \sin \theta \cos \theta \dot{\phi}^2 &= 0, \\ \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} - 2\frac{\sin \theta}{\cos \theta}\dot{\theta}\dot{\phi} &= 0.\end{aligned}$$

We see that this system is compatible with  $\phi = \text{const.}$  and becomes

$$\begin{aligned}\ddot{r} - r\dot{\theta}^2 &= 0, \\ \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} &= 0.\end{aligned}$$

(c) The Lagrangian of a freely moving particle is

$$\mathcal{L} = \dot{r}^2 + r^2\dot{\theta}^2 + r^2 \cos^2 \theta \dot{\phi}^2.$$

The Euler-Lagrange equations are thus

$$\begin{aligned} 0 &= \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 2\ddot{r} - 2r\dot{\theta}^2 - 2r \cos^2 \theta \dot{\phi}^2, \\ 0 &= \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 2r^2\ddot{\theta} + 4r\dot{r}\dot{\theta} + 2r^2 \sin \theta \cos \theta \dot{\phi}^2, \\ 0 &= \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = 2r^2 \cos^2 \theta \ddot{\phi} - 4r^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} + 4r \cos^2 \theta \dot{r} \dot{\phi}. \end{aligned}$$

The two systems are obviously equivalent.

4. (a) If we view the surfaces of the problem as surfaces embedded into the 3-dimensional Euclidean space then it is clear that the metrics can be derived from the 3-dimensional flat metric with certain constraints. The 3-dimensional flat metric is

$$ds^2 = dx^2 + dy^2 + dz^2.$$

To derive the metrics on the surfaces we use adapted coordinates. For the sphere these are naturally spherical coordinates:

$$\begin{aligned} x &= r \cos \theta \cos \phi, \\ y &= r \cos \theta \sin \phi, \\ z &= r \sin \theta, \end{aligned}$$

and thus

$$\begin{aligned} dx &= dr \cos \theta \cos \phi - r \sin \theta \cos \phi d\theta - r \cos \theta \sin \phi d\phi, \\ dy &= dr \cos \theta \sin \phi - r \sin \theta \sin \phi d\theta + r \cos \theta \cos \phi d\phi, \\ dz &= dr \sin \theta + r \cos \theta d\theta. \end{aligned}$$

On the sphere it is clear that  $dr$  vanishes. Hence we get

$$\begin{aligned} dx &= -r \sin \theta \cos \phi d\theta - r \cos \theta \sin \phi d\phi, \\ dy &= -r \sin \theta \sin \phi d\theta + r \cos \theta \cos \phi d\phi, \\ dz &= r \cos \theta d\theta. \end{aligned}$$

Inserting these expressions into the Euclidean metric we get

$$ds^2 = dx^2 + dy^2 + dz^2 = r^2 d\theta^2 + r^2 \cos^2 \theta d\phi^2.$$

The metric is thus

$$(g)_{ik} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \cos^2 \theta \end{pmatrix}.$$

For the cylinder barrel the adapted coordinates are cylindrical coordinates:

$$x = r \cos \phi, \quad y = r \sin \phi, \quad z = \tilde{z},$$

what implies

$$dx = dr \cos \phi - r \sin \phi d\phi, \quad dy = dr \sin \phi + r \cos \phi d\phi, \quad dz = d\tilde{z}.$$

Again on the cylinder barrel  $dr$  vanishes and we get

$$ds^2 = dx^2 + dy^2 + dz^2 = r^2 d\phi^2 + d\tilde{z}^2.$$

The metric is thus

$$(g)_{ik} = \begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

For the rotational paraboloid we do not use adapted coordinates. Instead we use the constraint

$$z = a(x^2 + y^2),$$

that implies

$$\begin{aligned} dz &= 2axdx + 2aydy, \\ dz^2 &= 4a^2(x^2 dx^2 + 2xy dx dy + y^2 dy^2). \end{aligned}$$

The line element thus becomes

$$ds^2 = dx^2 + dy^2 + dz^2 = (1 + 4a^2 x^2) dx^2 + (1 + 4a^2 y^2) dy^2 + 8axy dx dy$$

This leads to the metric

$$(g)_{ik} = \begin{pmatrix} 1 + 4a^2 x^2 & 4axy \\ 4axy & 1 + 4a^2 y^2 \end{pmatrix}.$$

Instead of the coordinates  $x, y$  one may also use cylindrical coordinates  $r, \phi$ . In this case we have

$$\begin{aligned} dx &= dr \cos \phi - r \sin \phi d\phi, \quad dy = dr \sin \phi + r \cos \phi d\phi, \quad z = ar^2, \\ dz &= 2ardr, \end{aligned}$$

and hence

$$ds^2 = (1 + 4r^2) dr^2 + r^2 d\phi^2.$$

The metric is thus

$$(g)_{ik} = \begin{pmatrix} 1 + 4a^2 r^2 & 0 \\ 0 & r^2 \end{pmatrix}.$$

(b) For the sphere we get

$$\Gamma_{\theta\phi}^{\phi} = -\tan \theta, \quad \Gamma_{\phi\phi}^{\theta} = -\sin \theta \cos \theta,$$

where all other Christoffel symbols vanish.

For the cylinder barrel all Christoffel symbols are zero.

5. The arclength is the integral of the line element  $ds$  along the curve

$$s = \int ds = \int \frac{ds}{dt} dt = \int \dot{s} dt.$$

An invariant parametrisation of a curve has the property that the curve parameter is the arclength, and the tangent vector of a curve  $C$  has components  $dC/ds$ .

(a) The arclength of  $C_1$  is

$$\begin{aligned} s_1 &= \int dt \frac{ds}{dt} = \int dt \sqrt{a^2 \dot{\theta}^2} = \int dt a \dot{\theta} \\ &= a(\theta(t) - \theta(0)) = a\theta(t). \end{aligned}$$

An invariant parametrisation of  $C_1$  is thus

$$C_1 : s \mapsto \left( \frac{s}{a}, \phi_0 \right),$$

and the tangent vector of  $C_1$  is

$$\left( \frac{1}{a}, 0 \right).$$

The arclength of  $C_2$  is

$$\begin{aligned} s_2 &= \int dt \frac{ds}{dt} = \int dt \sqrt{a^2 \sin^2 \theta_0 \dot{\phi}^2} = \int dt a \sin \theta_0 \dot{\phi} \\ &= a \sin \theta_0 (\phi(t) - \phi(0)) = a \sin \theta_0 \phi(t). \end{aligned}$$

An invariant parametrisation of  $C_2$  is thus

$$C_2 : s \mapsto \left( \theta_0, \frac{s}{a \sin \theta_0} \right),$$

and the tangent vector of  $C_2$  is

$$\left( 0, \frac{1}{a \sin \theta_0} \right).$$

(b) The arclength of  $C_1$  is

$$s_1 = \int dt \frac{ds}{dt} = \int dt \sqrt{\frac{\dot{r}^2}{1 \pm r(t)^2}} = \int dt \frac{\dot{r}}{\sqrt{1 \pm r(t)^2}}.$$

For the + sign we make the substitution  $r = \sinh \psi$ ,  $\dot{r} = \cosh \psi \dot{\psi}$  and get

$$\begin{aligned} s_1 &= \int dt \frac{\cosh \psi \dot{\psi}}{\sqrt{1 + \sinh^2 \psi}} = \int dt \frac{\cosh \psi \dot{\psi}}{\cosh \psi} \\ &= \psi(t) - \psi(0) = \operatorname{arsinh} r(t). \end{aligned}$$

For the  $-$  sign we use the substitution  $r = \sin \psi$ ,  $\dot{r} = \cos \psi \dot{\psi}$  and get

$$\begin{aligned} s_1 &= \int dt \frac{\cos \psi \dot{\psi}}{\sqrt{1 - \sin^2 \psi}} = \int dt \frac{\cos \psi \dot{\psi}}{\cos \psi} \\ &= \psi(t) - \psi(0) = \arcsin r(t). \end{aligned}$$

An invariant parametrisation of  $C_1$  is thus

$$C_1 : s \mapsto \left( \begin{Bmatrix} \sinh s, & + \\ \sin s, & - \end{Bmatrix}, \phi_0 \right),$$

and the tangent vector of  $C_1$  is

$$\left( \begin{Bmatrix} \cosh s, & + \\ \cos s, & - \end{Bmatrix}, 0 \right).$$

The arclength of  $C_2$  is

$$\begin{aligned} s_2 &= \int dt \frac{ds}{dt} = \int dt \sqrt{r_0^2 \dot{\phi}^2} = \int dt r_0 \dot{\phi} \\ &= r_0(\phi(t) - \phi(0)) = r_0 \phi(t). \end{aligned}$$

An invariant parametrisation of  $C_2$  is thus

$$C_2 : s \mapsto \left( r_0, \frac{s}{r_0} \right),$$

and the tangent vector of  $C_2$  is

$$\left( 0, \frac{1}{r_0} \right).$$

6. At the point  $P$  the vector  $a^\lambda$  is tangent to the curve  $x(t)$ . This implies that

$$a^\lambda = a \dot{x}^\lambda \quad \text{at } P.$$

If  $a^\lambda$  is parallelly transported along  $x(t)$  then it satisfies everywhere along the curve the equation

$$\dot{a}^\lambda + \Gamma_{\mu\nu}^\lambda a^\mu \dot{x}^\nu = 0. \quad (1)$$

On the other hand  $x(t)$  is a geodesic curve. Its coordinate representation thus satisfies

$$\ddot{x}^\lambda + \Gamma_{\mu\nu}^\lambda \dot{x}^\mu \dot{x}^\nu = 0. \quad (2)$$

Hence, we get

$$\dot{a}^\lambda = a \ddot{x}^\lambda \quad \text{at } P,$$

and by comparison of (1) and (2) we see that  $a^\lambda = a\dot{x}^\lambda$  is a solution of (1) with initial values

$$a^\lambda(P) = a\dot{x}^\lambda(P), \quad \dot{a}^\lambda(P) = a\ddot{x}^\lambda(P).$$

From standard theorems on ordinary differential equations one then gets that  $a^\lambda = a\dot{x}^\lambda$  is the only solution of that problem, and thus  $a^\lambda$  is everywhere tangent to  $x(t)$ .

7. We want to prove that the Einstein tensor satisfies  $G^\mu{}_{\nu;\mu} = 0$ . To achieve that we first prove the Bianchi-identities:

$$R^\alpha{}_{\beta[\gamma\delta;\sigma]} = 0.$$

We may prove this in any coordinate system, and since the above expression is a tensor, it is then satisfied in every coordinate system.

Moreover, given a point  $P$  it is always possible to find a coordinate system where the Christoffel symbols vanish at  $P$ . At this point we thus need to prove

$$R^\alpha{}_{\beta[\gamma\delta;\sigma]} = 0.$$

Using the formula for the Riemann tensor we get

$$\begin{aligned} R^\alpha{}_{\beta[\gamma\delta;\sigma]} &= -\Gamma_{\beta[\gamma,\delta,\sigma]}^\alpha + \Gamma_{\beta[\delta,\gamma,\sigma]}^\alpha - (\Gamma_{\beta[\gamma}^\rho \Gamma_{|\rho|\delta]}^\alpha)_{,\sigma} - (\Gamma_{\beta[\delta}^\rho \Gamma_{|\rho|\gamma]}^\alpha)_{,\sigma} \\ &= 0. \end{aligned}$$

The first two terms vanish, because the second derivatives may be exchanged. Due to the antisymmetrisation these terms cancel. The third and the fourth term vanish, because the Christoffel symbols at  $P$  vanish. These terms have the form  $\Gamma\partial\Gamma$ .

Now, contracting the Bianchi identity twice with the metric one obtains

$$\begin{aligned} 0 &= g^{\beta\sigma} R^\alpha{}_{\beta[\gamma\alpha;\sigma]} \\ &= \frac{1}{6} (R^{\alpha\beta}{}_{\gamma\alpha;\beta} - R^{\alpha\beta}{}_{\alpha\gamma;\beta} + R^{\alpha\beta}{}_{\alpha\beta;\gamma} - R^{\alpha\beta}{}_{\beta\alpha;\gamma} + R^{\alpha\beta}{}_{\beta\gamma;\alpha} - R^{\alpha\beta}{}_{\gamma\beta;\alpha}) \\ &= \frac{1}{6} (R^\beta{}_{\gamma;\beta} + R^\beta{}_{\gamma;\beta} - R_{;\gamma} - R_{;\gamma} + R^\alpha{}_{\gamma;\alpha} + R^\alpha{}_{\gamma;\alpha}) \\ &= \frac{2}{3} R^\alpha{}_{\gamma;\alpha} - \frac{1}{3} R_{;\gamma} \\ &= \frac{2}{3} \left( R^\alpha{}_{\gamma} - \frac{1}{2} \delta^\alpha{}_{\gamma} R \right)_{;\alpha} = \frac{2}{3} G^\alpha{}_{\gamma;\alpha}. \end{aligned}$$

8. For  $S_{\mu\nu}$  we get

$$S_{\mu\nu} g^{\mu\nu} = R_{\mu\nu} g^{\mu\nu} - \frac{1}{4} R g_{\mu\nu} g^{\mu\nu} = R \left( 1 - \frac{1}{4} g_{\mu\nu} g^{\mu\nu} \right) = 0.$$

The Weyl tensor in four dimensions is defined as

$$C_{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho} + g_{\lambda[\nu}R_{\rho]\mu} - g_{\mu[\nu}R_{\rho]\lambda} - \frac{1}{3}Rg_{\lambda[\nu}g_{\rho]\mu}.$$

Thus, contracting the Weyl tensor with the inverse metric we get

$$\begin{aligned} g^{\lambda\rho}C_{\lambda\mu\nu\rho} &= R_{\mu\nu} + \frac{1}{2}\delta_{\nu}^{\rho}R_{\rho\mu} - 2R_{\nu\mu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{2}\delta_{\mu}^{\lambda}R_{\nu\lambda} \\ &\quad - \frac{1}{6}R\delta_{\nu}^{\rho}g_{\rho\mu} + \frac{2}{3}Rg_{\nu\mu} \\ &= R_{\mu\nu} \left( 1 + \frac{1}{2} - 2 + \frac{1}{2} \right) + g_{\mu\nu}R \left( -\frac{1}{2} - \frac{1}{6} + \frac{2}{3} \right) \\ &= 0. \end{aligned}$$

9. The tensor  $g^{\lambda\rho}C_{\lambda\mu\nu\rho}$  has two free indices  $(\mu, \nu)$ . In four dimension the system  $g^{\lambda\rho}C_{\lambda\mu\nu\rho} = 0$  is thus composed of 16 linear equations for the components of  $C_{\lambda\mu\nu\rho}$ .

Yet, this tensor is symmetric. That means if  $g^{\lambda\rho}C_{\lambda\mu\nu\rho} = 0$  then automatically  $g^{\lambda\rho}C_{\lambda\nu\mu\rho} = 0$ , i.e. at least six equations are linearly dependent.

Apart from this symmetry the Weyl tensor also satisfies  $C_{\lambda[\mu\nu\rho]} = 0$ , but this property does not induce symmetries on the contracted tensor  $g^{\lambda\rho}C_{\lambda\nu\mu\rho}$ . Hence,  $g^{\lambda\rho}C_{\lambda\mu\nu\rho} = 0$  indeed provides ten linearly independent equations.