In many cases we know the values of a function $f(x)$ at a set of points $x_1, x_2, \ldots, x_N$, but we don’t have the analytic expression of the function that lets us calculate its value at an arbitrary point. We will try to estimate $f(x)$ for arbitrary $x$ by “drawing” a curve through the $x_i$ and sometimes beyond them.

The procedure of estimating the value of $f(x)$ for $x \in [x_1, x_N]$ is called interpolation while if the value is for points $x \notin [x_1, x_N]$ extrapolation.

The form of the function that approximates the set of points should be a convenient one and should be applicable to a general class of problems.
Polynomial functions are the most common ones while rational and trigonometric functions are used quite frequently.

We will study the following methods for polynomial approximations:

- Lagrange’s Polynomial
- Hermite Polynomial
- Taylor Polynomial
- Cubic Splines
Let's assume the following set of data:

<table>
<thead>
<tr>
<th></th>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>3.2</td>
<td>2.7</td>
<td>1.0</td>
<td>4.8</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>22.0</td>
<td>17.8</td>
<td>14.2</td>
<td>38.3</td>
</tr>
<tr>
<td>$f_0$</td>
<td>$f_1$</td>
<td>$f_2$</td>
<td>$f_3$</td>
<td></td>
</tr>
</tbody>
</table>

Then the interpolating polynomial will be of 4th order i.e. $ax^3 + bx^2 + cx + d = P(x)$. This leads to 4 equations for the 4 unknown coefficients and by solving this system we get $a = -0.5275$, $b = 6.4952$, $c = -16.117$ and $d = 24.3499$ and the polynomial is:

$$P(x) = -0.5275x^3 + 6.4952x^2 - 16.117x + 24.3499$$

It is obvious that this procedure is quite laboureus and Lagrange developed a direct way to find the polynomial

$$P_n(x) = f_0L_0(x) + f_1L_1(x) + \ldots + f_nL_n(x) = \sum_{i=0}^{n} f_iL_i(x) \quad (1)$$

where $L_i(x)$ are the Lagrange coefficient polynomials.
\[ L_j(x) = \frac{(x - x_0)(x - x_1) \ldots (x - x_{j-1})(x - x_{j+1}) \ldots (x - x_n)}{(x_j - x_0)(x_j - x_1) \ldots (x_j - x_{j-1})(x_j - x_{j+1}) \ldots (x_j - x_n)} \]  

and obviously:

\[ L_j(x_k) = \delta_{jk} = \begin{cases} 
0 & \text{if } j \neq k \\
1 & \text{if } j = k 
\end{cases} \]

where \( \delta_{jk} \) is Kronecker’s symbol.

**ERRORS**

The error when the Lagrange polynomial is used to approximate a continuous function \( f(x) \) is:

\[ E(x) = (x - x_0)(x - x_1) \ldots (x - x_n) \frac{f^{(n+1)}(\xi)}{(n + 1)!} \quad \text{where} \quad \xi \in [x_0, x_N] \]  

**NOTE**

- Lagrange polynomial applies for **evenly** and **unevenly** spaced points.
- If the points are evenly spaced then it reduces to a much simpler form.
EXAMPLE: Find the Lagrange polynomial that approximates the function \( y = \cos(\pi x) \).

We create the table

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_i )</td>
<td>1</td>
<td>0.0</td>
<td>-1</td>
</tr>
</tbody>
</table>

The Lagrange coefficient polynomials are:

\[
\begin{align*}
L_1(x) &= \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} = \frac{(x - 0.5)(x - 1)}{(0 - 0.5)(0 - 1)} = 2x^2 - 3x + 1, \\
L_2(x) &= \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} = \frac{(x - 0)(x - 1)}{(0.5 - 0)(0.5 - 1)} = -4x(x - 1), \\
L_3(x) &= \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} = \frac{(x - 0)(x - 0.5)}{(1 - 0)(1 - 0.5)} = 2x^2 - x
\end{align*}
\]

thus

\[
P(x) = 1 \cdot (2x^2 - 3x + 1) - 0 \cdot 4x(x - 1) + (-1) \cdot (2x^2 - x) = -2x + 1
\]
The error will be:

\[ E(x) = x \cdot (x - 0.5) \cdot (x - 1) \frac{\pi^3 \sin(\pi \xi)}{3!} \]

e.g. for \( x = 0.25 \) is \( E(0.25) \leq 0.24 \).
**Newton Polynomial**

**Forward Newton-Gregory**

\[ P_n(x_s) = f_0 + s \cdot \Delta f_0 + \frac{s(s - 1)}{2!} \cdot \Delta^2 f_0 + \frac{s(s - 1)(s - 2)}{3!} \cdot \Delta^3 f_0 + \ldots \]

\[ = f_0 + \left( \begin{pmatrix} s \\ 1 \end{pmatrix} \right) \cdot \Delta f_0 + \left( \begin{pmatrix} s \\ 2 \end{pmatrix} \right) \cdot \Delta^2 f_0 + \left( \begin{pmatrix} s \\ 3 \end{pmatrix} \right) \cdot \Delta^3 f_0 + \ldots \]

\[ = \sum_{i=0}^{n} \left( \begin{pmatrix} s \\ i \end{pmatrix} \right) \cdot \Delta^i f_0 \quad (4) \]

where \( x_s = x_0 + s \cdot h \) and

\[ \Delta f_i = f_{i+1} - f_i \quad (5) \]

\[ \Delta^2 f_i = f_{i+2} - 2f_{i+1} + f_i \quad (6) \]

\[ \Delta^3 f_i = f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i \quad (7) \]

\[ \Delta^n f_i = f_{i+n} - nf_{i+n-1} + \frac{n(n-1)}{2!}f_{i+n-2} - \frac{n(n-1)(n-2)}{3!}f_{i+n-3} + \ldots \quad (8) \]
Newton Polynomial

**ERROR** The error is the same with the Lagrange polynomial:

\[ E(x) = (x - x_0)(x - x_1) \ldots (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!} \] where \( \xi \in [x_0, x_N] \) \hspace{1cm} (9)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( \Delta f(x) )</th>
<th>( \Delta^2 f(x) )</th>
<th>( \Delta^3 f(x) )</th>
<th>( \Delta^4 f(x) )</th>
<th>( \Delta^5 f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000</td>
<td>0.203</td>
<td>0.017</td>
<td>0.024</td>
<td>0.020</td>
<td>0.032</td>
</tr>
<tr>
<td>0.2</td>
<td>0.203</td>
<td>0.220</td>
<td>0.041</td>
<td>0.044</td>
<td>0.052</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.423</td>
<td>0.261</td>
<td>0.085</td>
<td>0.096</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.684</td>
<td>0.246</td>
<td>0.181</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>1.030</td>
<td>0.527</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>1.557</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table:** Example of a difference matrix

**Backward Newton-Gregory**

\[ P_n(x) = f_0 + \binom{s}{1} \Delta f_{-1} + \binom{s + 1}{2} \Delta^2 f_{-2} + \ldots + \binom{s + n - 1}{n} \Delta^n f_{-n} \] \hspace{1cm} (10)
\[ \begin{array}{|c|c|c|c|c|c|c|}
\hline
x & F(x) & \Delta f(x) & \Delta^2 f(x) & \Delta^3 f(x) & \Delta^4 f(x) & \Delta^5 f(x) \\
\hline
0.2 & 1.06894 & 0.11242 & 0.01183 & 0.00138 & 0.0015 & 0.00001 \\
0.5 & 1.18136 & 0.12425 & 0.01306 & 0.00138 & 0.0014 & \\
0.8 & 1.30561 & 0.13731 & 0.01444 & 0.0152 & & \\
1.1 & 1.44292 & 0.15175 & 0.01596 & & & \\
1.4 & 1.59467 & 0.16771 & & & & \\
1.7 & 1.76238 & & & & & \\
\hline
\end{array} \]

**Newton-Gregory forward** with \( x_0 = 0.5 \):

\[
 P_3(x) = 1.18136 + 0.12425 s + 0.01306 \left( \frac{s}{2} \right) + 0.00138 \left( \frac{s}{3} \right) \\
= 1.18136 + 0.12425 s + 0.01306 s (s - 1)/2 + 0.00138 s (s - 1) (s - 2)/6 \\
= 0.9996 + 0.3354 x + 0.052 x^2 + 0.085 x^3
\]

**Newton-Gregory backwards** with \( x_0 = 1.1 \):

\[
 P_3(x) = 1.44292 + 0.13731 s + 0.01306 \left( \frac{s + 1}{2} \right) + 0.00123 \left( \frac{s + 2}{3} \right) \\
= 1.44292 + 0.13731 s + 0.01306 s (s + 1)/2 + 0.00123 s (s + 1) (s + 2)/6 \\
= 0.99996 + 0.33374 x + 0.05433 x^2 + 0.007593 x^3
\]
This applies when we have information not only for the values of $f(x)$ but also on its derivative $f'(x)$

$$P_{2n-1}(x) = \sum_{i=1}^{n} A_i(x)f_i + \sum_{i=1}^{n} B_i(x)f'_i$$  \hspace{1cm} (11)

where

$$A_i(x) = [1 - 2(x - x_i)L'_i(x_i)] \cdot [L_i(x)]^2$$ \hspace{1cm} (12)

$$B_i(x) = (x - x_i) \cdot [L_i(x)]^2$$ \hspace{1cm} (13)

and $L_i(x)$ are the Lagrange coefficients.

**ERROR**: the accuracy is similar to that of Lagrange polynomial of order $2n!$

$$y(x) - p(x) = \frac{y^{(2n+1)}(\xi)}{(2n+1)!} [(x - x_1)(x - x_2) \ldots (x - x_n)]^2$$  \hspace{1cm} (14)
Hermite Polynomial: Example

Fit a Hermite polynomial to the data of the table:

<table>
<thead>
<tr>
<th>k</th>
<th>x_k</th>
<th>y_k</th>
<th>y'_k</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

The Lagrange coefficients are:

\[ L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 4}{0 - 4} = -\frac{x - 4}{4} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x}{4} \]

\[ L'_0(x) = \frac{1}{x_0 - x_1} = -\frac{1}{4} \quad L'_1(x) = \frac{1}{x_1 - x_0} = \frac{1}{4} \]

Thus

\[ A_0(x) = \left[ 1 - 2 \cdot L'_0(x - x_0) \right] \cdot L_0^2 = \left[ 1 - 2 \cdot \left( -\frac{1}{4} \right) (x - 0) \right] \cdot \left( \frac{x - 4}{4} \right)^2 \]

\[ A_1(x) = \left[ 1 - 2 \cdot L'_1(x - x_1) \right] \cdot L_1^2 = \left[ 1 - 2 \cdot \frac{1}{4} (x - 4) \right] \cdot \left( \frac{x}{4} \right)^2 = \left( 3 - \frac{x}{2} \right) \cdot \left( \frac{x}{4} \right)^2 \]

\[ B_0(x) = (x - 0) \cdot \left( \frac{x - 4}{4} \right)^2 = x \left( \frac{x - 4}{4} \right)^2 \quad B_1(x) = (x - 4) \cdot \left( \frac{x}{4} \right)^2 \]

And the Hermite polynomial is:

\[ P(x) = (6 - x) \frac{x^2}{16}. \]
Taylor Polynomial

It is an alternative way of approximating functions with polynomials. In the previous two cases we found the polynomial $P(x)$ that gets the same value with a function $f(x)$ at $N$ points or the polynomial that agrees with a function and its derivative at $N$ points. Taylor polynomial has the same value $x_0$ with the function but agrees also up to the $N$th derivative with the given function. That is:

$$P^{(i)}(x_0) = f^{(i)}(x_0) \quad \text{and} \quad i = 0, 1, ..., n$$

and the Taylor polynomial has the well known form from Calculus

$$P(x) = \sum_{i=0}^{N} \frac{f^{(i)}(x)}{i!} (x - x_0)^i \quad (15)$$

**ERROR**: was also estimated in calculus

$$E_N(x) = (x - x_0)^{N+1} \frac{f^{(N+1)}(\xi)}{(N + 1)!} \quad (16)$$
Taylor Polynomial: Example

We will show that for the calculation of $e = 2.718281828459...$ with 13-digit approximation we need 15 terms of the Taylor expansion. All the derivatives at $x = 1$ are:

$$y_0 = y_0^{(1)} = y_0^{(2)} = ... = y_0^{(n)} = 1$$

thus

$$p(x) = \sum_{i=1}^{n} \frac{1}{i!} x^n = 1 + x + \frac{x^2}{2} + ... + \frac{1}{n!} x^n$$

and the error will be

$$|E_n| = x^{n+1} \frac{e^\xi}{(n+1)!} = \frac{e^\xi}{16!} < \frac{3}{16!} = 1.433 \times 10^{-13}$$
Interpolation with Cubic Splines

In some cases the typical polynomial approximation cannot smoothly fit certain sets of data. Consider the function

\[
f(x) = \begin{cases} 
0 & -1 \leq x \leq -0.2 \\
1 - 5|x| & -0.2 < x < 0.2 \\
0 & 0.2 \leq x \leq 1.0 
\end{cases}
\]

We can easily verify that we cannot fit the above data with any polynomial degree!

The answer to the problem is given by the spline fitting. That is we pass a set of cubic polynomials (cubic splines) through the points, using a new cubic for each interval. But we require that the slope and the curvature be the same for the pair of cubics that join at each point.

\[
P(x) = 1 - 26x^2 + 25x^4
\]
Interpolation with Cubic Splines

(a) Original function

(b) Fitted with quadratic

(c) Fitted with $P_3(x)$

(d) Fitted with $P_6(x)$

(e) Fitted with $P_8(x)$
Interpolation with Cubic Splines

Let the cubic for the \(i\)th interval, which lies between the points \((x_i, y_i)\) and \((x_{i+1}, y_{i+1})\) has the form:

\[
y(x) = a_i \cdot (x - x_i)^3 + b_i \cdot (x - x_i)^2 + c_i \cdot (x - x_i) + d_i
\] (17)

Since it fits at the two endpoints of the interval:

\[
y_i = y(x_i) = a_i \cdot (x_i - x_i)^3 + b_i \cdot (x_i - x_i)^2 + c_i \cdot (x_i - x_i) + d_i = d_i
\]

\[
y_{i+1} = y(x_{i+1}) = a_i \cdot (x_{i+1} - x_i)^3 + b_i \cdot (x_{i+1} - x_i)^2 + c_i \cdot (x_{i+1} - x_i) + d_i = a_i h_i^3 + b_i h_i^2 + c_i h_i + d_i
\]

where \(h_i = x_{i+1} - x_i\).

We need the 1st and 2nd derivatives of the polynomial to relate the slopes and curvatures of the joining polynomials, by differentiation we get

\[
y'(x) = 3a_i \cdot (x - x_i)^2 + 2b_i \cdot (x - x_i) + c_i
\]

\[
y''(x) = 6a_i \cdot (x - x_i) + 2b_i
\]
The procedure is simplified if we write the equations in terms of the 2nd derivatives of the interpolating cubics. Let’s name \( S_i = y''(x_i) \) the 2nd derivative at the point \((x_i, y_i)\) then we can easily get:

\[
\begin{align*}
    b_i &= \frac{S_i}{2}, \quad a_i = \frac{S_{i+1} - S_i}{6 h_i}
\end{align*}
\]  

which means

\[
y_{i+1} = \frac{S_{i+1} - S_i}{6 h_i} h_i^3 + \frac{S_i}{2} h_i^2 + c_i h_i + y_i
\]

and finally

\[
c_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2 h_i S_i + h_i S_{i+1}}{6}
\]

Now we invoke the condition that the slopes of the two cubics joining at \((x_i, y_i)\) are the same:

\[
\begin{align*}
    y'_i &= 3 a_i \cdot (x_i - x_i)^2 + 2 b_i \cdot (x_i - x_i) + c_i = c_i \quad \text{for the } i \text{ interval} \\
    y'_{i-1} &= 3 a_{i-1} \cdot (x_i - x_{i-1})^2 + 2 b_{i-1} \cdot (x_i - x_{i-1}) + c_{i-1} \\
    &= 3 a_{i-1} h_{i-1}^2 + 2 b_{i-1} h_{i-1} + c_{i-1} \quad \text{for the } i - 1 \text{ interval}
\end{align*}
\]
By equating these and substituting \( a, b, c \) and \( d \) we get:

\[
y'_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i S_i + h_i S_{i+1}}{6}
\]

\[
= 3 \left( \frac{S_i - S_{i-1}}{6h_{i-1}} \right) h_{i-1}^2 + 2 \frac{S_{i-1}}{2} h_{i-1} + \frac{y_i - y_{i-1}}{h_{i-1}} - \frac{2h_{i-1} S_{i-1} + h_{i-1} S_i}{6}
\]

and by simplifying we get:

\[
h_{i-1} S_{i-1} + 2 (h_{i-1} + h_i) S_i + h_i S_{i+1} = 6 \left( \frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}} \right)
\] (20)

If we have \( n + 1 \) points the above relation can be applied to the \( n - 1 \) internal points.

Thus we create a system of \( n - 1 \) equations for the \( n + 1 \) unknown \( S_i \). This system can be solved if we specify the values of \( S_0 \) and \( S_n \).
The system of \( n - 1 \) equations with \( n + 1 \) unknown will be written as:

\[
\begin{pmatrix}
  h_0 & 2(h_0 + h_1) & \cdots & \cdots & S_0 \\
  h_1 & 2(h_1 + h_2) & \cdots & \cdots & S_1 \\
  h_2 & 2(h_2 + h_3) & \cdots & \cdots & S_2 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  h_{n-2} & 2(h_{n-2} + h_{n-1}) & \cdots & \cdots & S_{n-2} \\
  h_{n-1} & & & & S_{n-1} \\
\end{pmatrix}
\begin{pmatrix}
  S_0 \\
  S_1 \\
  S_2 \\
  \vdots \\
  S_{n-2} \\
  S_{n-1} \\
\end{pmatrix}
\equiv \vec{Y}
\]

From the solution of this linear systems we get the coefficients \( a_i, b_i, c_i \) and \( d_i \) via the relations:

\[
a_i = \frac{S_{i+1} - S_i}{6h_i}, \quad b_i = \frac{S_i}{2} \tag{21}
\]

\[
c_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_iS_i + h_iS_{i+1}}{6} \tag{22}
\]

\[
d_i = y_i \tag{23}
\]
**Choice I** Take, $S_0 = 0$ and $S_n = 0$ this will lead to the solution of the following $(n - 1) \times (n - 1)$ linear system:

$$
\begin{pmatrix}
2(h_0 + h_1) & h_1 \\
h_1 & 2(h_1 + h_2) & h_2 \\
h_2 & 2(h_2 + h_3) & h_3 \\
& & \vdots \\
h_{n-2} & 2(h_{n-2} + h_{n-1}) & \\
\end{pmatrix}
\begin{pmatrix}
S_1 \\
S_2 \\
S_3 \\
\vdots \\
S_{n-1}
\end{pmatrix}
= \vec{Y}
$$

**Choice II** Take, $S_0 = S_1$ and $S_n = S_{n-1}$ this will lead to the solution of the following $(n - 1) \times (n - 1)$ linear system:

$$
\begin{pmatrix}
3h_0 + 2h_1 & h_1 \\
h_1 & 2(h_1 + h_2) & h_2 \\
h_2 & 2(h_2 + h_3) & h_3 \\
& & \vdots \\
h_{n-2} & 2(h_{n-2} + h_{n-1}) & \\
\end{pmatrix}
\begin{pmatrix}
S_1 \\
S_2 \\
S_3 \\
\vdots \\
S_{n-1}
\end{pmatrix}
= \vec{Y}
$$
• **Choice III** Use linear extrapolation

\[
\frac{S_1 - S_0}{h_0} = \frac{S_2 - S_1}{h_1} \Rightarrow S_0 = \frac{(h_0 + h_1)S_1 - h_0S_2}{h_1}
\]

\[
\frac{S_n - S_{n-1}}{h_{n-1}} = \frac{S_{n-1} - S_{n-2}}{h_{n-2}} \Rightarrow s_n = \frac{(h_{n-2} + h_{n-1})S_{n-1} - h_{n-1}S_{n-2}}{h_{n-2}}
\]

this will lead to the solution of the following \((n - 1) \times (n - 1)\) linear system:

\[
\begin{pmatrix}
\frac{(h_0 + h_1)(h_0 + 2h_1)}{h_1} & \frac{h_1^2 - h_0^2}{h_1} & h_2 & \frac{h_2^2 - h_1^2}{h_1} \\
\frac{h_1}{h_1} & \frac{h_0}{h_0} & \frac{2(h_1 + h_2)}{h_2} & \frac{2(h_2 + h_3)}{h_2} \\
h_1 & h_2 & \frac{h_2}{h_2} & \frac{h_3}{h_3} \\
\cdots & \cdots & \cdots & \cdots \\
h_{n-2} & \frac{h_{n-2}^2 - h_{n-1}^2}{h_{n-2}} & \frac{h_{n-1}^2 - h_{n-2}^2}{h_{n-2}} & \frac{(h_{n-1} + h_{n-2})(h_{n-1} + 2h_{n-2})}{h_{n-2}}
\end{pmatrix}
\cdot
\begin{pmatrix}
S_1 \\
S_2 \\
S_3 \\
S_{n-1}
\end{pmatrix}
= \vec{Y}
\]
• **Choice IV** Force the slopes at the end points to assume certain values. If \( f'(x_0) = A \) and \( f'(x_n) = B \) then

\[
\begin{align*}
2h_0S_0 + h_1S_1 &= 6 \left( \frac{y_1 - y_0}{h_0} - A \right) \\
h_{n-1}S_{n-1} + 2h_nS_n &= 6 \left( B - \frac{y_n - y_{n-1}}{h_{n-1}} \right)
\end{align*}
\]

\[
\begin{pmatrix}
2h_0 & h_1 \\
h_0 & 2(h_0 + h_1) & h_1 \\
h_1 & 2(h_1 + h_2) & h_2 \\
\vdots & \vdots & \vdots \\
h_{n-2} & 2h_{n-1}
\end{pmatrix}
\begin{pmatrix}
S_1 \\
S_2 \\
S_3 \\
\vdots \\
S_{n-1}
\end{pmatrix}
= \vec{Y}
\]
Interpolation with Cubic Splines: Example

Fit a cubic spline in the data \( y = x^3 - 8 \):

\[
\begin{array}{c|cccc}
 x & 0 & 1 & 2 & 3 & 4 \\
 y & -8 & -7 & 0 & 19 & 56 \\
\end{array}
\]

Depending on the condition at the end we get the following solutions:

- **Condition I**: \( S_0 = 0, S_4 = 0 \)

\[
\begin{pmatrix}
4 & 1 & 0 \\
1 & 4 & 1 \\
0 & 1 & 4
\end{pmatrix}
\begin{pmatrix}
S_1 \\
S_2 \\
S_3
\end{pmatrix}
= 
\begin{pmatrix}
36 \\
72 \\
108
\end{pmatrix}
\Rightarrow
S_1 = 6.4285, \quad S_2 = 10.2857, \quad S_3 = 24.4285
\]

- **Condition II**: \( S_0 = S_1, S_4 = S_3 \)

\[
\begin{pmatrix}
5 & 1 & 0 \\
1 & 4 & 1 \\
0 & 1 & 5
\end{pmatrix}
\begin{pmatrix}
S_1 \\
S_2 \\
S_3
\end{pmatrix}
= 
\begin{pmatrix}
36 \\
72 \\
108
\end{pmatrix}
\Rightarrow
S_1 = S_0 = 4.8, \quad S_2 = 12, \quad S_3 = 19.2 = S_4
\]
• Condition III:

\[
\begin{pmatrix}
6 & 0 & 0 \\
1 & 4 & 1 \\
0 & 0 & 6
\end{pmatrix}
\begin{pmatrix}
S_1 \\
S_2 \\
S_3
\end{pmatrix}
= \begin{pmatrix}
36 \\
72 \\
108
\end{pmatrix}
\Rightarrow
S_0 = 0 \quad S_1 = 6 \\
S_2 = 12 \quad S_3 = 18 \\
S_4 = 24
\]

• Condition IV:

\[
\begin{pmatrix}
2 & 1 & 0 & 0 & 0 \\
1 & 4 & 1 & 0 & 0 \\
0 & 1 & 4 & 1 & 0 \\
0 & 0 & 1 & 4 & 1 \\
0 & 0 & 0 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
S_0 \\
S_1 \\
S_2 \\
S_3 \\
S_4
\end{pmatrix}
= \begin{pmatrix}
6 \\
36 \\
72 \\
108 \\
66
\end{pmatrix}
\Rightarrow
S_0 = 0 \quad S_1 = 6 \\
S_2 = 12 \quad S_3 = 18 \\
S_4 = 24
The following data are from astronomical observations and represent variations of the apparent magnitude of a type of variable stars called Cepheids.

<table>
<thead>
<tr>
<th>Time</th>
<th>0.0</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnitude</td>
<td>0.302</td>
<td>0.185</td>
<td>0.106</td>
<td>0.093</td>
<td>0.24</td>
<td>0.579</td>
<td>0.561</td>
<td>0.468</td>
<td>0.302</td>
</tr>
</tbody>
</table>

Use splines to create a new table for the apparent magnitude for intervals of time of 0.5.

From the following table find the acceleration of gravity at Tübingen (48° 31’) and the distance between two points with angular separation of 1’ of a degree.

<table>
<thead>
<tr>
<th>Latitude</th>
<th>Length of 1’ of arc on the parallel</th>
<th>local acceleration of gravity $g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>1855.4 m</td>
<td>9.7805 m/sec$^2$</td>
</tr>
<tr>
<td>15°</td>
<td>1792.0 m</td>
<td>9.7839 m/sec$^2$</td>
</tr>
<tr>
<td>30°</td>
<td>1608.2 m</td>
<td>9.7934 m/sec$^2$</td>
</tr>
<tr>
<td>45°</td>
<td>1314.2 m</td>
<td>9.8063 m/sec$^2$</td>
</tr>
<tr>
<td>60°</td>
<td>930.0 m</td>
<td>9.8192 m/sec$^2$</td>
</tr>
<tr>
<td>75°</td>
<td>481.7 m</td>
<td>9.8287 m/sec$^2$</td>
</tr>
<tr>
<td>90°</td>
<td>0.0 m</td>
<td>9.8322 m/sec$^2$</td>
</tr>
</tbody>
</table>
Here we introduce the notion of rational approximations for functions. We will constrain our discussion to the so called Padé approximation. A rational approximation of $f(x)$ on $[a, b]$, is the quotient of two polynomials $P_n(x)$ and $Q_m(x)$ with degrees $n$ and $m$:

$$f(x) = R_N(x) \equiv \frac{P_n(x)}{Q_m(x)} = \frac{a_0 + a_1x + a_2x^2 + \ldots + a_nx^n}{1 + b_1x + b_2x^2 + \ldots + b_mx^m}, \quad N = n + m$$

i.e. there are $N + 1 = n + m + 1$ constants to be determined.

The method of Padé requires that $f(x)$ and its derivatives are continuous at $x = 0$. This choice makes the manipulation simpler and a change of variable can be used to shift, if needed, the calculations over to an interval that contains zero.

We begin with the Maclaurin series for $f(x)$ (up to the term $x^N$), this can be written as:

$$f(x) \approx c_0 + c_1x + \ldots + c_Nx^N$$

where $c_i = f^{(i)}(0)/(i!)$.

\(^1\)Sometimes we write $R_N(x) \equiv R_{n,m}(x)$. 
Then we create the difference

\[ f(x) - R_N(x) \approx \left( c_0 + c_1 x + \ldots + c_N x^N \right) - \frac{a_0 + a_1 x + \ldots + a_n x^n}{1 + b_1 x + \ldots + b_m x^m} \]

\[ = \left( c_0 + c_1 x + \ldots + c_N x^N \right) \left( 1 + b_1 x + \ldots + b_m x^m \right) - \frac{a_0 + a_1 x + \ldots + a_n x^n}{1 + b_1 x + \ldots + b_m x^m} \]

If \( f(0) = R_N(0) \) then \( c_0 - a_0 = 0 \).

In the same way, in order for the first \( N \) derivatives of \( f(x) \) and \( R_N(x) \) to be equal at \( x = 0 \) the coefficients of the powers of \( x \) up to \( x^N \) in the numerator must be zero also.
This gives additionally $N$ equations for the $a$'s and $b$'s

\begin{align*}
    b_1 c_0 + c_1 - a_1 &= 0 \\
    b_2 c_0 + b_1 c_1 + c_2 - a_2 &= 0 \\
    b_3 c_0 + b_2 c_1 + b_1 c_2 + c_3 - a_3 &= 0 \\
    &\vdots \\
    b_m c_{n-m} + b_{m-1} c_{n-m+1} + \cdots + c_n - a_n &= 0 \\
    b_m c_{n-m+1} + b_{m-1} c_{n-m+2} + \cdots + c_{n+1} &= 0 \\
    b_m c_{n-m+2} + b_{m-1} c_{n-m+3} + \cdots + c_{n+2} &= 0 \\
    &\vdots \\
    b_m c_{N-m} + b_{m-1} c_{N-m+1} + \cdots + c_N &= 0
\end{align*} (24)

Notice that in each of the above equations, the sum of subscripts on the factors of each product is the same, and is equal to the exponent of the $x$-term in the numerator.
Padé approximation: Example

For the $R_9(x)$ or $R_{5,4}(x)$ Padé approximation for the function $\tan^{-1}(x)$ we calculate the Maclaurin series of $\tan^{-1}(x)$:

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9$$

and then

$$f(x) - R_9(x) = \frac{(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9)(1 + b_1x + b_2x^2 + b_3x^3 + b_4x^4) - (a_0 + a_1x + a_2x^2 + \ldots + a_5x^5)}{1 + b_1x + b_2x^2 + b_3x^3 + b_4x^4}$$

and the coefficients will be found by the following system of equations:

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = b_1, \quad a_3 = -\frac{1}{3} + b_2, \quad a_4 = -\frac{1}{3}b_1 + b_3, \quad a_5 = \frac{1}{5} - \frac{1}{3}b_2 + b_4$$

$$\frac{1}{5}b_1 - \frac{1}{2}b_3 = 0, \quad -\frac{1}{7} + \frac{1}{5}b_2 - \frac{1}{3}b_4 = 0, \quad -\frac{1}{7}b_1 + \frac{1}{5}b_3 = 0, \quad \frac{1}{9} - \frac{1}{7}b_2 + \frac{1}{5}b_4 = 0$$

(from which we get)

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = \frac{7}{9}, \quad a_4 = 0, \quad a_5 = \frac{64}{945}, \quad b_1 = 0, \quad b_2 = \frac{10}{9}, \quad b_3 = 0, \quad b_4 = \frac{5}{21}.$$

$$\tan^{-1}x \approx R_9(x) = \frac{x + \frac{7}{9}x^3 + \frac{64}{945}x^5}{1 + \frac{10}{9}x^2 + \frac{5}{21}x^4}$$

For $x = 1$, exact 0.7854, $R_9(1) = 0.78558$ while from the Maclaurin series we get 0.8349!
If instead of the analytic form of a function \( f(x) \) we have a set of \( k \) points \( (x_i, f(x_i)) \) in order to find a rational function \( R_N(x) \) such that for every \( x_i \) we will get \( f(x_i) = R_N(x_i) \) i.e.

\[
R_N(x_i) = \frac{a_0 + a_1 x_i + a_2 x_i^2 + \ldots + a_n x_i^n}{1 + b_1 x_i + b_2 x_i^2 + \ldots + b_m x_i^m} = f(x_i)
\]

we will follow the approach used in constructing the approximate polynomial. In other words the problem will be solved by finding the solution of the following system of \( k \geq m + n + 1 \) equations:

\[
\begin{align*}
& a_0 + a_1 x_1 + \ldots + a_n x_1^n - (f_1 x_1) b_1 - \ldots - (f_1 x_1^m) b_m = f_1 \\
& \vdots \\
& a_0 + a_1 x_i + \ldots + a_n x_i^n - (f_i x_i) b_1 - \ldots - (f_i x_i^m) b_m = f_i \\
& \vdots \\
& a_0 + a_1 x_k + \ldots + a_n x_k^n - (f_k x_k) b_1 - \ldots - (f_k x_k^m) b_m = f_k
\end{align*}
\]

i.e. we get \( k \) equations for the \( k \) unknowns \( a_0, a_1, \ldots, a_n \) and \( b_1, b_2, \ldots, b_m \).
We will find the rational function approximations for the following set of data \((-1,1), (0,2)\) and \((1,-1)\).

It is obvious that the sum of degrees of the polynomials in the nominator and denominator must be \((n + m + 1 \leq 3)\). Thus we can write:

\[
R_{1,1}(x) = \frac{a_0 + a_1 x}{1 + b_1 x}
\]

which leads to the following system

\[
\begin{align*}
    a_0 + (-1) a_1 - (-1) b_1 &= 1 \\
    a_0 + 0 \cdot a_1 - 0 \cdot b_1 &= 2 \\
    a_0 + 1 \cdot a_1 - (-1) b_1 &= -1
\end{align*}
\]

\[
\begin{align*}
    \begin{cases}
        a_0 + (-1) a_1 - (-1) b_1 &= 1 \\
        a_0 + 0 \cdot a_1 - 0 \cdot b_1 &= 2 \\
        a_0 + 1 \cdot a_1 - (-1) b_1 &= -1
    \end{cases}
\end{align*}
\]

\Rightarrow

\[
\begin{align*}
    a_0 &= 2 \\
    a_1 &= -1 \\
    b_1 &= -2
\end{align*}
\]

and the rational function will be:

\[
R_{1,1}(x) = \frac{2 - x}{1 - 2x}.
\]

Alternatively, one may derive the following rational function:

\[
R_{0,1}(x) = \frac{a_0}{1 + b_1 x + b_2 x} \Rightarrow R(x) = \frac{2}{1 - 2x - x^2}
\]
1. Find the Padé approximation $R_{3,3}(x)$ for the function $y = e^x$. Compare with the Maclaurin series for $x = 1$.

2. Find the Padé approximation $R_{3,5}(x)$ for the functions $y = \cos(x)$ and $y = \sin(x)$. Compare with the Maclaurin series for $x = 1$.

3. Find the Padé approximation $R_{4,6}(x)$ for the function $y = 1/x \sin(x)$. Compare with the Maclaurin series for $x = 1$.

4. Find the rational approximation for the following set of points:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>0.83</td>
<td>1.06</td>
<td>1.25</td>
<td>4.15</td>
</tr>
</tbody>
</table>