Numerical Integration of ODEs

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The numerical solution of the ODE

\[ y' = f(x, y) \quad \text{with} \quad y(x_0) = y_0 \]  

(1)

at a given point \( x \) can be found by finding the coefficients of the Taylor series expansion about the initial point \( x_0 \). The actual solution will be:

\[ y(x) = y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2} y''(x_0) + \cdots \]  

(2)

where \( h = x - x_0 \).

Thus we need to provide the coefficients:

\[ y'' = f'(x, y), \quad y''' = f''(x, y), \ldots \]  

(3)
Taylor series : Example

We will find the $y(1)$ for the ODE

$$y' = x + y \quad \text{with} \quad y(0) = 1$$

(4)

$$y'(x_0) = y'(0) = 0 + 1 = 1$$
$$y'' = 1 + y' = 1 + x + y = 1 + y'$$
$$y''(x_0) = 1 + y'(0) = 2$$
$$y''' = 1 + y'$$
$$y'''(x_0) = 1 + y'(0) = 2$$
$$y^{(4)} = 1 + y'$$
$$y^{(4)}(x_0) = 1 + y'(0) = 2$$

Thus

$$y(0 + h) = 1 + h + h^2 + \frac{h^3}{3} + \frac{h^4}{12} + \cdots$$

(5)

and the error will be

$$E = \frac{y^{(5)}(\xi)}{5!} h^5 \quad \text{for} \quad 0 < \xi < h$$

(6)
## Taylor series: Example

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>Analytic</th>
<th>Error</th>
<th>Error *</th>
</tr>
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<td>1</td>
<td></td>
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<td>1.383649</td>
<td>$1.8 \times 10^{-4}$</td>
<td>$9.1 \times 10^{-7}$</td>
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<td>3.436564</td>
<td>$2 \times 10^{-2}$</td>
<td>$4.2 \times 10^{-6}$</td>
</tr>
</tbody>
</table>
Single Step Methods : Euler-Heun

- **EULER** We keep only the 1st term of the Taylor series:

\[ y(x_0 + h) = y(x_0) + hy'(x_0) \quad (7) \]

with and obvious error:

\[ E = \frac{y''(\xi)}{2} h^2 \quad \text{for} \quad 0 < \xi < h \]

- **EULER - HEUN** : This is a predictor - corrector method

1st step : \[ y_{n+1} = y_n + hy'_n + O(h^2) \quad (8) \]

2nd step : \[ y_{n+1} = y_n + \frac{h}{2} (y'_n + y'_{n+1}) + O(h^3) \quad (9) \]

Can you explain the smaller error?
Propagation of Errors

If we consider the first order ODE $y' = f(x, y)$, with $y(x_0) = y_0$ then if $Y_n$ is the calculated value at $x_n$ and $y_n$ the true value at $x_n$ the error at $x_n$ will be

$$\varepsilon_n = y_n - Y_n \quad (10)$$

Let’s try to study the propagation of the error for the Euler method:

$$\varepsilon_{n+1} = y_{n+1} - Y_{n+1} = y_n + h \cdot f(x_n, y_n) - Y_n - h \cdot f(x_n, Y_n)$$

$$= \varepsilon_n + h \left(\frac{f(x_n, y_n) - f(x_n, Y_n)}{y_n - Y_n}\right) \varepsilon_n = \varepsilon_n \left[1 + h \cdot f_y(x_n, y_n)\right]$$

$$= (1 + h \cdot k) \varepsilon_n + \frac{1}{2} h^2 y''(\xi_n) \quad \text{where} \quad k = \frac{\partial f}{\partial y}. \quad (11)$$

I.e. the propagation of the error is linear. If $|1 + hk| \geq 1$ the error increases while $|1 + hk| \leq 1$ decreases.

This leads to the necessary condition for absolute convergence:

$$-2 < hk < 0 \quad \text{or} \quad \frac{\partial f}{\partial y} < 0. \quad (12)$$
Convergence

Let's assume an ODE of the form

$$\frac{dy}{dx} = Ay \quad (13)$$

which has an obvious solution of the form $y = e^{Ax}$.

Euler's method gives the approximate solution via the recurrence relation:

$$y_{n+1} = y_n + hAy_n = (1 + hA)y_n \quad \text{for} \quad n = 0, 1, 2, ... \quad (14)$$

Thus if we use this relation $n$-times we will get:

$$y_{n+1} = (1 + hA)^{n+1}y_0 \quad \text{for} \quad n = 0, 1, 2, ... \quad (15)$$

But for small $h$ we know that $1 + hA \approx e^{hA}$ thus

$$y_{n+1} = (1 + hA)^{n+1}y_0 \approx e^{(n+1)hA}y_0 = e^{(x_{n+1} - x_0)A}y_0 = e^{Ax_{n+1}} \quad (16)$$

where $h = (x_{n+1} - x_0)/(n + 1)$.

This means that the numerical solution for small $h$ converges to the analytic solution: $y = e^{Ax}$. 

Numerical Integration of ODEs
Let’s assume the ODE

\[ \frac{dy}{dx} = f(x, y) \quad (17) \]

A possible recurrence relation for the estimation of \( y_n \) can be of the form

\[ y_{n+1} = y_n + ak_1 + bk_2 \quad (18) \]

where

\[
\begin{align*}
    k_1 &= hf(x_n, y_n) \quad (19) \\
    k_2 &= hf(x_n + Ah, y_n + Bk_1) \quad (20)
\end{align*}
\]

Let’s try to find the equivalent Taylor expansion we get

\[ y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2} f'(x_n, y_n) + \ldots \quad (21) \]

But since

\[
\frac{df}{dx} = f_x + f_y \frac{dy}{dx} = f_x + f_y f \quad (22)
\]

we get

\[ y_{n+1} = y_n + hf_n + \frac{h^2}{2} (f_x + ff_y)_{x=x_n} \quad (23) \]
By comparing with (18), which can be written as:

\[ y_{n+1} = y_n + ahf (x_n, y_n) + bhf [x_n + Ah, y_n + Bhf (x_n, y_n)] \]  

(24)

we can get the Taylor series \(^1\) expansion

\[ y_{n+1} = y_n + h (a + b) f_n + h^2 (A b f_x + B b f_y)_n \]  

(25)

and we come to the following relations for the arbitrary constants \(a\), \(b\), \(A\) and \(B\):

\[ a + b = 1, \quad A \cdot b = \frac{1}{2} \quad \text{and} \quad B \cdot b = \frac{1}{2} \]  

(26)

By setting \(a = \frac{2}{3}, \frac{1}{2}, \frac{5}{4} \ldots\) we can estimate the other 3 unknowns.

here we choose \(a = \frac{1}{2}\) then \(b = \frac{1}{2}\) and \(A = B = 1\), i.e.

\[ y_{n+1} = y_n + \frac{1}{2} (k_1 + k_2) \]  

(27)

where

\[ k_1 = hf (x_n, y_n), \quad k_2 = hf (x_n + h, y_n + k_1) \]  

(28)

which is Euler-Heun’s method.

\(^1\)Here we use: \(f (x_n + Ah, y_n + Bhf (x_n, y_n)) \approx (f + f_x Ah + f_y Bhf)_{x=x_n}\)
4th order Runge - Kutta Method

- If we repeat the same procedure for a Taylor series up to $h^4$, we will create a system of 11 equations with 13 unknowns.
- Then with the appropriate choice of two of them we come to a recurrence relation of the form:

\[ y_{n+1} = y_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \]  

(29)

where

\[ k_1 = hf (x_n, y_n) \]  

(30)

\[ k_2 = hf \left( x_n + \frac{1}{2} h, y_n + \frac{1}{2} k_1 \right) \]  

(31)

\[ k_3 = hf \left( x_n + \frac{1}{2} h, y_n + \frac{1}{2} k_2 \right) \]  

(32)

\[ k_4 = hf (x_n + h, y_n + k_3) \]  

(33)

This the 4th order Runge - Kutta with local error $E \approx O(h^5)$ and global error after $n$ steps $E \approx O(h^4)$. 

Numerical Integration of ODEs
4th order Runge - Kutta - Merson Method

In this method the order is not given by the number of \( k \)'s, but by the global error:

\[
k_1 = hf \left( x_n, y_n \right) \quad (34)
\]
\[
k_2 = hf \left( x_n + \frac{1}{3} h, y_n + \frac{1}{3} k_1 \right) \quad (35)
\]
\[
k_3 = hf \left( x_n + \frac{1}{2} h, y_n + \frac{1}{6} k_1 + \frac{1}{6} k_2 \right) \quad (36)
\]
\[
k_4 = hf \left( x_n + \frac{1}{2} h, y_n + \frac{1}{8} k_1 + \frac{3}{8} k_3 \right) \quad (37)
\]
\[
k_5 = hf \left( x_n + h, y_n + \frac{1}{2} k_1 - \frac{3}{3} k_3 + 2 k_4 \right) \quad (38)
\]

\[
y_{n+1} = y_n + \frac{1}{6} \left( k_1 + 4 k_4 + k_5 \right) + O(h^5) \quad (39)
\]

**ERROR:**

\[
\mathcal{E} = \frac{1}{30} \left( 2k_1 - 9k_3 + 8k_4 - k_5 \right) \quad (40)
\]
Runge - Kutta - Fehlberg Method

\[ k_1 = hf(x_n, y_n) \]  \hspace{1cm} (41)

\[ k_2 = hf \left( x_n + \frac{1}{4} h, y_n + \frac{1}{4} k_1 \right) \]  \hspace{1cm} (42)

\[ k_3 = hf \left( x_n + \frac{3}{8} h, y_n + \frac{3}{32} k_1 + \frac{9}{32} k_2 \right) \]  \hspace{1cm} (43)

\[ k_4 = hf \left( x_n + \frac{12}{13} h, y_n + \frac{1932}{2197} k_1 - \frac{7200}{2197} k_2 + \frac{7296}{2197} k_3 \right) \]  \hspace{1cm} (44)

\[ k_5 = hf \left( x_n + h, y_n + \frac{439}{216} k_1 - 8k_2 + \frac{3680}{513} k_3 - \frac{845}{4104} k_4 \right) \]  \hspace{1cm} (45)

\[ k_6 = hf \left( x_n + \frac{1}{2} h, y_n - \frac{8}{27} k_1 + 2k_2 - \frac{3544}{2565} k_3 + \frac{1859}{4104} k_4 - \frac{11}{40} k_5 \right) \]  \hspace{1cm} (46)

Then a first estimation for \( y_{n+1} \) is:

\[ \bar{y}_{n+1} = y_n + \left( \frac{25}{216} k_1 + \frac{1408}{2565} k_3 + \frac{2197}{4104} k_4 - \frac{1}{5} k_5 \right) \]  \hspace{1cm} (47)

here the local error is \( \approx h^5 \).
The next step will include $k_6$ and gives:

$$y_{n+1} = y_n + \left( \frac{16}{135} k_1 + \frac{6656}{12825} k_3 + \frac{28561}{56430} k_4 - \frac{9}{50} k_5 + \frac{2}{55} k_6 \right)$$

(48)

with local error $\approx h^6$ and global $\approx h^5$.

A formula for the estimation of error of the Runge - Kutta - Fehlberg method is

$$E = \frac{1}{360} k_1 - \frac{128}{4275} k_3 - \frac{2197}{7524} k_4 + \frac{1}{50} k_5 + \frac{2}{55} k_6$$

(49)

Since the $k_1, k_2, \ldots, k_6$ are known in every step we can always test the accuracy of the method and if is worst than the demanded we subdivide the step $h$. 

Numerical Integration of ODEs
• The principle behind a multistep method is to utilize the past values of \( y \) and/or \( y' \) to construct a polynomial that approximates the derivative function, and extrapolate this into the next interval.
• The number of past points that are used sets the degree of the polynomial and is therefore responsible for the truncation error.
• The order of the method is equal to the power of \( h \) in the global error term of the formula, which is also equal to one more than the degree of the polynomial.
• We will write the ODE in the form:

\[
\frac{dy}{dx} = f(x, y) \quad \text{then} \quad dy = f(x, y) \, dx
\]  

(50)

and then we integrate between \( x_n \) and \( x_{n+1} \):

\[
\int_{x_n}^{x_{n+1}} dy = y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} f(x, y) \, dx
\]  

(51)
We can use the trapezoidal rule for the integration

\[ y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} f(x, y) \, dx = \frac{h}{2} [f(x_n, y(x_n)) + f(x_n + h, y(x_n + h))] + \frac{h^3}{12} f''(\xi) \]  

(52)

thus the new scheme will be:

\[ y_{n+1} = y_n + \frac{h}{2} (y'_n + y'_{n+1}) - \frac{h^3}{12} y'''(\xi) \]  

(53)

alternatively:

\[ y_{n+1} = y_n + hy'_n + \frac{1}{2} y''_n h^2 + \frac{h^3}{6} y'''(\xi) \]

\[ = y_n + hy'_n + \frac{1}{2} \left[ \frac{y'_{n+1} - y'_n}{h} - \frac{h}{2} y'''(\xi) \right] h^2 + \frac{h^3}{6} y'''(\xi) \]

\[ = y_n + \frac{1}{2} (y'_n + y'_{n+1}) - \frac{h^3}{12} y'''(\xi) \]  

(54)

Did you ever met this relation again?
NOTE: Can you prove the following expression?

\[ y_{n+1} = y_n + \frac{h}{12} \left( 5y'_n + 8y'_n + y'_{n-1} \right) - \frac{h^4}{24} y^{(4)} \]  
(55)

or

\[ y_{n+1} = y_n + \frac{h}{12} \left( 5f'_{n+1} + 8f_n + f_{n-1} \right) - \frac{h^4}{24} y^{(4)} \]  
(56)

HINT
The previous relations suggest that we can use (appropriately) the methods that we developed for the numerical evaluation of integrals in order to derive more accurate relations.
The typical way was to integrate the term on the right of (51) by using an approximate form of \( f(x, y) \) e.g. using an interpolating polynomial in \( x \).
NOTE: In deriving this we have to use several "past points".
Here we use quadratic approximation

\[
\int_{x_n}^{x_{n+1}} dy = y_{n+1} - y_n
\]

\[
= \int_{x_n}^{x_{n+1}} \left[ f_n + s \Delta f_{n-1} + \frac{(s + 1)s}{2} \Delta^2 f_{n-2} + \text{Error} \right] dx
\]

\[
= h \int_{s=0}^{s=1} \left[ f_n + s \Delta f_{n-1} + \frac{(s + 1)s}{2} \Delta^2 f_{n-2} \right] ds
\]

\[
+ h \int_{s=0}^{s=1} \frac{s(s + 1)(s + 2)}{6} h^3 f'''(\xi) ds
\]

then

\[
y_{n+1} - y_n = h \left[ f_n + \frac{1}{2} \Delta f_{n-1} + \frac{5}{12} \Delta^2 f_{n-2} \right] + O(h^4)
\]

\[
= h \left[ f_n + \frac{1}{2} (f_n - f_{n-1}) + \frac{5}{12} (f_n - 2f_{n-1} + f_{n-2}) \right] + O(h^4)
\]

\[
y_{n+1} = y_n + \frac{h}{12} [23f_n - 16f_{n-1} + 5f_{n-2}] + \frac{3}{8} h^4 f'''(\xi) \quad (57)
\]
If we use cubic approximation we get:

\[ y_{n+1} = y_n + \frac{h}{24} [55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] + O(h^5) \]  \hspace{1cm} (58)

**ERROR TERM**: We get the error term of the 4th order Adams formula by integrating the error of the cubic interpolating polynomial

\[ \mathcal{E} \approx \frac{251}{720} h^5 y^{(5)}(\xi), \quad x_{n-3} \leq \xi \leq x_{n+1} \]  \hspace{1cm} (59)
This is an alternative method to Adams. We follow the same procedure but we use different limits of integration. Thus the ODE

$$\frac{dx}{dy} = f(x, y)$$

(60)

can be written as:

$$\int_{x_{n-3}}^{x_{n+1}} dy = y_{n+1} - y_{n-3} = \int_{x_{n-3}}^{x_{n+1}} f(x, y) \, dx$$

(61)

and if we replace $f(x, y)$ with a 2nd order polynomial

$$y_{n+1} = y_{n-3} + \frac{4h}{3} (2f_n - f_{n-1} + 2f_{n-2}) + O(h^5)$$

(62)

ERROR TERM

$$\mathcal{E} \approx \frac{28}{90} h^5 y^{(5)}(\xi), \quad x_{n-3} \leq \xi \leq x_{n+1}$$

(63)
We estimate the value of $y_{k+1} = y(x_{k+1})$ by using two successive approximations.

**Prediction**: we use the $m$ previous points

$$y_{k+1} = \mathcal{P}(y_{k-m}, y_{k-m+1}, \ldots, y_k) \quad (64)$$

**Correction**: we include the "predicted" value of $y_{k+1}$

$$y_{k+1} = \mathcal{C}(y_{k-m}, y_{k-m+1}, \ldots, y_k, y_{k+1}) \quad (65)$$

This 2nd step can be repeated a few times until the two "corrected" values converge i.e. until

$$\left| y_{k+1}^{(j)} - y_{k+1}^{(j+1)} \right| < \mathcal{E}$$

where $j$ is the number of iterations and $\mathcal{E}$ is the maximum allowed error.
A corrector formula for Milne’s method can be derived by just shifting the integration limits

\[ \int_{x_{n-1}}^{x_{n+1}} dy = y_{n+1} - y_{n-1} = \int_{x_{n-1}}^{x_{n+1}} f(x, y) \, dx \]  (66)

and by using Simpson’s rule for the integral we get

\[ y_{n+1} = y_{n-1} + \frac{h}{3} (f_{n+1} + 4f_n + f_{n-1}) \]  (67)

with error:

\[ E \approx \frac{h^5}{90} y^{(5)}(\xi) \quad \text{where} \quad x_{n-1} < \xi < x_{n+1} \]  (68)

**Milne’s Predictor-Corrector formulas**

\[ y_{k+1} = y_{k-3} + \frac{4h}{3} (2f_{k-2} - f_{k-1} + 2f_k) + \frac{28}{90} h^5 y^{(5)}(\xi_1) \quad (x_{k-3} < \xi_1 < x_{k+1}) \]  (69)

\[ y_{k+1} = y_{k-1} + \frac{h}{3} (f_{k-1} + 4f_k + f_{k+1}) - \frac{1}{90} h^5 y^{(5)}(\xi_2) \quad (x_{k-1} < \xi_2 < x_{k+1}) \]  (70)
**Adams-Multon Method**

**PREDICTION** \((x_{k-3} < \xi_1 < x_{k+1})\)

\[
y_{k+1} = y_k + \frac{h}{24} (55f_k - 59f_{k-1} + 37f_{k-2} - 9f_{k-3}) + \frac{251}{720} h^5 y^{(5)}(\xi_1) \quad (71)
\]

**CORRECTION** \((x_{k-2} < \xi_2 < x_{k+1})\)

\[
y_{k+1} = y_k + \frac{h}{24} (9f_{k+1} + 19f_k - 5f_{k-1} + f_{k-2}) - \frac{19}{720} h^5 y^{(5)}(\xi_2) \quad (72)
\]

**Hamming Method**

**PREDICTION** \((x_{k-3} < \xi_1 < x_{k+1})\)

\[
y_{k+1} = y_{k-3} + \frac{4h}{3} (2f_{k-2} - f_{k-1} + 2f_k) + \frac{28}{90} h^5 y^{(5)}(\xi_1) \quad (73)
\]

**CORRECTION** \((x_{k-2} < \xi_2 < x_{k+1})\)

\[
y_{k+1} = \frac{1}{8} (9y_k - y_{k-2}) + \frac{3h}{8} (-f_{k-1} + 2f_k + f_{k+1}) \quad (74)
\]
For the numerical solution of systems of ODEs we follow the methods
developed earlier. For example, the 4th order Runge-Kutta will look like:

\[ y_{n+1} = y_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \] (75)

\[ z_{n+1} = z_n + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4) \] (76)

where

\[ k_1 = hf(x_n, y_n, z_n) \] (77)

\[ k_2 = hf \left( x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1, z_n + \frac{1}{2}l_1 \right) \] (78)

\[ k_3 = hf \left( x_n + \frac{h}{2}, y_n + \frac{1}{2}k_2, z_n + \frac{1}{2}l_2 \right) \] (79)

\[ k_4 = hf \left( x_n + h, y_n + k_3, z_n + l_3 \right) \] (80)

\[ l_1 = hg(x_n, y_n, z_n) \] (81)

\[ l_2 = hg \left( x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1, z_n + \frac{1}{2}l_1 \right) \] (82)

\[ l_3 = hg \left( x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2, z_n + \frac{1}{2}l_2 \right) \] (83)

\[ l_4 = hg \left( x_n + h, y_n + k_3, z_n + l_3 \right) \] (84)
We will demonstrate that Milne's method can become unstable even for simple ODEs.
Consider the ODE
\[ \frac{dy}{dx} = Ay \quad \text{with general solution for } x = x_n \quad \text{the} \quad y_n = y_0 e^{A(x_n-x_0)} \]
If we solve this ODE with Milne's method we will use the corrector formula
\[ y_{n+1} = y_{n-1} + \frac{h}{3} \left( y'_n + 4y'_n + y'_{n-1} \right) \]
where by substituting \( y'_n = Ay_n \) we get
\[ y_{n+1} = y_{n-1} + \frac{h}{3} \left( Ay_{n+1} + 4Ay_n + Ay_{n-1} \right) \]
and by rearranging, we get
\[ \left( 1 - \frac{hA}{3} \right) y_{n+1} - \frac{4hA}{3} y_n - \left( 1 + \frac{hA}{3} \right) y_{n-1} = 0 \]
This is a 2nd order difference equation.
The difference equation admits a solution of the form

\[ y_n = c_1 Z_1^n + c_2 Z_2^n \]

where \( Z_1 \) and \( Z_2 \) are the roots of the quadratic equation:

\[
\left( 1 - \frac{hA}{3} \right) Z^2 - \frac{4hA}{3} Z - \left( 1 + \frac{hA}{3} \right) = 0
\]

If we set \( r = hA/3 \) the roots of the above quadratic equation will be

\[
Z_1 = \frac{2r + \sqrt{3r^2 + 1}}{1 - r} \approx 1 + 3r \approx e^{3r} + O(r^2) = e^{hA}
\]

\[
Z_2 = \frac{2r - \sqrt{3r^2 + 1}}{1 - r} \approx -1 + r \approx -e^{-r} + O(r^2) = -e^{-hA/3}
\]

Hence the Milne solution is represented by

\[ y_n = c_1 e^{nhA} + c_2 e^{-nhA/3} = c_1 e^{A(x_n-x_0)} + c_2 e^{-A(x_n-x_0)/3} , \; x_n - x_0 = nh \]

- The 1st term is as expected
- The 2nd term is parasitic: For \( A > 0 \) dies out as \( x_n \) increases, but for \( A < 0 \) it will grow exponentially with \( x_n \).
Solve numerically the ODE
\[ y' = -10y, \quad \text{with } y(0) = 1 \]
from \( x = 0 \) to \( x = 2 \) by using Milne’s and Adams-Multon’s methods. What are your conclusions about the error?

Use Milne’s method to solve numerically the ODE
\[ y' = 2(x + 1), \quad \text{with } y(1) = 3 \]
Is there any indication of instability?
Convergence Criteria

We will look for a criterion to show how small \( h \) should be in the Adams-Multon method so that re-corrections are not necessary. Let:

- \( y_p = \) value of \( y_{n+1} \) from predictor formula
- \( y_c = \) value of \( y_{n+1} \) from corrector formula
- \( y_{cc}, y_{ccc}, \ldots = \) value of \( y_{n+1} \) if successive re-corrections are made,
- \( y_\infty = \) values to which successive re-corrections converge
- \( D = y_c - y_p \)

The change of \( y_c \) by re-correcting would be

\[
y_{cc} - y_c = \left( y_n + \frac{h}{24} \left( 9y'_c + 19y'_n - 5y'_{n-1} + y'_{n-2} \right) \right) - \left( y_n + \frac{h}{24} \left( 9y'_p + 19y'_n - 5y'_{n-1} + y'_{n-2} \right) \right) = \frac{9h}{24} (y'_c - y'_p)
\]

By manipulating the difference \( (y'_c - y'_p) \) we get:

\[
y'_c - y'_p = f(x_{n+1}, y_c) - f(x_{n+1}, y_p) = \frac{f(x_{n+1}, y_c) - f(x_{n+1}, y_p)}{y_c - y_p} (y_c - y_p)
\]

\[
= f_y(\xi_1) D \quad \text{where} \quad y_c \leq \xi_1 \leq y_p.
\]

(85)
Hence
\[ y_{cc} - y_c = \frac{9h}{24} (y'_c - y'_p) = \frac{9h}{24} f_y(\xi_1)D \] (86)

If recorrected again, the result is
\[ y_{ccc} - y_{cc} = \frac{9h}{24} (y'_{cc} - y'_c) = \frac{9h}{24} f_y(\xi_2)(y_{cc} - y_c) = \frac{9h}{24} f_y(\xi_2) \left[ \frac{9hD}{24} f_y(\xi_1) \right] \]
\[ = \left( \frac{9h}{24} \right)^2 [f_y(\xi)]^2 D \] (87)

On further recorrections we will have a similar relation. Finally, we get \( y_\infty \) by adding all the corrections of \( y_p \) together:
\[ y_\infty = y_p + (y_c - y_p) + (y_{cc} - y_c) + (y_{ccc} - y_{cc}) + \ldots \]
\[ = y_p + D + \frac{9hf_y(\xi)}{24} D + \left( \frac{9hf_y(\xi)}{24} \right)^2 D + \left( \frac{9hf_y(\xi)}{24} \right)^3 D + \ldots \]
\[ = y_p + D \left[ 1 + r + r^2 + r^3 + \ldots \right] = y_p + \frac{D}{1 - r}, \quad r = \frac{9hf_y(\xi)}{24} \]

The above geometrical series will converge only if the ratio is smaller than unity:
\[ |r| = \frac{h|f_y(\xi)|}{24/9} = \frac{h|f_y(x_n, y_n)|}{24/9} < 1 \]

Numerical Integration of ODEs
Convergence Criteria - Adams-Multon Method

Thus the 1st convergence criterion will be:

$$h < \frac{24/9}{|f_y(x_n, y_n)|}$$

If we wish to have $y_c$ and $y_{\infty}$ the same to within one in the $N$th decimal places, then

$$y_{\infty} - y_c = \left( y_p + \frac{D}{1 - r} \right) - (y_p + D) = \frac{rD}{1 - r} < 10^{-N}$$

If $r \ll 1$ the fraction will be approximately $r/(1 - r) \approx r$ we can write a 2nd convergence criterion which ensures that the 1st corrected value is adequate (i.e. it will not changed in the $N$th decimal place by further corrections)

$$D \cdot 10^N < \left| \frac{1}{r} \right| = \frac{24/9}{h|f_y(x_n, y_n)|}$$
The previous criteria were derived for the Adams - Moulton method. Similar criteria can be found for the Milne method:

1st convergence criterion:
\[ h < \frac{3}{|f_y(x_n, y_n)|} \]

2nd convergence criterion:
\[ D \cdot 10^N < \left| \frac{1}{r} \right| = \frac{3}{h|f_y(x_n, y_n)|} \]

**NOTE**: These criteria are for a single 1st order ODE only. A similar analysis for a system is much more complicated.
• The previous error analysis examined the error of a single step only, the so called **local truncation error**.
• The accumulation of these errors, termed as the **global truncation error** is important.

There are other sources of error in addition to the truncation error.

• **Original data errors**: If the initial conditions are not known exactly or expressed inexactly as terminated decimal number, the solution will be affected to a greater or lesser degree depending on the sensitivity of the equation.
• **Round-off errors**: Both floating-point and fixed-point calculations in computers are subject to round off errors.
• **Truncation errors of the method**: These are the types of error due to the use of truncated series for approximation in our work, when infinite series is needed for exactness.
Error Propagation

When we solve ODEs numerically we must worry about the propagation of the previous errors through the subsequent steps. If we consider the 1st order ODE \( y' = f(x, y) \), with \( y(x_0) = y_0 \) then if \( Y_n \) is the calculated value at \( x_n \) and \( y_n \) the true value, then the error in \( Y_n \) will be:

\[
\epsilon_n = y_n - Y_n
\]  \hspace{1cm} (88)

By the Euler algorithm:

\[
\begin{align*}
\epsilon_{n+1} &= y_{n+1} - Y_{n+1} = y_n + hf(x_n, y_n) - Y_n - hf(x_n, Y_n) + \frac{1}{2} h^2 y''(\xi_n) \\
&= \epsilon_n + h \frac{f(x_n, y_n) - f(x_n, Y_n)}{y_n - Y_n} \epsilon_n + \frac{1}{2} h^2 y''(\xi_n) \\
&= \epsilon_n (1 + h f_y(x_n, \eta_n)) + \frac{1}{2} h^2 y''(\xi_n) \quad \text{where } \eta_n \text{ between } y_n, Y_n \\
&= (1 + hk) \epsilon_n + \frac{1}{2} h^2 y''(\xi_n) \quad \text{where } k = \frac{\partial f}{\partial y}.
\end{align*}
\]  \hspace{1cm} (89)

I.e. the error propagation is linear. If \( |1 + hk| \geq 1 \) the error increases while for \( |1 + hk| \leq 1 \) decreases. This observation leads to the condition:

\[
-2 < hk < 0 \quad \text{or} \quad \frac{\partial f}{\partial y} < 0 \]  \hspace{1cm} (90)
\[ \varepsilon_{n+1} = (1 + h k) \varepsilon_n + \frac{1}{2} h^2 y''(\xi_n) \] (91)

Since \( Y_0 = y_0 \) we get:

\[
\begin{align*}
\varepsilon_0 &= 0 \\
\varepsilon_1 &\leq (1 + h k) \varepsilon_0 + \frac{1}{2} h^2 y''(\xi_0) = \frac{1}{2} h^2 y''(\xi_0) \\
\varepsilon_2 &\leq (1 + h k) \left[\frac{1}{2} h^2 y''(\xi_0)\right] + \frac{1}{2} h^2 y''(\xi_1) = \frac{1}{2} h^2 \left[(1 + h k)y''(\xi_0) + y''(\xi_1)\right] \\
\varepsilon_3 &\leq \frac{1}{2} h^2 \left[(1 + h k)^2 y''(\xi_0) + (1 + h k)y''(\xi_1) + y''(\xi_2)\right] \\
\varepsilon_n &\leq \frac{1}{2} h^2 \left[(1 + h k)^{n-1} y''(\xi_0) + (1 + h k)^{n-2} y''(\xi_1) + \ldots + y''(\xi_{n-1})\right]
\end{align*}
\]

- If \( k \geq 0 \) the truncation error at every step is propagated to the next step after being "amplified" by the factor \((1 + h k)\). Finally, for \( h \to 0 \) the error at any point is the sum of all previous errors.
- If the \( f_y \) is negative and of magnitude such that \( kh \leq 2 \) the errors are propagated with diminishing effect.
We now show that the accumulated error after \( n \) steps is \( O(h) \); that is the global error of the simple Euler method after \( n \) steps is \( O(h) \).

If \( |y''(x)| < M \) with \( M > 0 \) then the previous equation will be written as:

\[
|\varepsilon_{n+1}| \leq (1 + hk)|\varepsilon_n| + \frac{1}{2} h^2 M
\]

which is a difference equations of the form:

\[
Z_{n+1} = (1 + hk)Z_n + \frac{1}{2} h^2 M \quad \text{with} \quad Z_0 = 0
\]

With obvious solution

\[
Z_n = \frac{hM}{2k} (1 + hk)^n - \frac{hM}{2k}
\]

since \( 1 + hk < e^{hk} \) for \( k > 0 \) we get

\[
Z_n < \frac{hM}{2k} (e^{hk})^n - \frac{hM}{2k} = \frac{hM}{2k} (e^{nhk} - 1) = \frac{hM}{2k} \left( e^{(x_n-x_0)k} - 1 \right) = O(h)
\]

It follows that the global error \( \varepsilon_n \) is \( O(h) \)