Boundary and Characteristic - Value Problems

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Introduction to BVPs i

- The 2nd order ODEs must have two initial conditions for their numerical solution. Typically both conditions are given at the beginning (**initial value problems**).
- This may not always be the case; the given conditions may be at different points, usually the endpoints of the domain of interest (**boundary value problems**).

Thus for an ODE of the form :

$$y'' + p(x)y(x) + q(x)y = g(x) \tag{1}$$

the following combinations of boundary condition are possible (integration $x_0 \to x_1$)

1. $y(x_0) = y_0$ & $y(x_1) = y_1$
2. $y'(x_0) = y_0$ & $y'(x_1) = y_1$

Introduction to BVPs ii

3. $y'(x_0) = y_0$ & $y(x_1) = y_1$
4. $y(x_0) = y_0$ & $y'(x_1) = y_1$

PROBLEM: Solve the BVP

$$y'' + 9y = 0 \quad \text{with BC} \quad y(0) = 1 \quad \text{and} \quad y(\pi/12) = 0 \tag{2}$$

Solution: The general solution of the ODE is:

$$y(x) = A\cos(3x) + B\sin(3x) \tag{3}$$

From the **1st BC** we get:

$$y(0) = 1 = A\cos(0) + B\sin(0) \rightarrow A = 1$$

and from the **2nd BC**:

$$y(\pi/12) = 0 = A\cos(\pi/4) + B\sin(\pi/4) \rightarrow (A+B)\frac{\sqrt{2}}{2} = 0 \rightarrow B = -1$$

BVP: Shooting Method i

Solve the BVP

$$u'' - \left(1 - \frac{x}{5}\right) u = x, \quad u(1) = 2, \quad u(3) = -1 \tag{4}$$

We can integrate the above ODE (using Runge-Kutta) by assuming for example :

- $u(1) = 2$ and $u'(1) = -1.5 \quad \rightarrow \quad u(3) = 4.7876$ and $u'(3) = 5.1119$

if we try again with

- $u(1) = 2$ and $u'(1) = -3.0 \quad \rightarrow \quad u(3) = 0.4360$ and $u'(1) = 1.6773$

Finally, we assume (why?):

- $u(1) = 2$ and $u'(1) = -3.495 \quad \rightarrow \quad u(3) = -1.0$ and $u'(1) = 0.5439$

BVP: Shooting Method ii

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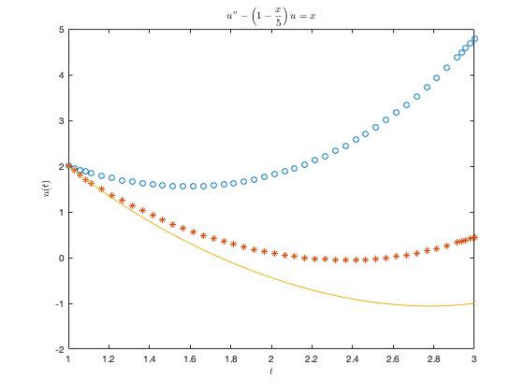

Figure 1: Integration of eqn (4) for the 3 guess values of $u'(x)$ defined earlier. The *circles* correspond to $u'(1) = -1.5$, the *stars* correspond to $u'(1) = -3$. Finally, by choosing$u'(1) = -3.495$, (continuous line) we get $u(3) = -1$.

BVP: Shooting Method iii

The success in selecting the 3rd trial value was not accidental.

The ODE is **linear** and for the linear ones the interpolation from the first two trials gives always the correct solution.

If
G=guess,
R=result, and
DR= desired result,
then:

$$G_3 = G_2 + (DR - R_2)\,\frac{G_1 - G_2}{R_1 - R_2}$$

BVP: Shooting Method iv

NON-LINEAR ODEs

If we have to solve the **nonlinear** ODE

$$u'' - \left(1 - \frac{x}{5}\right) uu' = x, \quad u(1) = 2, \quad u(3) = -1$$

We will not converge to the solution after two iterations, instead the convergence maybe slow.

For example:

$u'(1)$	-1.5	-3.0	-2.2137	-1.9460	-2.0215	-2.0162	-2.0161
$u(3)$	-0.0282	-2.0705	-1.2719	-0.8932	-1.0080	-1.0002	-1.0000

BVP: Finite Difference Method i

In this case we will write the ODE (4) as:

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} - \left(1 - \frac{x_i}{5}\right) u_i = x_i$$

which will be written as:

$$u_{i-1} - \left[2 + h^2 \left(1 - \frac{x_i}{5}\right)\right] u_i + u_{i+1} = h^2 x_i$$

I.e. in this way we create a system of $N - 2$ linear equations which together with the boundary conditions will give the unique solution to the problem, i.e. the $N - 2$ values for u_i.

COMMENT: In principle, this method can be implemented faster than the shooting method but typically for the same number of points is less accurate, because it is only $O(h^2)$ while one can used much more accurate methods in solving the ODEs via the shooting method.

BVP: Finite Difference Method ii

For just 2 points in the interval $(1,3)$ we get two linear equations:

$$u_0 - \left[2 + h^2\left(1 - \frac{x_1}{5}\right)\right] u_1 + u_2 = h^2 x_1 \quad (5)$$

$$u_1 - \left[2 + h^2\left(1 - \frac{x_2}{5}\right)\right] u_2 + u_3 = h^2 x_2 \quad (6)$$

Then by using $u_0 = 2$, $u_3 = -1$, $x_1 = 5/3$, $x_2 = 7/3$ and $h = 2/3$ we get:

$u_1 = +0.188$ exact value $+0.165$

$u_2 = -0.826$ exact value -0.847

In a similar way for **three** interior points we get:

$u_1 = +0.552$ exact value $+0.540$

$u_2 = -0.424$ exact value -0.438

$u_3 = -0.964$ exact value -0.974

BVP: Finite Difference Method iii

In a similar way for **four** interior points we get:

$u_1 = +0.7900$ exact value $+0.7968$

$u_2 = -0.0997$ exact value -0.0905

$u_3 = -0.7076$ exact value -0.6992

$u_3 = -1.0237$ exact value -1.0185

BVP: Existence Theorem

Theorem I:

The BVP

$$y'' = f(x, y) \quad \text{with BCs} \quad y(0) = 0 \quad x(1) = 0$$

has a unique solution if $\partial f / \partial y$ is continuous, no-negative and bounded in the infinite strip defined by the inequalities:

$$0 \leq x \leq 1 \quad \text{and} \quad -\infty < y < \infty$$

EXAMPLE Show that the 2-point BVP has a unique solution:

$$y'' = (5y + \sin 3y)\, e^x \quad x(0) = x(1) = 0$$

Solution

$$\frac{\partial f}{\partial y} = (5 + 3\cos 3y)\, e^x$$

This is continuous in the strip $0 \leq x \leq 1$ and $-\infty < y < \infty$. Also it is bounded above by $8e$ and it is nonnegative. Therefore, admits a unique solution.

Characteristic - Value Problems : Introduction

Consider the homogeneous 2nd order ODE with boundary conditions:

$$\frac{d^2u}{dx^2} + k^2 u = 0\,, \quad u(0) = 0\,, \quad u(1) = 0, \tag{7}$$

where k^2 is a parameter. Then the question is what are the **characteristic** values of k for which the above equation has a solution. These characteristic values are called **eigenvalues**. It is obvious that for this simple wave equation the general solution is:

$$u = A\sin(kx) + B\cos(kx)$$

which for the given boundary conditions leads to the following eigensolution and eigenvalues

$$u = A\sin(n\pi x), \quad k = \pm n\pi, \quad n = 1, 2, 3, \ldots$$

The e-values are the most important information for a characteristic value problem. In the case of a vibrating string, these give the natural frequencies of the system which are important. To every e-value corresponds an e-vector $u(x)$, which determines the possible patterns of vibration for the string. Most of the times the smallest e-value is the most important.

Numerical solution

The following eqn can be written in its equivalent finite difference form

$$\frac{d^2u}{dx^2} + k^2 u = 0\,, \quad u(0) = 0\,, \quad u(1) = 0, \tag{8}$$

that is N equations of the form:

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + k^2 u_i = 0$$

This leads to the following system of linear equations in matrix form:

$$\begin{pmatrix} 2 - h^2k^2 & -1 & \cdots & & 0 \\ -1 & 2 - h^2k^2 & -1 & & 0 \\ \vdots & & & & \vdots \\ 0 & & \cdots & -1 & 2 - h^2k^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

or better:

$$(\mathbf{A} - \lambda \mathbf{I})\, u = 0 \ \text{ where } \ \lambda = h^2k^2$$

which for $h = 0.2$ gives $k = 3.09$ (1.6%), $k = 5.88$ (6.3%), $k = 8.09$ (14%)and $k = 9.51$ (24.3%).

Tridiagonal Matrices

The eigenvalues-eigenvectors of $N \times N$ tridiagonal matrices

$$A = \begin{pmatrix} b & c & 0 & \cdots & & 0 \\ a & b & c & 0 & & 0 \\ 0 & a & b & c & & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & & \cdots & a & b \end{pmatrix}$$

can be found analytically from the following relations:

$$\lambda_j = b + 2c\sqrt{\frac{a}{c}}\cos\frac{j\pi}{N+1} \quad \text{where} \quad j = 1, \ldots, N$$

and $\vec{u}_j = (u_1, u_2, \ldots, u_k, \ldots, u_N)^T$ where:

$$u_k = 2\left(\sqrt{\frac{a}{c}}\right)^k \sin\frac{kj\pi}{N+1} \quad \text{where} \quad j = 1, \ldots, N\,.$$