Numerical Integration of PDEs [a]

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[a] J.W Thomas *Numerical PDEs*, Springer 1995

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Numerical Solution of PDEs

Introduction i

- A differential equation involving more than one independent variable is called a **partial differential equations** (PDEs) For example, a PDE for the function $u(x, y, z)$ may look like:

$$F\left(x, y, z; u; \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}; \frac{\partial^2 u}{\partial x \partial x}, \frac{\partial^2 u}{\partial x \partial y}, \dots, \frac{\partial^2 u}{\partial z \partial z}; \dots\right) = 0 \qquad (1)$$

- Many problems in applied science, physics and engineering are modeled mathematically with PDE.

- Here we will study finite-difference methods in solving numerically PDEs, which are based on formulas for approximating the 1st and the 2nd derivatives of a function.

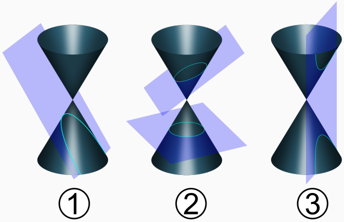
Introduction ii

The 2nd order linear PDEs (are of direct importance for physics) and are classified with terminology borrowed from the conic sections.
For a 2nd-degree polynomial in x and y

$$Ax^2 + 2Bxy + Cy^2 + \cdots = 0 \tag{2}$$

the graph is a quadratic curve, and when

- $B^2 - 4AC = 0$ the curve is a **parabola**,
- $B^2 - 4AC < 0$ the curve is an **ellipse**,
- $B^2 - 4AC > 0$ the curve is a **hyperbola**



Introduction iii

Now if one replaces x with ∂_x, y with ∂_y ... [1] converts equation (2) into a PDE of the form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right) = 0 \tag{3}$$

where A, B and C are constants, is called **quasilinear**. There are 3 types of quasilinear equations:

- If $B^2 - 4AC = 0$, the equation is called **parabolic**,
- If $B^2 - 4AC < 0$, the equation is called **elliptic**,
- If $B^2 - 4AC > 0$, the equation is called **hyperbolic**

Introduction iv

The **heat equation**

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \quad \text{for} \quad 0 < x < L \tag{4}$$

assuming that the initial temperature distribution at $t = 0$ is

- $u(x, 0) = f(x)$ for $t = 0$ and $0 \leq x \leq L$

and the boundary conditions at the ends of the rod are

- $u(x, t) = c_1$ for $x = 0$ and $0 \leq t \leq \infty$
- $u(L, t) = c_2$ for $x = L$ and $0 \leq t \leq \infty$

is an example of **parabolic** PDE.

Introduction v

Two classic examples of PDEs are the 2-D Laplace and Poisson eqns:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,, \quad \text{or} \tag{5}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x, y) \tag{6}$$

with boundary conditions:

- $u(x, 0) = f_1(x)$ for $y = 0$ and $0 \leq x \leq L_1$
- $u(x, L_2) = f_2(x)$ for $y = L_2$ and $0 \leq x \leq L_1$
- $u(0, y) = g_1(y)$ for $x = 0$ and $0 \leq y \leq L_2$
- $u(L_1, y) = g_2(y)$ for $x = L_1$ and $0 \leq y \leq L_2$

for which $B = 0$, $A = C = 1$ and thus they are **elliptic** PDEs.

The **wave equation**

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \quad \text{for} \quad 0 < x < L \quad \text{and} \quad 0 < t < \infty \tag{7}$$

with a given initial position and velocity functions

- $u(x, 0) = f(x)$ for $t = 0$ and $0 \leq x \leq L$
- $u_t(x, 0) = g(x)$ for $t = 0$ and $0 \leq x \leq L$

is a classic example of **hyperbolic** PDE.

[1] Formally via a Fourier transform

Elliptic PDEs (numerical solution) i

We will try to solve the Laplace equation in 2-dimensions

$$u_{xx} + u_{yy} = 0 \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad 0 < y < 1 \tag{8}$$

with boundary conditions:

- $u(x,0) = f_1(x)$ for $y = 0$ and $0 \leq x \leq 1$
- $u(x,0) = f_2(x)$ for $y = 1$ and $0 \leq x \leq 1$
- $u(x,0) = f_3(x)$ for $x = 0$ and $0 \leq y \leq 1$
- $u(x,0) = f_4(x)$ for $x = 1$ and $0 \leq y \leq 1$

Elliptic PDEs (numerical solution) ii

since

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2)$$

This means that u_{xx} at the point (x_i, y_j) will be

$$[u_{xx}]_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \tag{9}$$

and u_{yy} will be written as:

$$[u_{yy}]_{i,j} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} \tag{10}$$

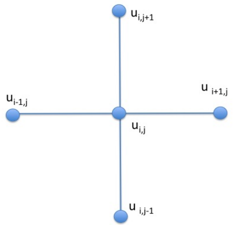
Thus Laplace's equation can be approximately written as

$$\nabla^2 u \approx \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} = 0 \tag{11}$$

where $i = 2, ..., n-1$ & $j = 2, ..., m-1$.

Elliptic PDEs (numerical solution) iii

This is the **5-point difference formula** for Laplace's equation and relates the function value $u_{i,j}$ to its 4 neighbouring values $u_{i-1,j}$, $u_{i+1,j}$, $u_{i,j-1}$ and $u_{i,j+1}$.



Which leads to the following Laplacian computational formula:

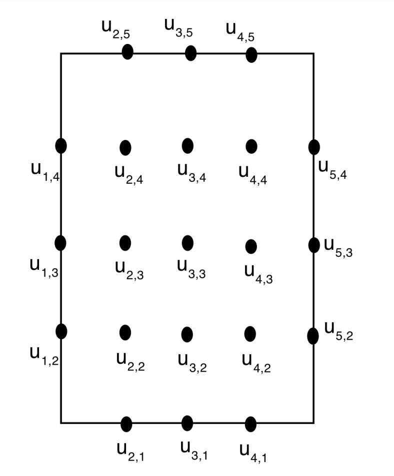
$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 0 \tag{12}$$

Elliptic PDEs (numerical solution) iv

Assume that the values of $u(x, y)$ are known at the following boundary grid points:

$$
\begin{aligned}
u(x_1, y_j) &= u_{1,j} \quad \text{for} \quad 2 \le j \le m-1 \\
u(x_i, y_1) &= u_{i,1} \quad \text{for} \quad 2 \le i \le n-1 \\
u(x_n, y_j) &= u_{n,j} \quad \text{for} \quad 2 \le j \le m-1 \\
u(x_i, y_m) &= u_{i,m} \quad \text{for} \quad 2 \le i \le n-1
\end{aligned}
$$

Then we can estimate the values of the function $u(x, y)$ at the interior grid points by solving a system of $(n-2) \times (n-2)$ equations for $(n-2)^2$ unknowns.



Elliptic PDEs (numerical solution) v

For the above 5×5 grid the solution of the Laplacian equation will be given by the following linear system:

$$\begin{array}{ccccccccccc}
-4u_{2,2} & +u_{3,2} & & +u_{2,3} & & & & & & = -u_{2,1} - u_{1,2} \\
u_{2,2} & -4u_{3,2} & +u_{4,2} & & +u_{3,3} & & & & & = -u_{3,1} \\
& u_{3,2} & -4u_{4,2} & & & +u_{4,3} & & & & = -u_{4,1} - u_{5,2} \\
u_{2,2} & & & -4u_{2,3} & +u_{3,3} & & +u_{2,4} & & & = -u_{1,3} \\
& u_{3,2} & & +u_{2,3} & -4u_{3,3} & +u_{4,3} & & +u_{3,4} & & = 0 \\
& & u_{4,2} & & +u_{3,3} & -4u_{4,3} & & & +u_{4,4} & = -u_{5,3} \\
& & & u_{2,3} & & & -4u_{2,4} & +u_{3,4} & & = -u_{2,5} - u_{1,4} \\
& & & & u_{3,3} & & +u_{2,4} & -4u_{3,4} & +u_{4,4} & = -u_{3,5} \\
& & & & & & u_{4,3} & +u_{3,4} & -4u_{4,4} & = -u_{4,5} - u_{5,4}
\end{array} \tag{13}$$

EXAMPLE 1

If the rectangle has dimensions $0 \leq x \leq 4$ and $0 \leq y \leq 4$ with boundary conditions

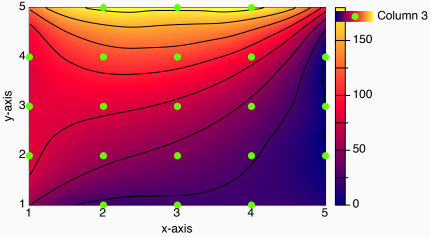
$$\begin{aligned} u(x,0) &= 20 \quad \text{and} \quad u(x,4) = 180 \quad \text{for} \quad 0 < x < 4 \\ u(0,y) &= 80 \quad \text{and} \quad u(4,x) = 0 \quad \text{for} \quad 0 < y < 4 \end{aligned}$$

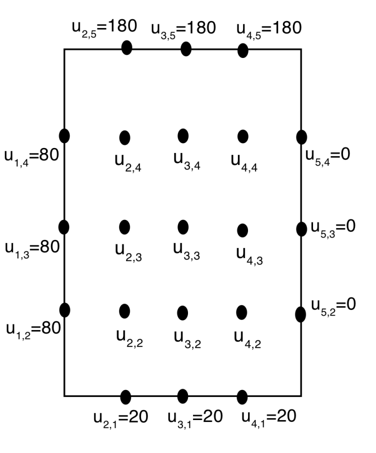
we create the following grid

$$\begin{array}{cccccccccl}
-4u_{2,2} & +u_{3,2} & & +u_{2,3} & & & & & & = -100 \\
u_{2,2} & -4u_{3,2} & +u_{4,2} & & +u_{3,3} & & & & & = -20 \\
& u_{3,2} & -4u_{4,2} & & & +u_{4,3} & & & & = -20 \\
u_{2,2} & & & -4u_{2,3} & +u_{3,3} & & +u_{2,4} & & & = -80 \\
& u_{3,2} & & +u_{2,3} & -4u_{3,3} & +u_{4,3} & & +u_{3,4} & & = 0 \\
& & u_{4,2} & & +u_{3,3} & -4u_{4,3} & & & +u_{4,4} & = 0 \\
& & & u_{2,3} & & & -4u_{2,4} & +u_{3,4} & & = -260 \\
& & & & u_{3,3} & & +u_{2,4} & -4u_{3,4} & +u_{4,4} & = -180 \\
& & & & & & u_{4,3} & +u_{3,4} & -4u_{4,4} & = -180
\end{array}$$

which admits the solution:

$u_{2,4} = 112.857,\ u_{3,4} = 111.786,\ u_{4,4} = 84.2857$
$u_{2,3} = 79.6429,\ u_{3,3} = 70.000,\ u_{4,3} = 45.3571,$
$u_{2,2} = 55.7143,\ u_{3,2} = 43.2143,\ u_{4,2} = 27.1429.$





Elliptic PDEs : S.O.R. i

The solution of the linear systems that derived earlier can be found according to the methods discussed in Section 2. For the 3-diagonal systems that we have here the iterative methods are the best choice.

Assuming some initial values for the internal unknown grid points $u_{i,j}$ we can use the following iterative scheme:

$$u_{i,j}{}^{(N+1)} = \frac{1}{4}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right)^{(N)} \tag{14}$$

A fasted approach is to use the following **successive over relaxation** (S.O.R) scheme

$$\begin{aligned} u_{i,j}{}^{(N+1)} &= u_{i,j}{}^{(N)} + \frac{\omega}{4}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}\right)^{(N)} \\ &= u_{i,j}{}^{(N)} + \omega r_{i,j}{}^{(N)} \end{aligned} \tag{15}$$

This procedure will be repeated until $|r_{i,j}|^{(N)} < \epsilon$. The optimal value for the overrelaxation factor ω is not always predictable.

For rectangular regions with Dirichlet boundary conditions there is a formula for the optimal ω which is the root of the quadratic equation

$$\left[\cos\left(\frac{\pi}{n-1}\right)+\cos\left(\frac{\pi}{m-1}\right)\right]^2\omega^2-16\omega+16=0 \qquad (16)$$

EXAMPLE 2

- Solve EXAMPLE 1 using S.O.R. method for different values of ω check if your numerical findings agree with the outcome of the previous relation.
- Test the speed of the method in comparison to the standard method - *for reliable comparison increase the grid points in each side by 100 times*.

Hyperbolic PDEs i

A typical example of hyperbolic equation is the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad \text{for} \quad 0 < x < a \quad \text{and} \quad 0 < t < b \tag{17}$$

with the **boundary**

$$u(0,t) = 0 \quad \text{and} \quad u(a,t) = 0 \qquad \text{for} \quad 0 \leq t \leq b \tag{18}$$

and **initial conditions**

$$\begin{aligned} u(x,0) &= f(x) \qquad && \text{for} \quad 0 \leq x \leq a \\ u_t(x,0) &= g(x) \qquad && \text{for} \quad 0 < x < a \end{aligned} \tag{19}$$

This equation typically describes 1D-waves propagating on a string with velocity

$$c^2 = \frac{T}{\rho A} \tag{20}$$

where T is the force (tension) in the string, ρ is the density of the string material, and A is the cross-sectional area of the string.

Hyperbolic PDEs ii

Furthermore, for **longitudinal vibration of a bar** $u(t,x)$ represents the longitudinal displacement with

$$c^2 = \frac{E}{\rho} \tag{21}$$

where E is the modulus of elasticity and ρ the density of the bar.

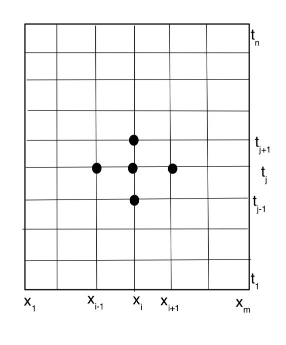
For the 2D case of **transverse vibrations of a membrane** $u(t,x,y)$ is the transverse displacement of the membrane with

$$c^2 = \frac{T}{\rho\,\Delta h} = \frac{T}{m} \tag{22}$$

where T is the tension per unit length, ρ the density and Δh the membrane thickness, $m = \rho\Delta h$ os the mass per unit area.

Derivation of Difference Equation

Partition the rectangle $R = (x, t) : 0 \leq x \leq a\,,\, 0 \leq t \leq b$ into a grid consisting of $(n-1)$ by $(m-1)$ rectangles with sides $\Delta x = h$ and $\Delta t = k$.



Hyperbolic PDEs iv

Then the central difference formulas will be:

$$u_{tt}(x,t) = \frac{u(x,t+\Delta t) - 2u(x,\Delta t) + u(x,t-\Delta t)}{\Delta t^2} + O(\Delta t^2) \tag{23}$$

$$u_{xx}(x,t) = \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} + O(\Delta x^2) \tag{24}$$

Because $x_{i+1} = x_i + \Delta x$ and $t_{j+1} = t_j + \Delta t$ we can write

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta t^2} = c^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \tag{25}$$

and if for simplicity we get $r = c\Delta t / \Delta x$ then

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = r^2 \left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right). \tag{26}$$

which finally becomes

$$u_{i,j+1} = 2(1 - r^2)u_{i,j} + r^2 \left(u_{i+1,j} + u_{i-1,j} \right) - u_{i,j-1} \tag{27}$$

Hyperbolic PDEs : Starting values

Two starting values corresponding to $j = 1$ and $j = 2$ must be supplied in order to use formula (27) to compute the 3rd row.

Since, the values of the 2nd row usually are not known we estimate them numerically from the information that we have for $u_t(x, 0)$.

The value of $u(x_i, \Delta t)$ satisfies

$$u(x_i, \Delta t) = u(x_i, 0) + \Delta t \, u_t(x_i, 0) + O(\Delta t^2) \tag{28}$$

But since $u(x_i, 0) = f(x_i) = f_i$ and $u_t(x_i, 0) = g(x_i) = g_i$ the above relation will be written:

$$u_{i,2} = f_i + \Delta t \, g_i \quad \text{for} \quad i = 2, 3, ..., n - 1. \tag{29}$$

Hyperbolic PDEs : Stability

Numerical methods suffer from **instabilities** which grow as we evolve the equation in time.

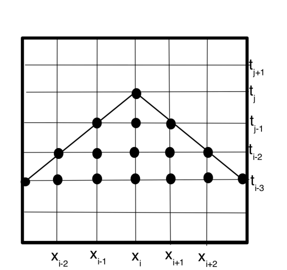
For the hyperbolic equation, under discussion, there exists a sufficient criterion which ensures the stability of the evolution is $r = c\Delta t/\Delta x \leq 1$.

This is called Courant-Friedrichs-Lewy (CFL).

In practice the CFL criterion demands

$$|c| \leq \frac{\Delta x}{\Delta t} \tag{30}$$

That is the propagation speed of the waves c to be smaller that the speed of propagation of the information in our grid.



The condition can be viewed as a sort of discrete "light cone" condition, namely that the time step must be kept small enough so that information has enough time to propagate through the space discretization.

The 1-dimension wave equation $u_{tt} = c^2 u_{xx}$ might not always have reflective boundaries i.e. $u(x_0) = u(x_n) = 0$ but instead one or both boundaries might "absorb" and not reflect the waves.

The d'Alembert solution will be written as:

$$u(x,t) = F_A(x - ct) + F_R(x + ct) \tag{31}$$

F_A is the wave advancing towards the boundary and F_R the reflecting wave. [2] Here $F_R = 0$ and thus $u_x = F'_A$ and $u_t = -cF'_A$ which means that we get the wave equation representing waves traveling in one only direction $u_t = -cu_x$.

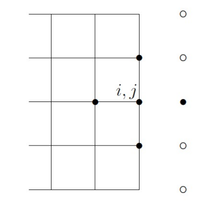


Figure 1: Finite Difference Mesh at non-reflecting boundary

The central difference approximation for the non-reflecting boundary will be

$$c\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} = -\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta t} \quad \Rightarrow$$

$$r\left(u_{i+1,j} - u_{i-1,j}\right) = -\left(u_{i,j+1} - u_{i,j-1}\right) \tag{32}$$

Hyperbolic PDEs : Nonreflecting Boundaries ii

So

$$u_{i+1,j} = u_{i-1,j} - r^{-1}\left(u_{i,j+1} - u_{i,j-1}\right) \tag{33}$$

We may substitute (35) into equation (27)

$$u_{i,j+1} = 2(1 - r^2)u_{i,j} + r^2\left(u_{i+1,j} + u_{i-1,j}\right) - u_{i,j-1} \tag{34}$$

in order to eliminate the point $u_{i+1,j}$ and we get:

$$(1 + r)u_{i,j+1} = 2r^2 u_{i-1,j} + 2(1 - r^2)u_{i,j} - (1 - r)u_{i,j-1} \tag{35}$$

If we assume $r = 1$ then we get the extremely simple form

$$u_{i,j+1} = u_{i-1,j} \tag{36}$$

At the initial time step the last term will be evaluating using (29).

[2]NOTE: The wave equation can be written as

$$\left(\partial_t - c\partial_x\right)\left(\partial_t + c\partial_x\right)u = 0$$

Parabolic PDEs (numerical solution) i

We will consider the 1D heat equation as an example of parabolic PDE

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{37}$$

for $\quad 0 \leq x \leq 1$ and $0 \leq t < \infty$

with boundary conditions:

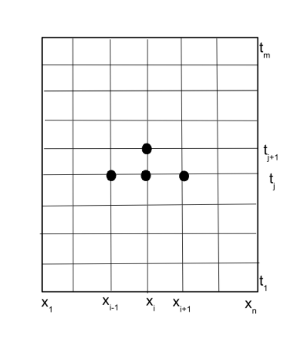
$$u(0,t) = c_1, \quad u(1,t) = c_2 \quad \text{for} \quad 0 \leq t < \infty \tag{38}$$

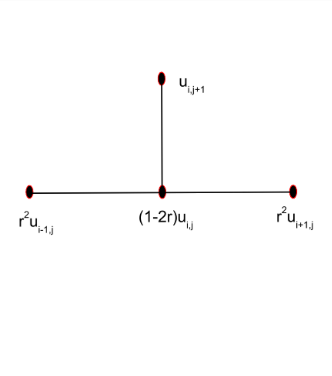
and initial conditions : $u(x,0) = f(x), \quad$ for $\quad 0 \leq x \leq 1\,.$

The heat equation models the temperature in an insulated rod with ends held at constant temperatures c_1 and c_2.

Parabolic PDEs (numerical solution) ii

We assume that the rectangle $R = \{(x,t) : 0 \leq x \leq 1,\ 0 \leq t < b\}$ is subdivided into $n-1$ by $m-1$ rectangles with sides $\Delta x = h$ and $\Delta t = k$.





Parabolic PDEs (numerical solution) iii

Then the finite difference formulas will be:

$$u_t(x,t) = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} + O(\Delta t) \quad (39)$$

$$u_{xx}(x,t) = \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{h^2} + O(\Delta x^2) \quad (40)$$

and the difference equation becomes

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \alpha^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \quad (41)$$

by setting $r = \alpha^2 \Delta t / \Delta x^2$ we get

$$u_{i,j+1} = (1-2r)\, u_{i,j} + r\,(u_{i-1,j} + u_{i+1,j}) \quad (42)$$

Parabolic PDEs : Stability

- The simplicity of eqn (42) makes it appealing to use. However, *it is important to use numerical techniques that are stable*.
- If any error made at one stage of the calculation is eventually damped out, the method is called **stable**
- The explicit forward-difference equation (42) is stable if and only if

$$0 \leq r \leq \frac{1}{2} \quad \text{or} \quad \Delta t \leq \frac{\Delta x^2}{2\alpha^2} \tag{43}$$

  If this condition is not fulfilled, errors, committed at one row might be magnified in subsequent rows.
- The difference eqn (42) has accuracy of the order $O(\Delta t) + O(\Delta x^2)$
- If we choose $r = 1/2$ the difference eqn (42) becomes even simpler:

$$u_{i,j+1} = \frac{u_{i-1,j} + u_{i+1,j}}{2} \tag{44}$$

Parabolic PDEs : Implicit methods (Crank - Nicholson) i

The implicit method of Crank - Nicholson is based on using the spatial derivative on both the point (i, j) and $(i, j+1)$. That is:

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}\alpha^2 \left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2} \right)$$

which after rearrangement leads to:

$$-r u_{i-1,j+1} + 2(1+r)u_{i,j+1} - r u_{i+1,j+1} = 2(1-r)u_{i,j} + r\left(u_{i-1,j} + u_{i+1,j}\right) \quad (45)$$

which for $r = 1$ leads to

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j} \quad (46)$$

Parabolic PDEs : Implicit methods (Crank - Nicholson) ii

The previous relation leads to the solution of the following linear system of equations

$$\begin{pmatrix} 4 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} u_{2,j+1} \\ u_{3,j+1} \\ \dots \\ u_{k,j+1} \\ \dots \\ u_{n-2,j+1} \\ u_{n-1,j+1} \end{pmatrix} = \begin{pmatrix} 2c_1 + u_{3,j} \\ u_{2,j} + u_{4,j} \\ \dots \\ u_{k-1,j} + u_{k+1,j} \\ \dots \\ u_{n-3,j} + u_{n-1,j} \\ u_{n-2,j} + 2c_2 \end{pmatrix}$$

It is obvious that this procedure has to be repeated in every time step, but the advandage is that it is **stable for every value of** r.

Convergence i

The finite difference schemes are used because their solutions approximate the solutions to certain PDEs.

The solution of the difference equations can be made to approximate the solution of the PDE to any desired accuracy.

Definition I:
A finite difference scheme (FDS) $\mathcal{L}_k^n u_k^n = \mathcal{G}_k^n$ approximating the PDE $Lv = F$ is a convergent scheme of order (p, q) if for any x and t, as $(k\Delta x, (n+1)\Delta t)$ converges to (x, t), $u_{n,k}$ converges to $v(x, t)$ as Δx and Δt converge to 0.

EXAMPLE

Let's consider the scheme used for the parabolic equations

$$\frac{u_{n,k+1} - u_{n,k}}{\Delta t} = \alpha^2 \frac{u_{n-1,k} - 2u_{n,k} + u_{n+1,k}}{\Delta x^2} \tag{47}$$

which can be written as:

$$u_{n,k+1} = (1 - 2r)\, u_{n,k} + r\,(u_{n-1,k} + u_{n+1,k}) \quad \text{with} \quad u_{k,0} = f(k\Delta x) \tag{48}$$

Convergence ii

we will show that converges to the solution of the initial value problem

$$v_t = \alpha^2 v_{xx}\,, \quad x \in \mathbb{R} \quad \text{and} \quad t > 0 \tag{49}$$

$$v(x,0) = f(x)\,, \quad x \in \mathbb{R} \tag{50}$$

SOLUTION

We will denote the exact solution of the initial value problem (49)-(50) as $v = v(x,t)$ and we will set

$$v\left(k\Delta x, n\Delta t\right) = u_{n,k} - z_{n,k}\,. \tag{51}$$

If we insert v into the following equation

$$\begin{aligned} v_t\left(k\Delta x, n\Delta t\right) \quad - \quad & \alpha^2 v_{xx}\left(k\Delta x, n\Delta t\right) = \frac{v_{n+1,k} - v_{n,k}}{\Delta t} \\ - \quad & \frac{\alpha^2}{\Delta x^2}\left(v_{n,k+1} - 2v_{n,k} + v_{n,k-1}\right) + \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2) \end{aligned} \tag{52}$$

and multiply with Δt then the left hand side will be zero, we see that $v_{n,k} = v\left(k\Delta x, n\Delta t\right)$ satisfies the equation

$$v_{n,k+1} = (1-2r)\,v_{n,k} + r\left(v_{n-1,k} + v_{n+1,k}\right) + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta t \Delta x^2) \tag{53}$$

Then by substracting (53) from (48), we see that $z_{n,k}$ satisfies

$$z_{n,k+1} = (1-2r)\, z_{n,k} + r\,(z_{n-1,k} + z_{n+1,k}) + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta t \Delta x^2) \tag{54}$$

for $0 < r \leq 1/2$ the coefficients on the right hand side are non-negative and

$$\begin{aligned} |z_{n,k+1}| &\leq (1-2r)\,|z_{n,k}| + r\,(|z_{n-1,k}| + |z_{n+1,k}|) + \mathcal{A}\left(\Delta t^2 + \Delta t \Delta x^2\right) \\ &\leq Z_n + \mathcal{A}\left(\Delta t^2 + \Delta t \Delta x^2\right) \quad \text{where} \quad Z_n = \sup_k\{|z_{n,k}|\} \end{aligned}$$

or

$$Z_{n+1} \leq Z_n + \mathcal{A}\left(\Delta t^2 + \Delta t \Delta x^2\right) \tag{55}$$

If we assume that the term $\mathcal{A}\left(\Delta t^2 + \Delta t \Delta x^2\right)$ is approximately constant, se can generate the sequence

$$\begin{aligned} Z_{n+1} &\leq Z_n + \mathcal{A}\left(\Delta t^2 + \Delta t \Delta x^2\right) \\ &\leq Z_{n-1} + 2\mathcal{A}\left(\Delta t^2 + \Delta t \Delta x^2\right) \\ &\dots \\ &\leq Z_0 + (n+1)\mathcal{A}\left(\Delta t^2 + \Delta t \Delta x^2\right) \end{aligned}$$

Convergence iv

Since $Z_0 = 0$, $|u_{n+1,k} - v(k\Delta x, (n+1)\Delta t)| \leq Z_{n+1}$ and $(n+1)\Delta t \to t$ we get

$$\begin{aligned} |u_{n+1,k} - v(k\Delta x, (n+1)\Delta t)| &\leq (n+1)\Delta t \mathcal{A}(\Delta t + \Delta x^2) \\ &\to 0 \quad \text{as} \quad \Delta t\,, \Delta x \to 0 \end{aligned} \tag{56}$$

Definition II

A finite difference scheme (FDS) $\mathcal{L}_k^n u_k^n = \mathcal{G}_k^n$ approximating the PDE $Lv = F$ is a convergent scheme at time t if as $(n+1)\Delta t \to t$,

$$||\mathbf{u}^{n+1} - \mathbf{v}^{n+1}|| = \mathcal{O}(\Delta x^p) + \mathcal{O}(\Delta t^q) \tag{57}$$

as $\Delta x \to 0$ and $\Delta t \to 0$.

Convergence (ODE)

Lets assume an ODE of the form

$$\frac{dy}{dx} = Ay \tag{58}$$

which has an obvious solution of the form $y = e^{Ax}$.
Euler's method gives the approximate solution via the recurrence relation:

$$y_{n+1} = y_n + hAy_n = (1 + hA)y_n \quad \text{for} \quad n = 0, 1, 2, ... \tag{59}$$

Thus if we use this relation n-times we will get:

$$y_{n+1} = (1 + hA)^{n+1} y_0 \quad \text{for} \quad n = 0, 1, 2, ... \tag{60}$$

But for small h we know that $1 + hA \approx e^{hA}$ thus

$$y_{n+1} = (1 + hA)^{n+1} y_0 \approx e^{(n+1)hA} y_0 = e^{(x_{n+1} - x_0)A} y_0 = e^{Ax_{n+1}} \tag{61}$$

where $h = (x_{n+1} - x_0)/(n + 1)$.

This means that the numerical solution for small h converges to the analytic solution: $y = e^{Ax}$.

Consistency i

Given a PDE $Lv = F$ and a FDS $\mathcal{L}_k^n u_k^n = \mathcal{G}_k^n$, we say that the FDS is **consistent** with the PDE if for any smooth function $\phi(x, t)$

$$L\phi - \mathcal{L}\phi \rightarrow 0 \quad \text{as} \quad \Delta t \, \Delta x \rightarrow 0 \tag{62}$$

Lax Equivalence Theorem For consistent schemes (in linear problems) convergence is equivalent to stability

Consider the wave equation defined by the operator $L = \frac{\partial}{\partial t} + a \frac{\partial}{\partial x}$ i.e. $Lu = u_t + au_x$

We will evaluate the FDS - FSFT

$$\mathcal{L}u = \frac{u_i^{n+1} - u_i^n}{\Delta t} + a \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

Consistency ii

By using Taylor expansion we get

$$u_i^{n+1} = u_i^n + u_t \Delta t + \frac{1}{2} u_{tt} \Delta t^2 + O(\Delta t^3) \tag{63}$$

$$u_{i+1}^n = u_i^n + u_x \Delta x + \frac{1}{2} u_{xx} \Delta x^2 + O(\Delta x^3) \tag{64}$$

This leads to

$$\mathcal{L}u = Lu + \frac{1}{2}\left(\Delta t \cdot u_{tt} + \Delta x \cdot u_{xx}\right) + O(\Delta t^2) + O(\Delta x^2)$$

Thus

$$\mathcal{L}u - Lu = \frac{1}{2}\left(\Delta t \cdot u_{tt} + \Delta x \cdot u_{xx}\right) + O(\Delta t^2) + O(\Delta x^2) \ \rightarrow \ 0 \text{ as } (\Delta x, \Delta t) \rightarrow 0$$

Stability - Initial Value Problems i

One interpretation of stability of difference scheme is that *for a stable difference scheme small errors in the initial conditions cause small errors in the solution*.

This definition allows the errors to grow, but limits them **to grow no faster than exponential.**

A difference scheme for solving a given (two level) initial-value problem is of the form

$$\mathbf{u}^{n+1} = Q\mathbf{u}^{n}\,, n \geq 0\,. \tag{65}$$

Definition The difference scheme (65) is said to be stable if there exist positive constants Δx_0 and Δt_0, and non-negative constants K and β so that

$$||\mathbf{u}^{n+1}|| \leq Ke^{\beta t}||\mathbf{u}^{0}|| \tag{66}$$

for $0 \leq t = (n+1)\Delta t, \quad 0 < \Delta x \leq \Delta x_0$ and $0 < \Delta t \leq \Delta t_0$.

Stability - Initial Value Problems ii

Another, more common, definition that is used is one that does not allow for exponential growth. Inequality (66) is replaced by

$$||\mathbf{u}^{n+1}|| \leq K||\mathbf{u}^0|| \tag{67}$$

which implies (66).

Where we define the **Euclidean norm**

$$||\mathbf{u}||_2 = \sqrt{\sum_{k=1}^{N} |u_k|^2}\,. \tag{68}$$

$$||\mathbf{u}||_{2,\Delta x} = \sqrt{\sum_{k=1}^{N} |u_k|^2 \Delta x}\,. \tag{69}$$

and the **sup-norm**

$$||\mathbf{u}||_\infty = \sup |u_k| \quad \text{for} \quad 1 \leq k \leq N \tag{70}$$

Preposition The difference scheme (65) is stable if and only if there exist positive constants Δx_0 and Δt_0, and non-negative constants K and β so that

$$||Q^{n+1}|| \leq Ke^{\beta t} \tag{71}$$

for $0 \leq t = (n+1)\Delta t$, $0 < \Delta x \leq \Delta x_0$ and $0 < \Delta t \leq \Delta t_0$.

Proof:

$$\mathbf{u}^{n+1} = Q\mathbf{u}^{n} = Q\left(Q\mathbf{u}^{n-1}\right) = Q^2\mathbf{u}^{n-1} = ... = Q^{n+1}\mathbf{u}^0$$

expression (66) can be written as

$$||\mathbf{u}^{n+1}|| = ||Q^{n+1}\mathbf{u}^0|| \leq Ke^{\beta t}||\mathbf{u}^0||$$

or

$$\frac{||Q^{n+1}\mathbf{u}^0||}{||\mathbf{u}^0||} \leq Ke^{\beta t}$$

by taking the supremum over both sides over all non-zero vectors $\mathbf{u}^0$ we get (71).

Stability - Initial Value Problems iv

Example Show that following difference scheme is stable with respect to the sup-norm.

$$u_k^{n+1} = (1-2r)\,u_k^n + r\left(u_{k+1}^n + u_{k-1}^n\right) \tag{72}$$

Solution We note that if $r \leq 1/2$

$$|u_k^{n+1}| \leq (1-2r)|u_k^n| + r|u_{k+1}^n| + r|u_{k-1}^n| \leq ||\mathbf{u}^n||_\infty$$

If we take the supremum over both sides (with respect to k), we get

$$||\mathbf{u}^{n+1}||_\infty \leq ||\mathbf{u}^n||_\infty$$

Hence the inequality (67) is satisfied with $K = 1$, or inequality (66) is satisfied with $K = 1$ and $\beta = 0$.

NOTES: For the stability of the scheme (72) we have required that $r \leq 1/2$.
In this case we say that the scheme is **conditionally stable**.
In the case where no-restrictions on the relationship between Δt and Δx are needed for stability, we say the scheme is **unconditionally stable**.

Stability - Initial Value Problems v

When solving initial-value problems a common analytical tool is to use the Fourier transform. For example, consider the problem:

$$v_t = v_{xx}\,, \quad \text{with} \quad v(x,0) = f(x)\,. \tag{73}$$

If we define the Fourier transform of v to be

$$\hat{v}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} v(x,t)dx \tag{74}$$

and take the Fourier transform of PDE (73) we get

$$\begin{aligned} \hat{v}_t(\omega, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} v_t(x,t)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} v_{xx}(x,t)dx \\ &= -\omega^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} v(x,t)dx = -\omega^2 \hat{v}(\omega, t). \end{aligned} \tag{75}$$

Here we integrated by parts twice.

Hence we see that the Fourier transform reduces the PDE to an ODE in transform space.

The technique then is to solve the ODE in transformed space and return our solution space.

We can return to our solution by using the inverse Fourier transform

$$v(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{v}(\omega, t) d\omega \tag{76}$$

The **discrete Fourier transform** can be written as:

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} v_k \tag{77}$$

for $\xi \in [-\pi, \pi]$.

Parseval's identity says that the norms of the function and its transform are equal in their respective spaces:

$$||v||_2 = ||\hat{v}||_2 \tag{78}$$

In a stability analysis we will use the Fourier transform and Perseval's Identity.

Recall that for the definition of stability we have used the inequality

$$||\mathbf{u}^{n+1}||_{2,\Delta x} \leq Ke^{\beta(n+1)\Delta t}||\mathbf{u}^0||_{2,\Delta x} \tag{79}$$

which can be now written as $(||\mathbf{u}||_{2,\Delta x} = \sqrt{\Delta x}||\mathbf{u}||_2 = \sqrt{\Delta x}||\hat{u}||_2)$

$$||\hat{u}^{n+1}||_2 \leq Ke^{\beta(n+1)\Delta t}||\hat{u}^0||_2 \tag{80}$$

then the same K and β will also satisfy (74).

When inequality (80) holds, we say that the sequence $\{\hat{u}^n\}$ *is* **stable** *in the transform space and this applies also to sequence* $\{u^n\}$.

Stability - Initial Value Problems - Example (****)

Analyze the stability of the difference scheme

$$u_k^{n+1} = ru_{k-1}^n + (1-2r)u_k^n + ru_{k+1}^n, \quad -\infty < k < \infty \tag{81}$$

where $r = \nu\Delta t/\Delta x^2$.

If we take the discrete Fourier transform of both sides of equation (81)

$$\begin{aligned}
\hat{u}^{n+1}(\xi) &= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} u_k^{n+1} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} \left[ru_{k-1}^n + (1-2r)u_k^n + ru_{k+1}^n\right] \\
&= r\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} u_{k-1}^n + (1-2r)\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} u_k^n \\
&\quad + r\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} u_{k+1}^n \\
&= r\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} u_{k-1}^n + (1-2r)\hat{u}^n(\xi) + r\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} u_{k+1}^n
\end{aligned}$$

By making the change of variables $m = k \pm 1$ we get,

$$\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} u_{k\pm 1}^{n} = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-i(m\mp 1)\xi} u_m^n \tag{82}$$

$$= e^{\pm i\xi} \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} u_m^n = e^{\pm i\xi} \hat{u}(\xi).$$

Then we get

$$\begin{aligned} \hat{u}^{n+1}(\xi) &= re^{-i\xi}\hat{u}^n(\xi) + (1-2r)\hat{u}^n(\xi) + re^{i\xi}\hat{u}^n(\xi) \\ &= \left[re^{-i\xi} + (1-2r) + re^{i\xi}\right] \hat{u}^n(\xi) \\ &= [2r\cos\xi + (1-2r)]\, \hat{u}^n(\xi) \\ &= \left[1 - 4r\sin^2(\xi/2)\right] \hat{u}^n(\xi) \end{aligned} \tag{83}$$

The term

$$\rho(\xi) = 1 - 4r\sin^2\frac{\xi}{2} \tag{84}$$

is called **the symbol** of the difference scheme (81).

Thus by taking the discrete Fourier transform, we get rid of the x derivatives and simplify the equation.

If we apply the result (83) $n+1$ times, we get

$$\hat{u}^{n+1}(\xi) = \left(1 - 4r\sin^2(\xi/2)\right)^{n+1} \hat{u}^0(\xi) \tag{85}$$

Thus if we restrict r so that

$$|1 - 4r\sin^2(\xi/2)| \leq 1 \tag{86}$$

Then we can choose $K = 1$ and $\beta = 0$ and satisfy inequality (67).

Thus our scheme will be stable if

$$-1 \leq 1 - 4r\sin^2(\xi/2) \leq 1 \tag{87}$$

or

$$4r\sin^2(\xi/2) \leq 2 \tag{88}$$

which is true for $r \leq 1/2$.

This is the necessary and sufficient condition for convergence of the scheme (81).

Stability - Initial Value Problems - Examples

For the hyperbolic PDE

$$u_t + au_x = 0 \tag{89}$$

study the stability of the following schemes ($|R| = |a|\Delta t/\Delta x \leq 1$)

$$\begin{aligned} u_k^{n+1} &= u_k^n - R\left(u_{k+1}^n - u_k^n\right) \quad \text{(FTFS)} && (90)\\ u_k^{n+1} &= u_k^n - R\left(u_k^n - u_{k-1}^n\right) \quad \text{(FTBS)} && (91)\\ u_k^{n+1} &= u_k^n - \frac{R}{2}\left(u_{k+1}^n - u_{k-1}^n\right) \quad \text{(FTCS)} && (92)\end{aligned}$$

The following abbreviations might be used later:

$$\begin{aligned} \delta_+ u_k &= u_{k+1} - u_k && (93)\\ \delta_- u_k &= u_k - u_{k-1} && (94)\\ \delta_0 u_k &= u_{k+1} - u_{k-1} && (95)\\ \delta^2 u_k &= u_{k+1} - 2u_k + u_{k-1} && (96)\end{aligned}$$

Stability : Example I

For the hyperbolic PDE

$$u_t + au_x = 0, \quad \text{with} \quad a < 0 \tag{97}$$

study the stability of the following scheme (FTFS) ($|R| = |a|\Delta t/\Delta x \leq 1$)

$$u_k^{n+1} = (1 + R)u_k^n - Ru_{k+1}^n \tag{98}$$

We begin by taking the discrete Fourier transform of the scheme

$$\begin{aligned} \hat{u}^{n+1} &= (1 + R)\hat{u}^n - Re^{i\xi}\hat{u}^n \\ &= [(1 + R) - R(\cos\xi + i\sin\xi)]\,\hat{u}^n \end{aligned} \tag{99}$$

Then because the symbol is complex and is given by

$$\rho(\xi) = (1 + R) - R\cos\xi - iR\sin\xi \tag{100}$$

we must bound the **magnitude** of ρ by 1 to satisfy the inequality (80) (with $K = 1$ and $\beta = 0$). Thus we calculate

$$|\rho(\xi)|^2 = (1 + R)^2 - 2R(1 + R)\cos\xi + R^2$$

Then we determine the maximum and minimum value of $|\rho(\xi)|^2$ for $\xi \in [-\pi, \pi]$ and we find that we have a potential maximum at $\xi = 0$ and $\xi = \pm\pi$.

If we evaluate $|\rho(\xi)|$ at these values, we see that

$$|\rho(0)| = 1 \quad \text{and} \quad |\rho(\pm\pi)| = |1 + 2R|$$

To bound $|\rho(\pm\pi)|$ by 1, we require that R satisfies $-1 \leq 1 + 2R \leq 1$.

Then since $1 + 2R \leq 1$ since $R < 0$ we see that the scheme is **conditionally stable** with condition $R \geq -1$.

Stability : Example II

For the hyperbolic PDE

$$u_t + au_x = 0, \quad \text{with} \quad a < 0 \tag{101}$$

study the stability of the following scheme (FTCS) ($|R| = |a|\Delta t/\Delta x \leq 1$)

$$u_k^{n+1} = u_k^n - \frac{R}{2}\delta_0 u_k^n \tag{102}$$

We begin by taking the discrete Fourier transform of the scheme

$$\hat{u}^{n+1} = \hat{u}^n - \frac{R}{2}\left(e^{i\xi} - e^{-i\xi}\right)\hat{u}^n = \left[1 - iR\sin\xi\right]\hat{u}^n \tag{103}$$

Thus the symbol is

$$|\rho|^2 = 1 + R^2 \sin\xi^2 \geq 1$$

So the difference scheme (102) is **unstable** for all $R \neq 0$.

Lax-Wendroff Scheme i

For the PDE $u_t + au_x = 0$ we can write:

$$u_{tt} = (-au_x)_t = -au_{xt} = -a(u_t)_x = -a(-au_x)_x = a^2 u_{xx} \tag{104}$$

Thus since

$$\begin{aligned}
u_k^{n+1} &= u_k^n + (u_t)_k^n \Delta t + (u_{tt})_k^n \frac{\Delta t^2}{2} + O\left(\Delta t^3\right) \\
&= u_k^n + (-au_x)_k^n \Delta t + (a^2 u_{xx})_k^n \frac{\Delta t^2}{2} + O\left(\Delta t^3\right) \\
&= u_k^n - a\left(\frac{u_{k+1}^n - u_{k-1}^n}{2\Delta x} + O(\Delta x^2)\right)\Delta t \\
&\quad + a^2\left(\frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2} + O(\Delta x^2)\right)\frac{\Delta t^2}{2} + O\left(\Delta t^3\right)
\end{aligned}$$

i.e. we approximate the PDE $u_t + au_x = 0$ with the difference scheme

$$u_k^{n+1} = u_k^n - \frac{R}{2}\delta_0 u_k^n + \frac{R^2}{2}\delta^2 u_k^n \text{ with } R = a\Delta t/\Delta x. \tag{105}$$

Lax-Wendroff Scheme ii

The Lax-Wendorff scheme is $O(\Delta t^2) + O(\Delta x^2)$ and its **symbol** is (why?)

$$\rho(\xi) = 1 - 2R^2 \sin^2(\xi/2) - iR \sin \xi \tag{106}$$

Since

$$|\rho(\xi)|^2 = 1 - 4R^2 \sin^4(\xi/2) + 4R^4 \sin^4(\xi/2) \tag{107}$$

if we differentiate with respect to ξ we can find the critical values at $\xi = \pm\pi$ and 0. For which we get that

$$|\rho(0)|^2 = 1 \quad \text{and} \quad |\rho(\pm\pi)|^2 = |\rho(\pi)|^2 = (1 - 2R^2)^2. \tag{108}$$

Then for $R^2 \leq 1$ we get $(1 - 2R^2)^2 \leq 1$ and thus the Lax-Wendroff scheme is conditionally stable for

$$|R| = |a| \frac{\Delta t}{\Delta x} \leq 1\,. \tag{109}$$

and it is **2nd order in both time and space**.

- Notice that the stability condition is independent of the sign of a.

Lax-Friedrichs Scheme

It can be derived from the unstable FTCS $O(Dt, Dx^2)$ scheme:

$$u_k^{n+1} = u_k^n - \frac{R}{2}\left(u_{k+1}^n - u_{k-1}^n\right) \tag{110}$$

by replacing u_k^n with its spatial average: $u_k^n = (u_{k+1}^n + u_{k-1}^n)/2$.

$$u_k^{n+1} = \frac{1}{2}\left(u_{k+1}^n + u_{k-1}^n\right) - \frac{R}{2}\left(u_{k+1}^n - u_{k-1}^n\right) \tag{111}$$

which is stable for $|R| \leq 1$ (WHY?).

PROBLEM: Show that the above writing corresponds to the descritization of the following PDE:

$$u_t + au_x = \frac{\Delta x^2}{2\Delta t} u_{xx} \tag{112}$$

The last term acts as **numerical dissipation**.

Implicit Schemes

For the hyperbolic PDE

$$u_t + au_x = 0 \tag{113}$$

we have studied the following **explicit schemes**

$$
\begin{aligned}
u_k^{n+1} &= u_k^n - R\left(u_{k+1}^n - u_k^n\right) \quad \textbf{(FTFS)} && (114)\\
u_k^{n+1} &= u_k^n - R\left(u_k^n - u_{k-1}^n\right) \quad \text{(FTBS)} && (115)\\
u_k^{n+1} &= u_k^n - \frac{R}{2}\left(u_{k+1}^n - u_{k-1}^n\right) \quad \textbf{(FTCS)} && (116)
\end{aligned}
$$

These schemes can be written in the following form:

$$
\begin{aligned}
(1-R)u_k^{n+1} + Ru_{k+1}^{n+1} &= u_k^n \quad \textbf{(BTFS)} && (117)\\
-Ru_{k-1}^{n+1} + (1+R)u_k^{n+1} &= u_k^n \quad \text{(BTBS)} && (118)\\
-\frac{R}{2}u_{k-1}^{n+1} + u_k^{n+1} + \frac{R}{2}u_{k+1}^{n+1} &= u_k^n \quad \textbf{(BTCS)} && (119)
\end{aligned}
$$

Implicit Schemes - Stability I

We will study the stability of the BFTS scheme (117). By taking the discrete Fourier transform we get

$$(1-R)\hat{u}^{n+1} + Re^{i\xi}\hat{u}^{n+1} = \hat{u}^n \tag{120}$$

Thus the symbol will be

$$\rho(\xi) = \frac{1}{1 - R + R\cos\xi + iR\sin\xi} \tag{121}$$

and the magnitude squared of the symbol is:

$$|\rho(\xi)|^2 = \frac{1}{1 - 4R\sin^2\xi/2 + 4R^2\sin^2\xi/2} \tag{122}$$

Since $R \leq 0$ $(a < 0)$ implies that:

$$1 - 4R\sin^2\xi/2 + 4R^2\sin^2\xi/2 = 1 - 4R(1-R)\sin^2\xi/2 \geq 1 \tag{123}$$

i.e. $|\rho(\xi)|^2 \leq 1$.

NOTE that for $0 < R < 1$, the difference scheme is **unstable**.

Hense, we see that the difference scheme (117) is stable if and only if $R \leq 0$ or $R \geq 1$.

Implicit Schemes - Stability II

For the difference scheme (119):

$$-\frac{R}{2}u_{k-1}^{n+1} + u_k^{n+1} + \frac{R}{2}u_{k+1}^{n+1} = u_k^n \tag{124}$$

the symbol is (how?)

$$\rho(\xi) = \frac{1}{1 + iR\sin\xi} \tag{125}$$

Then since

$$|\rho(\xi)|^2 = \frac{1}{1 + R^2\sin^2\xi} \leq 1 \tag{126}$$

the difference scheme (124) is **unconditionally stable**, even though its explicit counterpart is unstable!

PROBLEM: Study the stability of the difference scheme (118).

The explicit Lax - Wendroff scheme was 2nd order both in space and time for the wave equation $v_t + av_x = 0$. We will try to examine its implicit version.The scheme that follows serves the purpose:

$$-\frac{R}{2}(1+R)\,u_{k-1}^{n+1} + \left(1+R^2\right)u_k^{n+1} + \frac{R}{2}(1-R)\,u_{k+1}^{n+1} = u_k^n \tag{127}$$

If we expand this equation about $(k, n+1)$ we see that the equation is still end order both in space and time. By taking the Fourier transform we get the symbol of the operator

$$\rho(\xi) = \frac{1}{1 + 2R^2\sin^2\xi/2 + \mathrm{i}R\sin\xi} \tag{128}$$

which leads to

$$|(\rho(\xi)|^2 = \frac{1}{\left(1 + 2R^2\sin^2\xi/2\right)^2 + R^2\sin^2\xi} \geq 1 \tag{129}$$

which means that the implicit Lax-Wendroff scheme is unconditionally stable

Implicit Schemes - Lax-Wendroff ii

NOTE: the stability condition is independent of the sign of a, this is a big advantage when the sign of a, may change through out the domain. This can happen if a is function of the independent variables of the problem or its sign might not be known a priori.

Implicit Schemes - Crank - Nicolson i

The implicit Crank-Nicolson scheme for the parabolic PDEs was already presented and without proof it was mentioned that it was unconditionally stable. Here we will study with the same implicit approach the stability of the wave equation $v_t + av_x = 0$:

$$-\frac{R}{4}u_{k-1}^{n+1} + u_k^{n+1} + \frac{R}{4}u_{k+1}^{n+1} = \frac{R}{4}u_{k-1}^{n} + u_k^{n} - \frac{R}{4}u_{k+1}^{n} \tag{130}$$

Here as for the case of parabolic PDEs the spatial difference has been averaged at the nth and $(n+1)$st time slices.

The scheme can be considered as a half explicit step (with time step $\Delta t/2$) and a half implicit step (with time step $\Delta t/2$).

If we expand the scheme about the point $(k, n+1/2)$, it is easy to see (HOW?) that the scheme is of order $\mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$. By taking the Fourier transform we get the symbol of the operator

$$\rho(\xi) = \frac{1 - \mathrm{i}R/2\sin\xi}{1 + \mathrm{i}R/2\sin\xi} \quad \Rightarrow \quad |(\rho(\xi)|^2 = 1 \tag{131}$$

This is quite tricky since any small perturbation (e.g. round-off errors) can make the magnitude greater than one and result in instability.

NOTE I: the stability condition is independent of the sign of a.

NOTE II : This is also a proof that the stability results obtained for parabolic PDEs they cannot be transfer automatically to hyperbolic PDES.

1D form of the wave equation i

We have studied the numerical solution of the wave equation earlier. Now we will demonstrate how one can treat it with the schemes that we discussed earlier. The equation is:

$$u_{tt} = c^2 u_{xx} \quad \text{for} \quad 0 < x < a \quad \text{and} \quad 0 < t < b \tag{132}$$

Then we can write it as a system of 1st order PDEs. We set:

$$h = c\, u_x \quad \text{and} \quad f = u_t \tag{133}$$

and we get:

$$\begin{aligned} h_t &= c\, f_x \\ f_t &= c\, h_x \\ u_t &= f \end{aligned} \tag{134}$$

In vector notation this can be written as:

$$\vec{U}_t + \mathbf{Q}\vec{U}_x = 0 \tag{135}$$

where

$$\mathbf{Q} = -\begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \quad \text{and} \quad \vec{U} = \begin{pmatrix} h \\ f \end{pmatrix} \tag{136}$$

Lax-Wendroff Scheme application

We can use the following system of evolution equations

$$\begin{aligned} h_j^{n+1} &= h_j^n + \frac{R}{2}\left[\left(f_{j+1}^n - f_{j-1}^n\right) + R\left(h_{j+1}^n - 2h_j^n + h_{j-1}^n\right)\right] + \mathcal{O}\left(\Delta x^2\right) & (137)\\ f_j^{n+1} &= f_j^n + \frac{R}{2}\left[\left(h_{j+1}^n - h_{j-1}^n\right) + R\left(f_{j+1}^n - 2f_j^n + f_{j-1}^n\right)\right] + \mathcal{O}\left(\Delta x^2\right) & (138)\end{aligned}$$

and the ODE for u:

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{2}\left(f_j^{n+1} + f_j^n\right) + \mathcal{O}\left(\Delta x^2\right) \tag{139}$$

This is a stable scheme and can be easily extended to systems of wave equations.

2D Hyperbolic Equations i

Consider the PDE

$$u_t + au_x + bu_y = 0 \tag{140}$$

with the initial condition $u(x, y, 0) = f(x, y)$. Then a will be the speed of propagation in the x-direction and b will be the speed of propagation in the y-direction.

An obvious, but unfortunately unconditionally unstable scheme is:

$$\begin{aligned} u_{jk}^{n+1} &= u_{jk}^n - R_x \left(u_{j+1k}^n - u_{j-1k}^n\right) - R_y \left(u_{jk+1}^n - u_{jk-1}^n\right) \\ &= \left(1 - R_x\delta_{x0} - R_y\delta_{y0}\right) u_{jk}^n \end{aligned} \tag{141}$$

where $R_x = a\Delta t/\Delta x$ and $R_y = b\Delta t/\Delta y$.

2D Hyperbolic Equations ii

A conditionally stable scheme is:

$$u_{jk}^{n+1} = \left(1 - R_x \delta_{x-} - R_y \delta_{y-}\right) u_{jk}^{n} \tag{142}$$

STABILITY: If we take a 2-dimensional Fourier transform of eqn (142) we get:

$$\hat{u}^{n+1} = \left[1 - R_x \left(1 - e^{-i\xi}\right) - R_y \left(1 - e^{-i\eta}\right)\right] \hat{u}^{n} \tag{143}$$

So the symbol of the difference scheme (142) is given by

$$\rho(\xi, \eta) = 1 - R_x \left(1 - e^{-i\xi}\right) - R_y \left(1 - e^{-i\eta}\right) \tag{144}$$

and

$$|\rho(\xi, \eta)|^2 = \left[1 - 2R_x \sin^2(\xi/2) - 2R_y \sin^2(\eta/2)\right]^2 + \left[R_x \sin \xi + R_y \sin \eta\right]^2 .$$

2D Hyperbolic Equations iii

By differentiating $|\rho|^2$ with respect to ξ and η and setting the derivatives equal to zero, we find that there are potential maximums at $(\pm\pi, \pm\pi)$, $(\pm\pi, 0)$, $(0, \pm\pi)$ and $(0, 0)$. It is also easy to find that

$$|\rho(0,0)| = 1\,, \quad |\rho(\pm\pi,0)| = (1-2R_x)^2\,, \quad |\rho(0,\pm\pi)| = (1-2R_y)^2$$

and

$$|\rho(\pm\pi,\pm\pi)| = (1-2R_y-2R_y)^2\,.$$

- The condition $(1-2R_x)^2 \leq 1$ requires that $0 \leq R_x \leq 1$.
- The condition $(1-2R_y)^2 \leq 1$ requires that $0 \leq R_y \leq 1$.
- The condition $(1-2R_x-2R_y)^2 \leq 1$ requires that $0 \leq R_x + R_y \leq 1$.

CONCLUSION: Therefore, we find that the difference scheme (142) is 1st order accurate in space and time, and conditionally stable with condition $0 \leq R_x + R_y \leq 1$, for $R_x \geq 0$ and $R_y \geq 0$.

The 2D Lax-Friedrichs scheme for the approximate solution of (140) is:

$$\begin{aligned} u_{jk}^{n+1} &= \frac{1}{4}\left(u_{j+1k}^{n} + u_{j-1k}^{n} + u_{jk+1}^{n} + u_{jk-1}^{n}\right) \\ &- \frac{R_x}{2}\delta_{x0} u_{jk}^{n} - \frac{R_y}{2}\delta_{y0} u_{jk}^{n} \end{aligned} \tag{145}$$

STABILITY: we compute the discrete Fourier transform to obtain the symbol for the scheme

$$\rho(\xi,\eta) = \frac{1}{2}\left(\cos\xi + \cos\eta\right) - i\left(R_x \sin\xi + Ry\sin\eta\right) \tag{146}$$

Then the expression $|\rho(\xi,\eta)|^2$ can be written as

$$\begin{aligned} |\rho(\xi,\eta)|^2 &= 1 - \left(\sin^2\xi + \sin^2\eta\right)\left[1/2 - \left(Rx^2 + R_y^2\right)\right] \\ &- \frac{1}{4}\left(\cos\xi - \cos\eta\right)^2 - \left(R_x\sin\eta - R_y\sin\xi\right)^2 \end{aligned} \tag{147}$$

2D-Wave Equation: Lax-Friedrichs scheme ii

Since the last two terms in the equation are negative, we have:

$$|\rho(\xi,\eta)|^2 \quad = \quad 1-\left(\sin^2\xi+\sin^2\eta\right)\left[1/2-\left(R_x^2+R_y^2\right)\right] \tag{148}$$

If $\left[1/2-\left(R_x^2+R_y^2\right)\right]\geq 0$, then $|\rho(\xi,\eta)|\leq 1$.

Hence if

$$R_x^2+R_y^2\leq\frac{1}{2} \tag{149}$$

the difference scheme is **stable**.

NOTE: The stability condition (149) is very restrictive. It is not obvious that we can always find a scheme with stability condition the same as the CFL condition, but at least what we should try to do.

2D-Wave Equation: ADI Schemes

$$u_t = Au = -au_x - bu_y \quad \text{with} \quad u(x, y, 0) = f(x, y) \tag{150}$$

We begin by considering a locally 1D scheme for solving the above PDE

$$\left(1 + \frac{R_x}{2}\delta_{x0}\right) u_{jk}^{n+1/2} = u_{jk}^{n} \tag{151}$$

$$\left(1 + \frac{R_y}{2}\delta_{y0}\right) u_{jk}^{n+1} = u_{jk}^{n+1/2} \tag{152}$$

STABILITY: The symbol is:

$$\rho(\xi, \eta) = \frac{1}{(1 + iR_x \sin\xi)(1 + iR_y \sin\eta)} \tag{153}$$

Then since

$$|\rho(\xi, \eta)|^2 = \frac{1}{\left(1 + R_x^2 \sin^2\xi\right)\left(1 + R_y^2 \sin^2\eta\right)} \tag{154}$$

it is clear the $0 \leq |\rho(\xi, \eta) \leq 1$ and the difference scheme (151)-(152) is **unconditionally stable** and $O(\Delta t) + O(\Delta x^2) + O(\Delta y^2)$ order accurate.

2D-Wave Equation: ADI Schemes - Beam-Warming i

An ADI scheme for the Crank-Nicolson can be written as **HOW??**:

$$\left(1+\frac{R_x}{4}\delta_{x0}\right)\left(1+\frac{R_y}{4}\delta_{y0}\right)u_{jk}^{n+1}=\left(1-\frac{R_x}{4}\delta_{x0}\right)\left(1-\frac{R_y}{4}\delta_{y0}\right)u_{jk}^{n} \tag{155}$$

The above scheme is referred to as the **Beam-Warming scheme** and is most often written as

$$\begin{aligned}\left(1+\frac{R_x}{4}\delta_{x0}\right)u_{jk}^{*} &= \left(1-\frac{R_x}{4}\delta_{x0}\right)\left(1-\frac{R_y}{4}\delta_{y0}\right)u_{jk}^{n} &\quad (156)\\ \left(1+\frac{R_y}{4}\delta_{y0}\right)u_{jk}^{n+1} &= u_{jk}^{*} &\quad (157)\end{aligned}$$

The symbol of the Beam-Warming scheme is

$$\rho(\xi,\eta)=\frac{(1-i\frac{R_x}{2}\sin\xi)(1-i\frac{R_y}{2}\sin\eta)}{(1+i\frac{R_x}{2}\sin\xi)(1+i\frac{R_y}{2}\sin\eta)} \tag{158}$$

Thus we see that (how?) $|\rho(\xi,\eta)|^2=1$ for all $\xi,\eta\in[-\pi,\pi]$ and the scheme is unconditionally stable and 2nd order.

2D-scheme for the wave equation

Let's consider the equation

$$u_t = Au = (A_1 + A_2)\, u \tag{159}$$

e.g. $Au = -au_x - bu_y$ with $A_1 u = -au_x$ & $A_2 u = -bu_y$

By using 1st order approximation to the time derivative we get

$$\begin{aligned} u^{n+1} &= u^n + \Delta t\, A\, u^n + O(\Delta t^2) \\ &= (1 + \Delta t\, A_1 + \Delta t\, A_2)u^n + O(\Delta t^2) \\ &= (1 + \Delta t\, A_1)(1 + \Delta t\, A_2)u^n - \Delta t^2\, A_1\, A_2\, u^n + O(\Delta t^2) \end{aligned} \tag{160}$$

by dropping terms of order Δt^2 we get the approximate scheme

$$u^{n+1} = (1 + \Delta t\, A_1)(1 + \Delta t\, A_2)u^n \tag{161}$$

or

$$u^{n+1/2} = (1 + \Delta t\, A_2)u^n \tag{162}$$

$$u^{n+1} = (1 + \Delta t\, A_1)u^{n+1/2} \tag{163}$$

2D-scheme for the wave equation ($\sim$ Lax-Wendroff)

Let's assume the equation

$$u_t = Au = -au_x - bu_y \tag{164}$$

with with $A_1 u = -au_x$ & $A_2 u = -bu_y$.

If we approximate the A_1 and A_2 by the **1-D Lax-Wendorff scheme**, we get

$$u_{jk}^{n+1/2} = u_{jk}^n - \frac{R_y}{2}\delta_{y0} u_{jk}^n + \frac{R_y^2}{2}\delta_y^2 u_{jk}^n \tag{165}$$

$$u^{n+1} = u_{jk}^{n+1/2} - \frac{R_x}{2}\delta_{x0} u_{jk}^{n+1/2} + \frac{R_x^2}{2}\delta_x^2 u_{jk}^{n+1/2} \tag{166}$$

It is obvious that the above scheme is 2nd order in time.

By following the standard analysis we can prove:

- it is conditionally stable if $\max\{|R_x|, |R_y|\} \leq 1$.
- and of order $O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2)$.

2D Parabolic PDEs

Let's consider the 2-D parabolic equation:

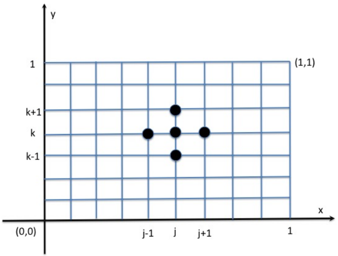
$$u_t = \nu\left(u_{xx} + u_{yy}\right) + F(x, y, t) \tag{167}$$

with $u(x, y, t) = g(x, y, t)$ on ∂R and $u(x, y, 0) = f(x, y)$.
The scheme will be

$$\frac{u_{jk}^{n+1} - u_{jk}^{n}}{\Delta t} = \frac{\nu}{\Delta x^2}\delta_x^2\, u_{jk}^{n} + \frac{\nu}{\Delta y^2}\delta_y^2\, u_{jk}^{n} + F_{jk}^{n} \tag{168}$$

which can be written in the explicit form ($r_x = \nu/\Delta x^2$ and $r_y = \nu/\Delta y^2$) :

$$u_{jk}^{n+1} = u_{jk}^{n} + \left(r_x\delta_x^2 + r_y\delta_y^2\right)u_{jk}^{n} + \Delta t F_{jk}^{n} \tag{169}$$



The symbol for equation (169) is

$$\begin{aligned} \rho &= 1 + 2r_x(\cos\xi - 1) + 2r_y(\cos\eta - 1) \\ &= 1 - 4r_x\sin^2(\xi/2) - 4r_y\sin^2(\eta/2) \end{aligned} \tag{170}$$

It is easy to see that :

The **maximum** of $\rho = 1$ occurs at $(\xi, \eta) = (0, 0)$

The **minimum** of $\rho = 1 - 4r_x - 4r_y$ occurs at $(\xi, \eta) = (\pi, \pi)$

The requirement that $\rho \geq -1$ yields the stability condition

$$r_x + r_y \leq \frac{1}{2} \tag{171}$$

Hence the difference scheme (169) is conditional stable.

For $\Delta x = \Delta y$ the condition for stability becomes $r \leq 1/4$.

2D Parabolic PDEs : Implicit scheme

The following scheme is the 2-D Crank-Nicolson implicit scheme for approximating the PDE (167)

$$\begin{aligned}\left(1-\frac{r_x}{2}\delta_x^2-\frac{r_y}{2}\delta_y^2\right)u_{jk}^{n+1} &= \left(1+\frac{r_x}{2}\delta_x^2+\frac{r_y}{2}\delta_y^2\right)u_{jk}^{n} \\ &+ \frac{\Delta t}{2}\left(F_{jk}^{n}+F_{jk}^{n+1}\right) \qquad (172)\end{aligned}$$

STABILITY: The symbol for the above difference scheme will be

$$\rho(\xi,\eta)=\frac{1-2r_x\sin^2(\xi/2)-2r_y\sin^2(\eta/2)}{1+2r_x\sin^2(\xi/2)+2r_y\sin^2(\eta/2)} \qquad (173)$$

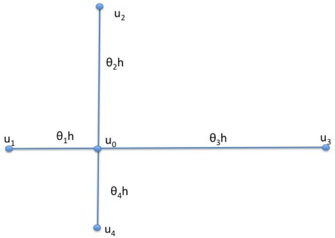
Since for any $r \geq 0$

$$\left|\frac{1-r}{1+r}\right| \leq 1$$

the difference scheme (172) is **unconditionally stable**.

Irregular Regions & Non-Rectangular Grids i

When the uniform grid does not fit to the boundaries, we must treat differently the points near the boundary. Consider 5 points with non-uniform spacing, with distances $\theta_1 h$, $\theta_2 h$, $\theta_3 h$, $\theta_4 h$ from the central point.



Then the derivatives can be approximated as

$$\left(\frac{\partial u}{\partial x}\right)_{1-0} = \frac{u_0 - u_1}{\theta_1 h} \tag{174}$$

$$\left(\frac{\partial u}{\partial x}\right)_{0-3} = \frac{u_3 - u_0}{\theta_3 h} \tag{175}$$

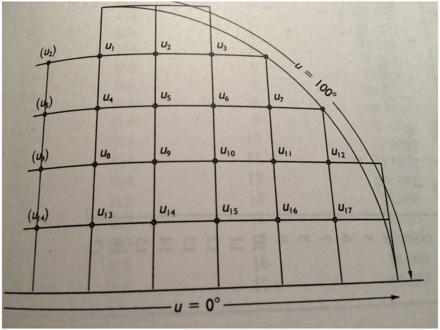
$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{(u_3 - u_0)/\theta_3 h - (u_0 - u_1)/\theta_1 h}{(\theta_1 + \theta_3)h/2} \\
&= \frac{2}{h^2}\left[\frac{u_1 - u_0}{\theta_1(\theta_1 + \theta_3)} + \frac{u_3 - u_0}{\theta_3(\theta_2 + \theta_3)}\right] + O(h) \qquad (176)
\end{aligned}
$$

$$
\frac{\partial^2 u}{\partial y^2} = \frac{2}{h^2}\left[\frac{u_2 - u_0}{\theta_2(\theta_2 + \theta_4)} + \frac{u_4 - u_0}{\theta_4(\theta_2 + \theta_4)}\right] + O(h) \qquad (177)
$$

Combining we get:

$$
\begin{aligned}
\nabla^2 u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
&= \frac{2}{h^3}\left[\frac{u_1}{\theta_1(\theta_1 + \theta_3)} + \frac{u_2}{\theta_2(\theta_2 + \theta_4)} + \frac{u_3}{\theta_3(\theta_1 + \theta_3)} + \frac{u_4}{\theta_4(\theta_2 + \theta_4)}\right] \\
&- \frac{2}{h^3}\left(\frac{1}{\theta_1\theta_3} + \frac{1}{\theta_2\theta_4}\right) u_0 \qquad (178)
\end{aligned}
$$

EXAMPLE

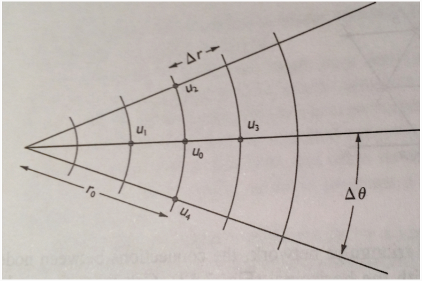


(u_2)
u_1
u_2
u_3
$u = 100°$
(u_5)
u_4
u_5
u_6
u_7
(u_9)
u_8
u_9
u_{10}
u_{11}
u_{12}
(u_{14})
u_{13}
u_{14}
u_{15}
u_{16}
u_{17}
$u = 0°$

Irregular Regions & Non-Rectangular Grids iv

For circular regions, one may derive a finite-difference approximation to the Laplacian in polar coordinates.

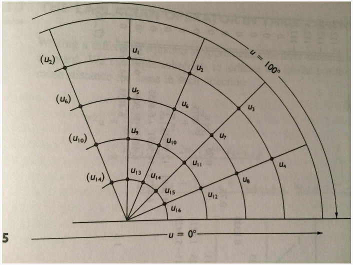
$$\begin{aligned}
\nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
&= \frac{u_3 - 2u_0 + u_1}{(\Delta r)^2} + \frac{1}{r_0}\frac{u_3 - u_1}{2\Delta r} + \frac{1}{r_0^2}\frac{u_2 - 2u_0 + u_4}{(\Delta\theta)^2} \qquad (179)
\end{aligned}$$



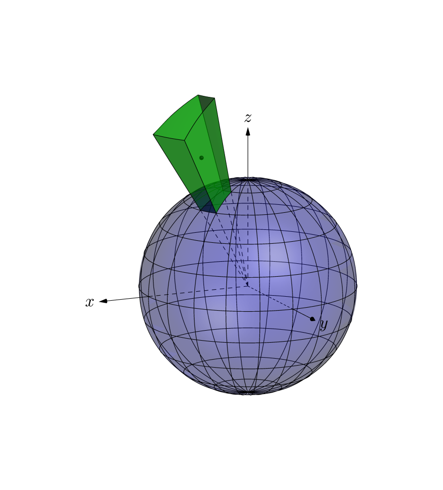
Irregular Regions & Non-Rectangular Grids v

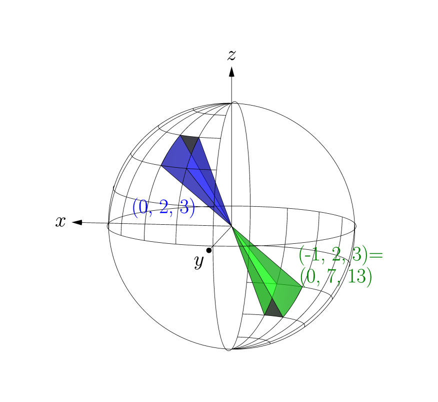
For circular regions, one may derive a finite-difference approximation to the Laplacian in polar coordinates.

$$
\begin{aligned}
\nabla^2 u \quad = \quad & \frac{1}{(\Delta r)^2}\left[\left(1-\frac{\Delta r}{2r_0}\right)u_1+\left(1+\frac{\Delta r}{2r_0}\right)u_3+\left(\frac{\Delta r}{r_0\Delta\theta}\right)^2(u_2+u_4)\right] \\
- \quad & \frac{2}{(\Delta r)^2}\left(1+\left(\frac{\Delta r}{r_0\Delta\theta}\right)^2\right)u_0 = 0 \qquad (180)
\end{aligned}
$$



Spherical Grids





Parabolic Eqns in Cylindrical & Spherical Polar Coordinates i

The heat conduction equation in cylindrical coordinates (r, θ, z) is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \tag{181}$$

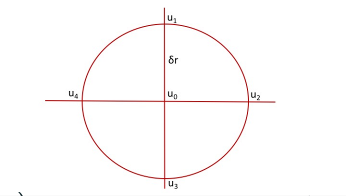
For simplicity we may assume that u is independent of z i.e.

$$\frac{\partial u}{\partial t} = \nabla^2 u \equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{182}$$

At $r = 0$ the right hand side appears to contain singularities, which can be approximated as follows:

Parabolic Eqns in Cylindrical & Spherical Polar Coordinates ii

Construct a circle of radius δr as in the figure then we name the value of the origin by u_0, and we write:



$$\nabla^2 u = \frac{4(u_m - u_0)}{(\delta r)^2} + O(\delta r^2) \quad \text{for} \quad u_m = \frac{u_1 + u_2 + u_3 + u_4}{4} \tag{183}$$

We may rotate the axis by $\delta\theta$ and get another prediction for u_m, the best mean value available is given by adding all values and dividing by their number.

When a 2D problem in cylindrical coordinates possesses circular symmetry $\frac{\partial^2 u}{\partial \theta^2} = 0$ we get the simpler form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\,. \tag{184}$$

Parabolic Equations in Spherical Polar Coordinates i

A similar problem arises at $r = 0$ with spherical polar coordinates in which the Laplacian operator assumes the form:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{\cot\theta}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} \tag{185}$$

By the same argument the previous equation can be replaced at $r = 0$ by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u^2}{\partial z^2} \tag{186}$$

which can be approximated by

$$\nabla^2 u = \frac{6(u_m - u_0)}{(\delta r)^2} + O(\delta r^2) \tag{187}$$

where u_m is the mean of u over the sphere of radius δr.

If the problem is symmetrical with respect to the origin, that is independent of θ and ϕ we get the simpler form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\,. \tag{188}$$

with $\frac{\partial u}{\partial r} = 0$ at $r = 0$

In the case of symmetrical heat flow problems for hallow cylinders and spheres that exclude $r = 0$ simpler equations than the above may be employed by suitable changes of variable.

- The change of variable $R = \log_r r$ transforms the cylindrical equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\,. \tag{189}$$

to

$$e^{2r}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial R^2}\,. \tag{190}$$

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- The change of dependent variable given by $u = w/r$ transforms the spherical equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\,. \tag{191}$$

to

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial r^2}\,. \tag{192}$$