Numerical Integration of PDEs

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aJ.W. Thomas Numerical PDEs, Springer 1995
A differential equation involving more than one independent variable is called a **partial differential equations** (PDEs). For example, a PDE for the function $u(x, y, z)$ may look like:

$$F \left( x, y, z; u; \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}; \frac{\partial^2 u}{\partial x \partial x}, \frac{\partial^2 u}{\partial x \partial y}, \ldots, \frac{\partial^2 u}{\partial z \partial z}; \ldots \right) = 0$$  \hspace{1cm} (1)

Many problems in applied science, physics, and engineering are modeled mathematically with PDE.

Here we will study **finite-difference methods** in solving numerically PDEs, which are based on formulas for approximating the 1st and the 2nd derivatives of a function.
The 2nd order linear PDEs (are of direct importance for physics) and are classified with terminology borrowed from the conic sections. For a 2nd-degree polynomial in $x$ and $y$

$$Ax^2 + 2Bxy + Cy^2 + \cdots = 0$$

the graph is a quadratic curve, and when

- $B^2 - 4AC = 0$ the curve is a **parabola**,
- $B^2 - 4AC < 0$ the curve is an **ellipse**,
- $B^2 - 4AC > 0$ the curve is a **hyperbola**
Now if one replaces $x$ with $\partial_x$, $y$ with $\partial_y$ ... \(^1\) converts equation (2) into a PDE of the form:

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \left( x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x} \right) = 0$$  \(3\)

where $A$, $B$ and $C$ are constants, is called \textit{quasilinear}. There are 3 types of quasilinear equations:

- If $B^2 - 4AC = 0$, the equation is called \textit{parabolic},
- If $B^2 - 4AC < 0$, the equation is called \textit{elliptic},
- If $B^2 - 4AC > 0$, the equation is called \textit{hyperbolic}
The **heat equation**

\[
\alpha^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad 0 < y < 1
\]  

(4)

the initial temperature distribution at \( t = 0 \) is

- \( u(x, 0) = f(x) \) for \( t = 0 \) and \( 0 \leq x \leq L \)

and the boundary conditions at the ends of the rod are

- \( u(x, t) = c_1 \) for \( x = 0 \) and \( 0 \leq t \leq \infty \)
- \( u(L, t) = c_2 \) for \( x = L \) and \( 0 \leq t \leq \infty \)

is an example of **parabolic** PDE.
Two classic examples of PDEs are the 2-D Laplace and Poisson eqns:

\[ \nabla^2 u = 0, \quad \nabla^2 u = g(x, y) \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad 0 < y < 1 \quad (5) \]

with boundary conditions:

- \( u(x, 0) = f_1(x) \) for \( y = 0 \) and \( 0 \leq x \leq 1 \)
- \( u(x, 0) = f_2(x) \) for \( y = 1 \) and \( 0 \leq x \leq 1 \)
- \( u(x, 0) = f_3(x) \) for \( x = 0 \) and \( 0 \leq y \leq 1 \)
- \( u(x, 0) = f_4(x) \) for \( x = 1 \) and \( 0 \leq y \leq 1 \)

for which \( B = 0, \ A = C = 1 \) and thus they are \textbf{elliptic} PDEs.
The wave equation

\[
\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{for} \quad 0 < x < L \quad \text{and} \quad 0 < t < \infty \quad (6)
\]

with a given initial position and velocity functions

- \( u(x, 0) = f(x) \) for \( t = 0 \) and \( 0 \leq x \leq L \)
- \( u_t(x, 0) = g(x) \) for \( t = 0 \) and \( 0 \leq x \leq L \)

is a classic example of hyperbolic PDE.

\(^1\)Formally via a Fourier transform
Elliptic PDEs

We will try to solve the Laplace equation in 2-dimensions

\[ u_{xx} + u_{yy} = 0 \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad 0 < y < 1 \quad (7) \]

with boundary conditions:

- \( u(x, 0) = f_1(x) \) for \( y = 0 \) and \( 0 \leq x \leq 1 \)
- \( u(x, 0) = f_2(x) \) for \( y = 1 \) and \( 0 \leq x \leq 1 \)
- \( u(x, 0) = f_3(x) \) for \( x = 0 \) and \( 0 \leq y \leq 1 \)
- \( u(x, 0) = f_4(x) \) for \( x = 1 \) and \( 0 \leq y \leq 1 \)

since

\[ u''(x) = \frac{u(x + h) - 2u(x) + u(x - h)}{h^2} + O(h^2) \]

This means that \( u_{xx} \) at the point \( (x_i, y_j) \) will be

\[ [u_{xx}]_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \quad (8) \]

and \( u_{yy} \) will be written as:

\[ [u_{yy}]_{i,j} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} \quad (9) \]
Thus Laplace’s equation can be approximately written as

\[ \nabla^2 u \approx \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} = 0 \]  

(10)

where \( i = 2, \ldots, n - 1 \) & \( j = 2, \ldots, m - 1 \).

This the **5-point difference formula** for Laplace’s equation and relates the function value \( u_{i,j} \) to its 4 neighbouring values \( u_{i-1,j}, u_{i+1,j}, u_{i,j-1} \) and \( u_{i,j+1} \).

This leads to the following Laplacian computational formula:

\[ u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 0 \]  

(11)
Assume that the values of $u(x, y)$ are known at the following boundary grid points:

\[
\begin{align*}
  u(x_1, y_j) &= u_{1,j} \quad \text{for} \quad 2 \leq j \leq m - 1 \\
  u(x_i, y_1) &= u_{i,1} \quad \text{for} \quad 2 \leq i \leq n - 1 \\
  u(x_n, y_j) &= u_{n,j} \quad \text{for} \quad 2 \leq j \leq m - 1 \\
  u(x_i, y_m) &= u_{i,m} \quad \text{for} \quad 2 \leq i \leq n - 1
\end{align*}
\]

Then we can estimate the values of the function $u(x, y)$ at the interior grid points by solving a system of $(n - 2) \times (n - 2)$ equations for $(n - 2)^2$ unknowns.
For the above $5 \times 5$ grid the solution of the Laplacian equation will be given by the following linear system:

\[
\begin{align*}
-4u_{2,2} + u_{3,2} + u_{2,3} + u_{3,3} + u_{4,3} + u_{2,4} + u_{3,4} + u_{4,4} &= -u_{2,1} - u_{1,2} \\
-4u_{3,2} + u_{4,2} + u_{3,3} + u_{4,3} + u_{3,4} + u_{4,4} &= -u_{3,1} \\
u_{2,2} - 4u_{3,2} + u_{4,2} + u_{3,3} + u_{4,3} + u_{3,4} + u_{4,4} &= -u_{4,1} - u_{5,2} \\
u_{2,2} - 4u_{3,2} + u_{4,2} + u_{3,3} + u_{4,3} + u_{3,4} + u_{4,4} &= -u_{1,3} \\
u_{2,2} - 4u_{3,2} + u_{4,2} + u_{3,3} + u_{4,3} + u_{3,4} + u_{4,4} &= 0 \\
u_{2,2} - 4u_{3,2} + u_{4,2} + u_{3,3} + u_{4,3} + u_{3,4} + u_{4,4} &= -u_{3,5} \\
u_{2,2} - 4u_{3,2} + u_{4,2} + u_{3,3} + u_{4,3} + u_{3,4} + u_{4,4} &= -u_{2,5} - u_{1,4} \\
u_{2,2} - 4u_{3,2} + u_{4,2} + u_{3,3} + u_{4,3} + u_{3,4} + u_{4,4} &= -u_{4,5} - u_{5,4} \\
\end{align*}
\]

\( (12) \)

**EXAMPLE 1**

If the rectangle has dimensions $0 \leq x \leq 4$ and $0 \leq y \leq 4$ with boundary conditions

\[
\begin{align*}
u(x, 0) &= 20 \quad \text{and} \quad u(x, 4) = 180 \quad \text{for} \quad 0 < x < 4 \\
u(0, y) &= 80 \quad \text{and} \quad u(4, x) = 0 \quad \text{for} \quad 0 < y < 4 \\
\end{align*}
\]

we create the following grid
\[\begin{align*}
-4u_{2,2} + u_{3,2} + u_{4,2} + u_{2,3} + u_{3,3} + u_{4,3} + u_{2,4} + u_{3,4} + u_{4,4} &= -100 \\
-4u_{3,2} - u_{2,3} - 4u_{4,2} - 4u_{3,3} - 4u_{4,3} - 4u_{2,4} - 4u_{3,4} - 4u_{4,4} &= -20 \\
-4u_{4,2} - u_{2,3} - 4u_{3,2} - 4u_{4,3} - 4u_{2,4} - 4u_{3,4} - 4u_{4,3} - 4u_{4,4} &= -80 \\
+u_{2,3} - u_{2,4} - u_{2,4} - u_{4,3} + u_{4,4} - u_{4,4} &= 0 \\
+u_{3,2} + u_{3,3} + u_{3,3} + u_{4,2} + u_{4,2} + u_{4,2} &= 0 \\
+u_{4,3} + u_{4,4} + u_{4,4} + u_{3,3} + u_{3,3} + u_{3,3} &= 0 \\
-4u_{2,2} - u_{2,3} - 4u_{3,2} - 4u_{4,3} - 4u_{2,4} - 4u_{3,4} - 4u_{4,3} - 4u_{4,4} &= -260 \\
-4u_{3,2} - u_{2,3} - 4u_{4,2} - 4u_{3,3} - 4u_{2,4} - 4u_{3,4} - 4u_{4,3} - 4u_{4,4} &= -180 \\
-4u_{4,2} - u_{2,3} - 4u_{3,2} - 4u_{4,3} - 4u_{2,4} - 4u_{3,4} - 4u_{4,3} - 4u_{4,4} &= -180 \\
+u_{2,3} - u_{2,4} - u_{2,4} - u_{4,3} + u_{4,4} - u_{4,4} &= 0 \\
+u_{3,2} + u_{3,3} + u_{3,3} + u_{4,2} + u_{4,2} + u_{4,2} &= 0 \\
+u_{4,3} + u_{4,4} + u_{4,4} + u_{3,3} + u_{3,3} + u_{3,3} &= 0 \\
-4u_{2,2} - u_{2,3} - 4u_{3,2} - 4u_{4,3} - 4u_{2,4} - 4u_{3,4} - 4u_{4,3} - 4u_{4,4} &= -260 \\
-4u_{3,2} - u_{2,3} - 4u_{4,2} - 4u_{3,3} - 4u_{2,4} - 4u_{3,4} - 4u_{4,3} - 4u_{4,4} &= -180 \\
-4u_{4,2} - u_{2,3} - 4u_{3,2} - 4u_{4,3} - 4u_{2,4} - 4u_{3,4} - 4u_{4,3} - 4u_{4,4} &= -180
\end{align*}\]

which admits the solution:

\(u_{2,4} = 112.857\), \(u_{3,4} = 111.786\), \(u_{4,4} = 84.2857\)

\(u_{2,3} = 79.6429\), \(u_{3,3} = 70.000\), \(u_{4,3} = 45.3571\),

\(u_{2,2} = 55.7143\), \(u_{3,2} = 43.2143\), \(u_{4,2} = 27.1429\).
The solution of the linear systems that derived earlier can be found according to the methods discussed in Section 2. For the 3-diagonal systems that we have here the iterative methods are the best choice. Assuming some initial values for the internal unknown grid points \( u_{i,j} \), we can use the following iterative scheme:

\[
 u_{i,j}^{(N+1)} = \frac{1}{4} \left( u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} \right)^{(N)}
\]  

(13)

A faster approach is to use the following successive over relaxation (S.O.R) scheme

\[
 u_{i,j}^{(N+1)} = u_{i,j}^{(N)} + \frac{\omega}{4} \left( u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} \right)^{(N)}
\]

\[
 = u_{i,j}^{(N)} + \omega r_{i,j}^{(N)}
\]

(14)

This procedure will be repeated until \( |r_{i,j}|^{(N)} < \epsilon \).
The optimal value for the overrelaxation factor $\omega$ is not always predictable.

For rectangular regions with Dirichlet boundary conditions there is a formula for the optimal $\omega$ which is the root of the quadratic equation

$$\left[ \cos \left( \frac{\pi}{n-1} \right) + \cos \left( \frac{\pi}{m-1} \right) \right]^2 \omega^2 - 16\omega + 16 = 0 \quad (15)$$

**EXAMPLE 2**
Solve EXAMPLE 1 using S.O.R. method for different values of $\omega$ check if your numerical findings agree with the outcome of the previous relation.
Test the speed of the method in comparison to the standard method - *for reliable comparison increase the grid points in each side by 100 times.*
Hyperbolic PDEs

A typical example of hyperbolic equation is the wave equation

$$\frac{\partial^2 u(x, t)}{\partial t^2} = c^2 \frac{\partial^2 u(x, t)}{\partial x^2} \text{ for } 0 < x < a \text{ and } 0 < t < b \quad (16)$$

with the boundary

$$u(0, t) = 0 \text{ and } u(a, t) = 0 \text{ for } 0 \leq t \leq b \quad (17)$$

and initial conditions

$$u(x, 0) = f(x) \text{ for } 0 \leq x \leq a \quad (18)$$
$$u_t(x, 0) = g(x) \text{ for } 0 < x < a$$
Derivation of Difference Equation

Partition the rectangle $R = (x, t): 0 \leq x \leq a, \; 0 \leq t \leq b$ into a grid consisting of $(n - 1)$ by $(m - 1)$ rectangles with sides $\Delta x = h$ and $\Delta t = k$.

Then the central difference formulas will be:

\[
\begin{align*}
    u_{tt}(x, t) &= \frac{u(x, t + k) - 2u(x, t) + u(x, t - k)}{k^2} + O(k^2) \quad (19) \\
    u_{xx}(x, t) &= \frac{u(x + h, t) - 2u(x, t) + u(x - h, t)}{h^2} + O(h^2) \quad (20)
\end{align*}
\]
Because $x_{i+1} = x_i + h$ and $t_{j+1} = t_j + k$ we can write

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \frac{c^2}{h^2} \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

(21)

and if for simplicity we get $r = ck/h$ then

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = r^2 (u_{i+1,j} - 2u_{i,j} + u_{i-1,j}).$$

(22)

which finally becomes

$$u_{i,j+1} = 2(1 - r^2)u_{i,j} + r^2 (u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}$$

(23)
Two starting values corresponding to $j = 1$ and $j = 2$ must be supplied in order to use formula (23) to compute the 3rd row. Since, the values of the 2nd row usually are not known we estimate them numerically from the information that we have for $u_t(x, 0)$.

The value of $u(x_i, k)$ satisfies

$$u(x_i, k) = u(x_i, 0) + k u_t(x_i, 0) + O(k^2)$$  \hspace{1cm} (24)

But since $u(x_i, 0) = f(x_i) = f_i$ and $u_t(x_i, 0) = g(x_i) = g_i$ the above relation will be written:

$$u_{i,2} = f_i + kg_i \quad \text{for} \quad i = 2, 3, \ldots, n - 1.$$  \hspace{1cm} (25)
Hyperbolic PDEs: Stability

Numerical methods suffer from **instabilities** which grow as we evolve the equation in time.

For the hyperbolic equation, under discussion, there exists a sufficient criterion which ensures the stability of the evolution is \( r = c k / h \leq 1 \).

This is called **Courant-Friedrichs-Lewy** (CFL).

In practice the CFL criterion demands

\[
|c| \leq \frac{\Delta x}{\Delta t}
\]

That is the propagation speed of the waves \( c \) to be smaller than the speed of propagation of the information in our grid.

The condition can be viewed as a sort of discrete "light cone" condition, namely that the time step must be kept small enough so that information has enough time to propagate through the space discretization.
Hyperbolic PDEs: Nonreflecting Boundaries

The 1-dimension wave equation \( u_{tt} = c^2 u_{xx} \) might not always have reflective boundaries i.e. \( u(x_0) = u(x_n) = 0 \) but instead one or both boundaries might “absorb” and not reflect the waves.

The d’Alembert solution will be written as:

\[
  u(x, t) = F_A(x - ct) + F_R(x + ct)
\]  

(27)

\( F_A \) is the wave advancing towards the boundary and \( F_R \) the reflecting wave.

Here \( F_R = 0 \) and thus \( u_x = F_A' \) and \( u_t = -cF_A' \) which means that we get the wave equation representing waves traveling in one only direction \( u_t = -cu_x \).

The central difference approximation for the non-reflecting boundary will be

\[
c \frac{u_{i+1,j} - u_{i-1,j}}{2h} = - \frac{u_{i,j+1} - u_{i,j-1}}{2k}
\]

\[
r (u_{i+1,j} - u_{i-1,j}) = -(u_{i,j+1} - u_{i,j-1})
\]  

(28)

Figure 1: Finite Difference Mesh at non-reflecting boundary
We may substitute (29) into equation (23) in order to eliminate the point \( u_{i+1,j} \) and we get:

\[
(1 + r)u_{i,j+1} = 2r^2 u_{i-1,j} + 2(1 - r^2)u_{i,j} - (1 - r)u_{i,j-1}
\]

At the initial time step the last term will be evaluating using (25) while if we assume \( r = 1 \) then we get the extremely simple form

\[
u_{i,j+1} = u_{i-1,j}
\]
We will consider the 1D heat equation as an example of parabolic PDE

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad 0 \leq x \leq 1 \quad \text{for} \quad 0 \leq t < \infty \quad (31)$$

with boundary conditions:

$$u(0, t) = c_1, \quad u(1, t) = c_2 \quad \text{for} \quad 0 \leq t < \infty \quad (32)$$

and initial conditions: $$u(x, 0) = f(x), \quad \text{for} \quad 0 \leq x \leq 1.$$ 

The heat equation models the temperature in an insulated rod with ends held at constant temperatures $c_1$ and $c_2$. 

Parabolic PDEs

We assume that the rectangle $R = \{(x, t) : 0 \leq x \leq 1, 0 \leq t < b\}$ is subdivided into $n - 1$ by $m - 1$ rectangles with sides $\Delta x = h$ and $\Delta t = k$.

Then the difference formulas will be:

\[
\begin{align*}
  u_t(x, t) &= \frac{u(x, t + k) - u(x, t)}{k} + O(k) \\
  u_{xx}(x, t) &= \frac{u(x + h, t) - 2u(x, t) + u(x - h, t)}{h^2} + O(h^2)
\end{align*}
\]
and the difference equation becomes

\[
\frac{u_{i,j+1} - u_{i,j}}{k} = \alpha^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}
\]  

(35)

by setting \( r = \alpha^2 k / h^2 \) we get

\[
u_{i,j+1} = (1 - 2r) u_{i,j} + r (u_{i-1,j} + u_{i+1,j})
\]  

(36)
• The simplicity of eqn (36) makes it appealing to use. However, it is important to use numerical techniques that are stable.

• If any error made at one stage of the calculation is eventually damped out, the method is called stable.

• The explicit forward-differnece eqn (36) is stable if and only if $0 \leq r \leq 1/2$. This means that the stepsize $k$ must satisfy $k \leq h^2/(2\alpha^2)$. If this condition is not fulfilled, errors, committed at one row might be magnified in subsequent rows.

• The difference eqn (36) has accuracy of the order $O(k) + O(h^2)$.

• If we choose $r = 1/2$ the difference eqn (36) becomes even simpler:

\[
u_{i,j+1} = \frac{u_{i-1,j} + u_{i+1,j}}{2}
\]  

(37)
The implicit method of Crank - Nicholson is based on using the spatial derivative on both the point \((i,j)\) and \((i,j+1)\). That is:

\[
\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2} \alpha^2 \left( \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2} \right)
\]

which after rearrangement leads to:

\[
-ru_{i-1,j+1} + 2(1 + r)u_{i,j+1} - ru_{i+1,j+1} = 2(1 - r)u_{i,j} + r \left( u_{i-1,j} + u_{i+1,j} \right) \quad (38)
\]

which for \(r = 1\) leads to

\[
-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j} \quad (39)
\]
The previous relation leads to the solution of the following linear system of equations

\[
\begin{pmatrix}
4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & ... & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & ... & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}
\begin{pmatrix}
u_{2,j+1} \\
u_{3,j+1} \\
... \\
u_{k,j+1} \\
... \\
u_{n-2,j+1} \\
u_{n-1,j+1} \\
\end{pmatrix}
= 
\begin{pmatrix}
2c_1 + u_{3,j} \\
u_{2,j} + u_{4,j} \\
... \\
u_{k-1,j} + u_{k+1,j} \\
... \\
u_{n-3,j} + u_{n-1,j} \\
u_{n-2,j} + 2c_2 \\
\end{pmatrix}
\]

It is obvious that this procedure has to be repeated in every time step, but the advantage is that it is stable for every value of \( r \).
The finite difference schemes are used because their solutions approximate the solutions to certain PDEs.

**I. Convergence** The solution of the difference equations can be made to approximate the solution of the PDE to any desired accuracy.

A finite difference scheme (FDS) \( \mathcal{L}_k^n u_k^n = G_k^n \) approximating the PDE \( Lv = F \) is a convergent scheme of order \((p, q)\) if for any \( t \), as \((n + 1)\Delta t \) converges to \( t \)

\[
\|u^{n+1} - v^{n+1}\| = O(\Delta x^p) + O(\Delta t^q) \tag{40}
\]

as \( \Delta x \) and \( \Delta t \) converge to 0.

**II. Consistency** Given a PDE \( Lv = F \) and a FDS \( \mathcal{L}_k^n u_k^n = G_k^n \), we say that the FDS is consistent with the PDE if for any smooth function \( \phi(x, t) \)

\[
L\phi - \mathcal{L}\phi \to 0 \quad \text{as} \quad \Delta t \Delta x \to 0 \tag{41}
\]

**Lax Equivalence Theorem** For consistent schemes (in linear problems) convergence is equivalent to stability

**III. Stability** ...
Convergence (ODE)

Let's assume an ODE of the form

$$\frac{dy}{dx} = Ay$$  \hspace{1cm} (42)

which has an obvious solution of the form $y = e^{Ax}$.

Euler's method gives the approximate solution via the recurrence relation:

$$y_{n+1} = y_n + hAy_n = (1 + hA)y_n \quad \text{for} \quad n = 0, 1, 2, ...$$  \hspace{1cm} (43)

Thus if we use this relation $n$-times we will get:

$$y_{n+1} = (1 + hA)^{n+1}y_0 \quad \text{for} \quad n = 0, 1, 2, ...$$  \hspace{1cm} (44)

But for small $h$ we know that $1 + hA \approx e^{hA}$ thus

$$y_{n+1} = (1 + hA)^{n+1}y_0 \approx e^{(n+1)hA}y_0 = e^{(x_{n+1} - x_0)A}y_0 = e^{Ax_{n+1}}$$  \hspace{1cm} (45)

where $h = (x_{n+1} - x_0)/(n + 1)$.

This means that the numerical solution for small $h$ converges to the analytic solution: $y = e^{Ax}$. 
Consider the wave equation defined by the operator \( L = \frac{\partial}{\partial t} + a \frac{\partial}{\partial x} \) i.e. \( Lu = u_t + au_x \)

We will evaluate the FDS - FSFT

\[
\mathcal{L}u = \frac{u_{i+1}^n - u_i^n}{\Delta t} + a \frac{u_{i+1}^n - u_i^n}{\Delta x}
\]

By using Taylor expansion we get

\[
u_{i+1}^n = u_i^n + u_t \Delta t + \frac{1}{2} u_{tt} \Delta t^2 + O(\Delta t^3)
\] (46)

\[
u_{i+1}^n = u_i^n + u_x \Delta x + \frac{1}{2} u_{xx} \Delta x^2 + O(\Delta x^3)
\] (47)

This leads to

\[
\mathcal{L}u = Lu + \frac{1}{2} (\Delta t \cdot u_{tt} + \Delta x \cdot u_{xx}) + O(\Delta t^2) + O(\Delta x^2)
\]

Thus

\[
\mathcal{L}u - Lu = \frac{1}{2} (\Delta t \cdot u_{tt} + \Delta x \cdot u_{xx}) + O(\Delta t^2) + O(\Delta x^2) \to 0 \text{ as } (\Delta x, \Delta t) \to 0
\]
One interpretation of stability of difference scheme is that for a stable difference scheme small errors in the initial conditions cause small errors in the solution.

This definition allows the errors to grow, but limits them to grow no faster than exponential.

A difference scheme for solving a given (two level) initial-value problem is of the form

\[ u^{n+1} = Qu^n, \quad n \geq 0. \]  

(48)

**Definition** The difference scheme (48) is said to be stable if there exist positive constants \( \Delta x_0 \) and \( \Delta t_0 \), and non-negative constants \( K \) and \( \beta \) so that

\[ ||u^{n+1}|| \leq Ke^{\beta t}||u^0|| \]  

(49)

for \( 0 \leq t = (n + 1)\Delta t, \quad 0 < \Delta x \leq \Delta x_0 \) and \( 0 < \Delta t \leq \Delta t_0. \)
Another, more common, definition that is used is one that does not allow for exponential growth. Inequality (49) is replaced by

$$||u^{n+1}|| \leq K||u^0||$$  \hspace{1cm} (50)$$

which implies (49).

Where we define the **Euclidean norm**

$$||u||_2 = \sqrt{\sum_{k=1}^{N} |u_k|^2}.$$  \hspace{1cm} (51)$$

$$||u||_{2,\Delta x} = \sqrt{\sum_{k=1}^{N} |u_k|^2 \Delta x}.$$  \hspace{1cm} (52)$$

and the **sub-norm**

$$||u||_\infty = \sup |u_k| \quad \text{for} \quad 1 \leq k \leq N$$  \hspace{1cm} (53)$$
Proposition The difference scheme (48) is stable if and only if there exist positive constants $\Delta x_0$ and $\Delta t_0$, and non-negative constants $K$ and $\beta$ so that

$$||Q^{n+1}|| \leq Ke^{\beta t}$$

(54)

for $0 \leq t = (n + 1)\Delta t$, $0 < \Delta x \leq \Delta x_0$ and $0 < \Delta t \leq \Delta t_0$.

Proof:

$$u^{n+1} = Qu^n = Q \left( Qu^{n-1} \right) = Q^2 u^{n-1} = \ldots = Q^{n+1} u^0$$

expression (49) can be written as

$$||u^{n+1}|| = ||Q^{n+1} u^0|| \leq Ke^{\beta t} ||u^0||$$

or

$$\frac{||Q^{n+1} u^0||}{||u^0||} \leq Ke^{\beta t}$$

by taking the supremum over both sides over all non-zero vectors $u^0$ we get (54).
Example Show that following difference scheme is stable with respect to the sup-norm.

\[ u^{n+1}_k = (1 - 2r) u^n_k + r (u^n_{k+1} + u^n_{k-1}) \]  

(55)

Solution We note that if \( r \leq 1/2 \)

\[ |u^{n+1}_k| \leq (1 - 2r)|u^n_k| + r|u^n_{k+1}| + r|u^n_{k-1}| \leq ||u^n||_\infty \]

If we take the supremum over both sides (with respect to \( k \)), we get

\[ ||u^{n+1}||_\infty \leq ||u^n||_\infty \]

Hence the inequality (50) is satisfied with \( K = 1 \), or inequality (49) is satisfied with \( K = 1 \) and \( \beta = 0 \).

NOTES: For the stability of the scheme (55) we have required that \( r \leq 1/2 \).
In this case we say that the scheme is conditionally stable.
In the case where no-restrictions on the relationship between \( \Delta t \) and \( \Delta x \) are needed for stability, we say the scheme is unconditionally stable.
When solving initial-value problems a common analytical tool is to use the Fourier transform. For example, consider the problem:

\[ v_t = v_{xx}, \quad \text{with} \quad v(x, 0) = f(x). \]  

(56)

If we define the Fourier transform of \( v \) to be

\[ \hat{v}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} v(x, t) \, dx \]  

(57)

and take the Fourier transform of PDE (56) we get

\[
\hat{v}_t(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} v_t(x, t) \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} v_{xx}(x, t) \, dx \\
= -\omega^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} v(x, t) \, dx = -\omega^2 \hat{v}(\omega, t). \]  

(58)

Here we integrated by parts twice.

Hence we see that the Fourier transform reduces the PDE to an ODE in transform space.
The technique then is to solve the ODE in transformed space and return our solution space.

We can return to our solution by using the inverse Fourier transform

\[ v(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{v}(\omega, t) d\omega \]  \hspace{1cm} (59)

The **discrete Fourier transform** can be written as:

\[ \hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} v_k \]  \hspace{1cm} (60)

for \( \xi \in [-\pi, \pi] \).

**Parseval’s identity** says that the norms of the function and its transform are equal in their respective spaces:

\[ \|v\|_2 = \|\hat{v}\|_2 \]  \hspace{1cm} (61)
In a stability analysis we will use the Fourier transform and Perseval’s Identity.

Recall that for the definition of stability we have used the inequality

\[ \| u^{n+1} \|_{2, \Delta x} \leq Ke^{\beta(n+1)\Delta t} \| u^0 \|_{2, \Delta x} \]  

which can be now written as (\( \| u \|_{2, \Delta x} = \sqrt{\Delta x} \| u \|_2 = \sqrt{\Delta x} \| \hat{u} \|_2 \))

\[ \| \hat{u}^{n+1} \|_2 \leq Ke^{\beta(n+1)\Delta t} \| \hat{u}^0 \|_2 \]  

then the same \( K \) and \( \beta \) will also satisfy (57).

*When inequality (63) holds, we say that the sequence \( \{ \hat{u}^n \} \) is **stable in the transform space** and this applies also to sequence \( \{ u^n \} \).*
Analyze the stability of the difference scheme

\[ u_{k}^{n+1} = ru_{k-1}^{n} + (1 - 2r)u_{k}^{n} + ru_{k+1}^{n}, \quad -\infty < k < \infty \]  

(64)

where \( r = \nu \Delta t/\Delta x^2 \).

If we take the discrete Fourier transform of both sides of equation (64)

\[ \hat{u}_{n+1}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} u_{k}^{n+1} \]

\[ = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} [ru_{k-1}^{n} + (1 - 2r)u_{k}^{n} + ru_{k+1}^{n}] \]

\[ = r \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} u_{k-1}^{n} + (1 - 2r) \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} u_{k}^{n} \]

\[ + r \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} u_{k+1}^{n} \]

\[ = r \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} u_{k-1}^{n} + (1 - 2r)\hat{u}_{n}(\xi) + r \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} u_{k+1}^{n} \]
By making the change of variables \( m = k \pm 1 \) we get,

\[
\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-ik\xi} u_{k\pm 1}^n = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-i(m\mp 1)\xi} u_m^n \tag{65}
\]

\[
= e^{\pm i\xi} \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} u_m^n = e^{\pm i\xi} \hat{u}(\xi).
\]

Then we get

\[
\hat{u}^{n+1}(\xi) = re^{-i\xi} \hat{u}^n(\xi) + (1 - 2r) \hat{u}^n(\xi) + re^{i\xi} \hat{u}^n(\xi)
\]

\[
= \left[ re^{-i\xi} + (1 - 2r) + re^{i\xi} \right] \hat{u}^n(\xi)
\]

\[
= \left[ 2r \cos \xi + (1 - 2r) \right] \hat{u}^n(\xi)
\]

\[
= \left[ 1 - 4r \sin^2(\xi/2) \right] \hat{u}^n(\xi) \tag{66}
\]

The term

\[
\rho(\xi) = 1 - 4r \sin^2 \frac{\xi}{2} \tag{67}
\]

is called the symbol of the difference scheme (64).
Thus by taking the discrete Fourier transform, we get rid of the $x$ derivatives and simplify the equation.

If we apply the result (66) $n + 1$ times, we get

$$\hat{u}^{n+1}(\xi) = \left(1 - 4r \sin^2(\xi/2)\right)^{n+1} \hat{u}^0(\xi)$$  \hspace{1cm} (68)

Thus if we restrict $r$ so that

$$|1 - 4r \sin^2(\xi/2)| \leq 1$$  \hspace{1cm} (69)

Then we can choose $K = 1$ and $\beta = 0$ and satisfy inequality (50).

Thus our scheme will be stable if

$$-1 \leq 1 - 4r \sin^2(\xi/2) \leq 1$$  \hspace{1cm} (70)

or

$$4r \sin^2(\xi/2) \leq 2$$  \hspace{1cm} (71)

which is true for $r \leq 1/2$.

This is the necessary and sufficient condition for convergence of the scheme (64).
For the hyperbolic PDE

\[ u_t + au_x = 0 \]  \hfill (72)

study the stability of the following schemes \((|R| = |a|\Delta t/\Delta x \leq 1)\)

\[ u_k^{n+1} = u_k^n - R(u_{k+1}^n - u_k^n) \quad \text{(FTFS)} \]  \hfill (73)
\[ u_k^{n+1} = u_k^n - R(u_k^n - u_{k-1}^n) \quad \text{(FTBS)} \]  \hfill (74)
\[ u_k^{n+1} = u_k^n - \frac{R}{2}(u_{k+1}^n - u_{k-1}^n) \quad \text{(FTCS)} \]  \hfill (75)

The following abbreviations might be used later:

\[ \delta_+ u_k = u_{k+1} - u_k \]  \hfill (76)
\[ \delta_- u_k = u_k - u_{k-1} \]  \hfill (77)
\[ \delta_0 u_k = u_{k+1} - u_{k-1} \]  \hfill (78)
\[ \delta^2 u_k = u_{k+1} - 2u_k + u_{k-1} \]  \hfill (79)
For the hyperbolic PDE

\[ u_t + a u_x = 0, \quad \text{with} \quad a < 0 \quad (80) \]

study the stability of the following scheme (FTFS) \((|R| = |a|\Delta t/\Delta x \leq 1)\)

\[ u_{k}^{n+1} = (1 + R)u_{k}^{n} - Ru_{k+1}^{n} \quad (81) \]

We begin by taking the discrete Fourier transform of the scheme

\[ \hat{u}_{k}^{n+1} = (1 + R)\hat{u}_{k}^{n} - \Re i\xi \hat{u}_{k}^{n} \]
\[ = [(1 + R) - R(\cos \xi + i \sin \xi)] \hat{u}_{k}^{n} \quad (82) \]

Then because the symbol is complex and is given by

\[ \rho(\xi) = (1 + R) - R \cos \xi - iR \sin \xi \quad (83) \]

we must bound the magnitude of \(\rho\) by 1 to satisfy the inequality (63) (with \(K = 1\) and \(\beta = 0\)). Thus we calculate

\[ |\rho(\xi)|^2 = (1 + R)^2 - 2R(1 + R) \cos \xi + R^2 \]
Then we determine the maximum and minimum value of $|\rho(\xi)|^2$ for $\xi \in [-\pi, \pi]$ and we find that we have a potential maximum at $\xi = 0$ and $\xi = \pm \pi$.

If we evaluate $|\rho(\xi)|$ at these values, we see that

$$|\rho(0)| = 1 \quad \text{and} \quad |\rho(\pm \pi)| = |1 + 2R|$$

To bound $|\rho(\pm \pi)|$ by 1, we require that $R$ satisfies $-1 \leq 1 + 2R \leq 1$.

Then since $1 + 2R \leq 1$ since $R < 0$ we see that the scheme is 

**conditionally stable** with condition $R \geq -1$. 

For the hyperbolic PDE

\[ u_t + au_x = 0, \quad \text{with} \quad a < 0 \]  \hspace{1cm} (84)

study the stability of the following scheme (FTCS) \((|R| = |a|\Delta t/\Delta x \leq 1)\)

\[ u^{n+1}_k = u^n_k - \frac{R}{2} \delta_0 u^{n+1}_{k+1} \]  \hspace{1cm} (85)

We begin by taking the discrete Fourier transform of the scheme

\[ \hat{u}^{n+1} = \hat{u}^n - \frac{R}{2} \left( e^{i\xi} - e^{-i\xi} \right) \hat{u}^n = \left[ 1 - iR \sin \xi \right] \hat{u}^n \]  \hspace{1cm} (86)

Thus the symbol is

\[ |\rho|^2 = 1 + R^2 \sin \xi^2 \geq 1 \]

So the difference scheme (85) is **unstable** for all \( R \neq 0 \).
Lax-Wendroff Scheme

For the PDE $u_t + au_x = 0$ we can write:

$$u_{tt} = (-au_x)_t = -au_{xt} = -a(u_t)_x = -a(-au_x)_x = a^2 u_{xx} \quad (87)$$

Thus since

$$u_{k}^{n+1} = u_{k}^{n} + (u_{t})_{k}^{n} \Delta t + (u_{tt})_{k}^{n} \frac{\Delta t^2}{2} + O(\Delta t^3)$$

$$= u_{k}^{n} + (-au_{x})_{k}^{n} \Delta t + (a^2 u_{xx})_{k}^{n} \frac{\Delta t^2}{2} + O(\Delta t^3)$$

$$= u_{k}^{n} - a \left( \frac{u_{k+1}^{n} - u_{k-1}^{n}}{2\Delta x} + O(\Delta x^2) \right) \Delta t$$

$$+ a^2 \left( \frac{u_{k+1}^{n} - 2u_{k}^{n} + u_{k-1}^{n}}{\Delta x^2} + O(\Delta x^2) \right) \frac{\Delta t^2}{2} + O(\Delta t^3)$$

i.e. we approximate the PDE $u_t + au_x = 0$ with the difference scheme

$$u_{k}^{n+1} = u_{k}^{n} - \frac{R}{2} \delta_0 u_{k}^{n} + \frac{R^2}{2} \delta^2 u_{k}^{n} \text{ with } R = a\Delta t/\Delta x. \quad (88)$$
The Lax-Wendroff scheme is $O(\Delta t^2) + O(\Delta x^2)$ and its symbol is (why?)

$$\rho(\xi) = 1 - 2R^2 \sin^2(\xi/2) - iR \sin \xi$$

(89)

Since

$$|\rho(\xi)|^2 = 1 - 4R^2 \sin^4(\xi/2) + 4R^4 \sin^4(\xi/2)$$

(90)

if we differentiate with respect to $\xi$ we can find the critical values at $\xi = \pm \pi$ and 0. For which we get that

$$|\rho(0)|^2 = 1 \quad \text{and} \quad |\rho(\pm \pi)|^2 = |\rho(\pi)|^2 = (1 - 2R^2)^2.$$  

(91)

Then for $R^2 \leq 1$ we get $(1 - 2R^2)^2 \leq 1$ and thus the Lax-Wendroff scheme is conditionally stable for

$$|R| = |a| \frac{\Delta t}{\Delta x} \leq 1.$$  

(92)

and it is 2nd order in both time and space.
It can be derived from the unstable FTCS $O(Dt, Dx^2)$ scheme:

$$u_{k+1}^{n+1} = u_k^n - \frac{R}{2} (u_{k+1}^n - u_{k-1}^n)$$ (93)

by replacing $u_k^n$ with its spatial average: $u_k^n = (u_{k+1}^n + u_{k-1}^n)/2$.

$$u_{k+1}^{n+1} = \frac{1}{2} (u_{k+1}^n + u_{k-1}^n) - \frac{R}{2} (u_{k+1}^n - u_{k-1}^n)$$ (94)

which is stable for $|R| \leq 1$ (WHY?).

**PROBLEM:** Show that the above writing corresponds to the discretization of the following PDE:

$$u_t + au_x = \frac{\Delta x^2}{2\Delta t} u_{xx}$$ (95)

The last term acts as **numerical dissipation**.
Implicit Schemes

For the hyperbolic PDE

$$u_t + au_x = 0$$  \hspace{1cm} (96)

we have studied the following **explicit schemes**

$$u_{n+1}^k = u_n^k - R (u_{k+1}^n - u_k^n) \quad \text{(FTFS)} \hspace{1cm} (97)$$

$$u_{n+1}^k = u_n^k - R (u_k^n - u_{k-1}^n) \quad \text{(FTBS)} \hspace{1cm} (98)$$

$$u_{n+1}^k = u_n^k - \frac{R}{2} (u_{k+1}^n - u_{k-1}^n) \quad \text{(FTCS)} \hspace{1cm} (99)$$

These schemes can be written in the following form:

$$(1 - R)u_{n+1}^k + Ru_{k+1}^n = u_k^n \quad \text{(BTFS)} \hspace{1cm} (100)$$

$$-Ru_{n+1}^k + (1 + R)u_k^n = u_k^n \quad \text{(BTBS)} \hspace{1cm} (101)$$

$$-\frac{R}{2} u_{k-1}^{n+1} + u_k^{n+1} + \frac{R}{2} u_{k+1}^{n+1} = u_k^n \quad \text{(BTCS)} \hspace{1cm} (102)$$
We will study the stability of the BFTS scheme (100). By taking the discrete Fourier transform we get

\[(1 - R)\hat{u}^{n+1} + Re^{i\xi}\hat{u}^{n+1} = \hat{u}^{n}\]  \hspace{1cm} (103)

Thus the symbol will be

\[\rho(\xi) = \frac{1}{1 - R + R \cos \xi + iR \sin \xi}\]  \hspace{1cm} (104)

and the magnitude squared of the symbol is:

\[|\rho(\xi)|^2 = \frac{1}{1 - 4R \sin^2 \xi/2 + 4R^2 \sin^2 \xi/2}\]  \hspace{1cm} (105)

Since \(R \leq 0 \quad (a < 0)\) implies that:

\[1 - 4R \sin^2 \xi/2 + 4R^2 \sin^2 \xi/2 = 1 - 4R(1 - R) \sin^2 \xi/2 \geq 1\]  \hspace{1cm} (106)

i.e. \(|\rho(\xi)|^2 \leq 1\).

**NOTE** that for \(0 < R < 1\), the difference scheme is **unstable**.

Hence, we see that the difference scheme (100) is stable if and only if \(R \leq 0\) or \(R \geq 1\).
For the difference scheme (102):

\[-\frac{R}{2} u_{k-1}^{n+1} + u_k^{n+1} + \frac{R}{2} u_{k+1}^{n+1} = u_k^n\]  (107)

the symbol is (how?)

\[\rho(\xi) = \frac{1}{1 + iR \sin \xi}\]  (108)

Then since

\[|\rho(\xi)|^2 = \frac{1}{1 + R^2 \sin^2 \xi} \leq 1\]  (109)

the difference scheme (107) is unconditionally stable, even though its explicit counterpart is unstable!

**PROBLEM:** Study the stability of the difference scheme (101).
We have studied the numerical solution of the wave equation earlier. Now we will demonstrate how one can treat it with the schemes that we discussed earlier. The equation is:

\[ u_{tt} = c^2 u_{xx} \quad \text{for} \quad 0 < x < a \quad \text{and} \quad 0 < t < b \quad (110) \]

Then we can write it as a system of 1st order PDEs. We set:

\[ h = c \ u_x \quad \text{and} \quad f = u_t \quad (111) \]

and we get:

\[ h_t = c f_x \]
\[ f_t = c h_x \quad (112) \]
\[ u_t = f \]

In vector notation this can be written as:

\[ \vec{U}_t + Q \vec{U}_x = 0 \quad (113) \]

where

\[ Q = - \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \quad \text{and} \quad \vec{U} = \begin{pmatrix} h \\ f \end{pmatrix} \quad (114) \]
Consider the PDE

\[ u_t + au_x + bu_y = 0 \]  \hspace{1cm} (115)

with the initial condition \( u(x, y, 0) = f(x, y) \). Then \( a \) will be the speed of propagation in the \( x \)-direction and \( b \) will be the speed of propagation in the \( y \)-direction.

An obvious, but unfortunately unconditionally unstable scheme is:

\[ u_{jk}^{n+1} = u_{jk}^n - R_x \left( u_{j+1k}^n - u_{j-1k}^n \right) - R_y \left( u_{jk+1}^n - u_{jk-1}^n \right) \]

\[ = (1 - R_x \delta x_0 - R_y \delta y_0) u_{jk}^n \]  \hspace{1cm} (116)

where \( R_x = a \Delta t / \Delta x \) and \( R_y = b \Delta t / \Delta y \).
A conditionally stable scheme is:

$$u_{jk}^{n+1} = (1 - R_x \delta_x - R_y \delta_y) \, u_{jk}^n$$  \hspace{1cm} (117)

**STABILITY:** If we take a 2-dimensional Fourier transform of eqn (117) we get:

$$\hat{u}_{jk}^{n+1} = \left[ 1 - R_x \left( 1 - e^{-i\xi} \right) - R_y \left( 1 - e^{-i\eta} \right) \right] \hat{u}_{jk}^n$$  \hspace{1cm} (118)

So the symbol of the difference scheme (117) is given by

$$\rho(\xi, \eta) = 1 - R_x \left( 1 - e^{-i\xi} \right) - R_y \left( 1 - e^{-i\eta} \right)$$  \hspace{1cm} (119)

and

$$|\rho(\xi, \eta)|^2 = \left[ 1 - 2R_x \sin^2(\xi/2) - 2R_y \sin^2(\eta/2) \right]^2 + [R_x \sin \xi + R_y \sin \eta]^2$$
By differentiating $|\rho|^2$ with respect to $\xi$ and $\eta$ and setting the derivatives equal to zero, we find that there are potential maximums at $(\pm \pi, \pm \pi)$, $(\pm \pi, 0)$, $(0, \pm \pi)$ and $(0, 0)$. It is also easy to find that

$$|\rho(0, 0)| = 1, \quad |\rho(\pm \pi, 0)| = (1 - 2R_x)^2, \quad |\rho(0, \pm \pi)| = (1 - 2R_y)^2$$

and

$$|\rho(\pm \pi, \pm \pi)| = (1 - 2R_y - 2R_y)^2.$$

- The condition $(1 - 2R_x)^2 \leq 1$ requires that $0 \leq R_x \leq 1$.
- The condition $(1 - 2R_y)^2 \leq 1$ requires that $0 \leq R_y \leq 1$.
- The condition $(1 - 2R_x - 2R_y)^2 \leq 1$ requires that $0 \leq R_y + R_y \leq 1$.

**CONCLUSION:** Therefore, we find that the difference scheme (117) is 1st order accurate in space and time, and conditionally stable with condition $R_x + R_y \leq 1$, for $R_x \geq 0$ and $R_y \geq 0$. 
The 2D Lax-Friedrichs scheme for the approximate solution of (115) is:

\[
\frac{u_{jk}^{n+1}}{u_{jk}} = \frac{1}{4} \left( u_{j+1k}^{n} + u_{j-1k}^{n} + u_{jk+1}^{n} + u_{jk-1}^{n} \right) - \frac{R_x}{2} \delta_{x0} u_{jk}^{n} - \frac{R_y}{2} \delta_{y0} u_{jk}^{n}
\]  

(120)

**STABILITY:** we compute the discrete Fourier transform to obtain the symbol for the scheme

\[
\rho(\xi, \eta) = \frac{1}{2} \left( \cos \xi + \cos \eta \right) - i \left( R_x \sin \xi + R_y \sin \eta \right)
\]

(121)

Then the expression \( |\rho(\xi, \eta)|^2 \) can be written as

\[
|\rho(\xi, \eta)|^2 = 1 - \left( \sin^2 \xi + \sin^2 \eta \right) \left[ 1/2 - \left( R_x^2 + R_y^2 \right) \right] - \frac{1}{4} \left( \cos \xi - \cos \eta \right)^2 - \left( R_x \sin \eta - R_y \sin \xi \right)^2
\]

(122)
Since the last two terms in the equation are negative, we have:

\[ |\rho(\xi, \eta)|^2 = 1 - \left(\sin^2 \xi + \sin^2 \eta\right) \left[\frac{1}{2} - \left(R_x^2 + R_y^2\right)\right] \]  \hspace{1cm} (123)

If \( \left[\frac{1}{2} - \left(R_x^2 + R_y^2\right)\right] \geq 0 \), then \( |\rho(\xi, \eta)| \leq 1 \).

Hence if

\[ R_x^2 + R_y^2 \leq \frac{1}{2} \]  \hspace{1cm} (124)

the difference scheme is stable.

**NOTE:** The stability condition (124) is very restrictive. It is not obvious that we can always find a scheme with stability condition the same as the CFL condition, but at least what we should try to do.
2D-Wave Equation: ADI Schemes

\[ u_t = Au = -au_x - bu_y \quad \text{with} \quad u(x, y, 0) = f(x, y) \quad (125) \]

We begin by considering a locally 1D scheme for solving the above PDE

\[
\left( 1 + \frac{R_x}{2} \delta x_0 \right) u_{jk}^{n+1/2} = u_{jk}^n \quad (126)
\]

\[
\left( 1 + \frac{R_y}{2} \delta y_0 \right) u_{jk}^{n+1} = u_{jk}^{n+1/2} \quad (127)
\]

**STABILITY:** The symbol is:

\[
\rho(\xi, \eta) = \frac{1}{(1 + iR_x \sin \xi)(1 + iR_y \sin \eta)} \quad (128)
\]

Then since

\[
|\rho(\xi, \eta)|^2 = \frac{1}{(1 + R_x^2 \sin^2 \xi)(1 + R_y^2 \sin^2 \eta)} \quad (129)
\]

it is clear the \(0 \leq |\rho(\xi, \eta) \leq 1\) and the difference scheme (126)-(127) is **unconditionally stable** and \(O(\Delta t) + O(\Delta x^2) + O(\Delta y^2)\) order accurate.
The above scheme is referred to as the Beam-Warming scheme and is most often written as

\[
\left(1 + \frac{R_x}{4} \delta_{x0}\right) u_{jk}^* = \left(1 - \frac{R_x}{4} \delta_{x0}\right) \left(1 - \frac{R_y}{4} \delta_{y0}\right) u_{jk}^n \tag{131}
\]

\[
\left(1 + \frac{R_y}{4} \delta_{y0}\right) u_{jk}^{n+1} = u_{jk}^* \tag{132}
\]

The symbol of the Beam-Warming scheme is

\[
\rho(\xi, \eta) = \frac{(1 - i \frac{R_x}{2} \sin \xi)(1 - i \frac{R_y}{2} \sin \eta)}{(1 + i \frac{R_x}{2} \sin \xi)(1 + i \frac{R_y}{2} \sin \eta)} \tag{133}
\]

Thus we see that \(|\rho(\xi, \eta)|^2 = 1\) for all \(\xi, \eta \in [-\pi, \pi]\) and the scheme is unconditionally stable and 2nd order.
2D-scheme for the wave equation

Let’s consider the equation

\[ u_t = Au = (A_1 + A_2) u \]  \hspace{1cm} (134)

e.g. \( Au = -au_x - bu_y \) with \( A_1 u = -au_x \) & \( A_2 u = -bu_y \)

By using 1st order approximation to the time derivative we get

\[
\begin{align*}
  u^{n+1} &= u^n + \Delta t A u^n + O(\Delta t^2) \\
  &= (1 + \Delta t A_1 + \Delta t A_2) u^n + O(\Delta t^2) \\
  &= (1 + \Delta t A_1)(1 + \Delta t A_2)u^n - \Delta t^2 A_1 A_2 u^n + O(\Delta t^2) \hspace{1cm} (135)
\end{align*}
\]

by dropping terms of order \( \Delta t^2 \) we get the approximate scheme

\[ u^{n+1} = (1 + \Delta t A_1)(1 + \Delta t A_2)u^n \]  \hspace{1cm} (136)

or

\[
\begin{align*}
  u^{n+1/2} &= (1 + \Delta t A_2)u^n \hspace{1cm} (137) \\
  u^{n+1} &= (1 + \Delta t A_1)u^{n+1/2} \hspace{1cm} (138)
\end{align*}
\]
2D-scheme for the wave equation (∼ Lax-Wendroff)

Let’s assume the equation

\[ u_t = Au = -au_x - bu_y \]  \hspace{1cm} (139)

with \( A_1 u = -au_x \) & \( A_2 u = -bu_y \).

If we approximate the \( A_1 \) and \( A_2 \) by the 1-D Lax-Wendroff scheme, we get

\[ u_{jk}^{n+1/2} = u_{jk}^n - \frac{R_y}{2} \delta_y u_{jk}^n + \frac{R_y^2}{2} \delta_y^2 u_{jk}^n \]  \hspace{1cm} (140)

\[ u^{n+1} = u_{jk}^{n+1/2} - \frac{R_x}{2} \delta_x u_{jk}^{n+1/2} + \frac{R_x^2}{2} \delta_x^2 u_{jk}^{n+1/2} \]  \hspace{1cm} (141)

It is obvious that the above scheme is 2nd order in time.

By following the standard analysis we can prove:

- it is conditionally stable if \( \max\{|R_x|, |R_y|\} \leq 1 \).
- and of order \( O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2) \).
Let’s consider the 2-D parabolic equation:

$$u_t = \nu (u_{xx} + u_{yy}) + F(x, y, t)$$

(142)

with \( u(x, y, t) = g(x, y, t) \) on \( \partial R \) and \( u(x, y, 0) = f(x, y) \).

The scheme will be

$$\frac{u_{jk}^{n+1} - u_{jk}^{n}}{\Delta t} = \frac{\nu}{\Delta x^2} \delta_x^2 u_{jk}^{n} + \frac{\nu}{\Delta y^2} \delta_y^2 u_{jk}^{n} + F_{jk}^{n}$$

(143)

which can be written in the explicit form (\( r_x = \nu/\Delta x^2 \) and \( r_y = \nu/\Delta y^2 \)):

$$u_{jk}^{n+1} = u_{jk}^{n} + \left( r_x \delta_x^2 + r_y \delta_y^2 \right) u_{jk}^{n} + \Delta t F_{jk}^{n}$$

(144)
The symbol for equation (144) is
\[
\rho = 1 + 2r_x \cos \xi - 1 + 2r_y \cos \eta - 1 \\
= 1 - 4r_x \sin^2(\xi/2) - 4r_y \sin^2(\eta/2)
\] (145)

It is easy to see that:

The maximum of \(\rho = 1\) occurs at \((\xi, \eta) = (0, 0)\)

The minimum of \(\rho = 1 - 4r_x - 4r_y\) occurs at \((\xi, \eta) = (\pi, \pi)\)

The requirement that \(\rho \geq -1\) yields the stability condition
\[
r_x + r_y \leq \frac{1}{2}
\] (146)

Hence the difference scheme (144) is conditional stable.

For \(\Delta x = \Delta y\) the condition for stability becomes \(r \leq 1/4\).
The following scheme is the 2-D Crank-Nicolson implicit scheme for approximating the PDE (142)

\[
\left( 1 - \frac{r_x}{2} \delta x^2 - \frac{r_y}{2} \delta y^2 \right) u_{jk}^{n+1} = \left( 1 + \frac{r_x}{2} \delta x^2 + \frac{r_y}{2} \delta y^2 \right) u_{jk}^n \\
+ \frac{\Delta t}{2} \left( F_{jk}^n + F_{jk}^{n+1} \right)
\]  

(147)

**STABILITY**: The symbol for the above difference scheme will be

\[
\rho(\xi, \eta) = \frac{1 - 2r_x \sin^2(\xi/2) - 2r_y \sin^2(\eta/2)}{1 + 2r_x \sin^2(\xi/2) + 2r_y \sin^2(\eta/2)}
\]  

(148)

Since for any \( r \geq 0 \)

\[
\left| \frac{1 - r}{1 + r} \right| \leq 1
\]

the difference scheme (147) is **unconditionally stable**.
When the uniform grid does not fit to the boundaries, we must treat differently the points near the boundary. Consider 5 points with non-uniform spacing, with distances $\theta_1 h$, $\theta_2 h$, $\theta_3 h$, $\theta_4 h$ from the central point.

Then the derivatives can be approximated as

\[
\left( \frac{\partial u}{\partial x} \right)_{1-0} = \frac{u_0 - u_1}{\theta_1 h}
\]

\[
\left( \frac{\partial u}{\partial x} \right)_{0-3} = \frac{u_3 - u_0}{\theta_3 h}
\]
\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{(u_3 - u_0)/\theta_3 h - (u_0 - u_1)/\theta_1 h}{(\theta_1 + \theta_3) h/2} \]

\[ = \frac{2}{h^2} \left[ \frac{u_1 - u_0}{\theta_1(\theta_1 + \theta_3)} + \frac{u_3 - u_0}{\theta_3(\theta_2 + \theta_3)} \right] + O(h) \quad (151) \]

\[ \frac{\partial^2 u}{\partial y^2} = \frac{2}{h^2} \left[ \frac{u_2 - u_0}{\theta_2(\theta_2 + \theta_4)} + \frac{u_4 - u_0}{\theta_4(\theta_2 + \theta_4)} \right] + O(h) \quad (152) \]

Combining we get:

\[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \]

\[ = \frac{2}{h^3} \left[ \frac{u_1}{\theta_1(\theta_1 + \theta_3)} + \frac{u_2}{\theta_2(\theta_2 + \theta_4)} + \frac{u_3}{\theta_3(\theta_1 + \theta_3)} + \frac{u_4}{\theta_4(\theta_2 + \theta_4)} \right] \]

\[ - \frac{2}{h^3} \left( \frac{1}{\theta_1 \theta_3} + \frac{1}{\theta_2 \theta_4} \right) u_0 \quad (153) \]
EXAMPLE
For circular regions, one may derive a finite-difference approximation to the Laplacian in polar coordinates.

\[
\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \\
= \frac{u_3 - 2u_0 + u_1}{(\Delta r)^2} + \frac{1}{r_0} \frac{u_3 - u_1}{2\Delta r} + \frac{1}{r_0^2} \frac{u_2 - 2u_0 + u_4}{(\Delta \theta)^2}
\]

(154)
For circular regions, one may derive a finite-difference approximation to the Laplacian in polar coordinates.

\[
\nabla^2 u = \frac{1}{(\Delta r)^2} \left[ \left( 1 - \frac{\Delta r}{2r_0} \right) u_1 + \left( 1 + \frac{\Delta r}{2r_0} \right) u_3 + \left( \frac{\Delta r}{r_0 \Delta \theta} \right)^2 (u_2 + u_4) \right] - \frac{2}{(\Delta r)^2} \left( 1 + \left( \frac{\Delta r}{r_0 \Delta \theta} \right)^2 \right) u_0 = 0 \quad (155)
\]
Spherical Grids
The heat conduction equation in cylindrical coordinates \((r, \theta, z)\) is:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2}
\]  

(156)

For simplicity we may assume that \(u\) is independent of \(z\) i.e.

\[
\frac{\partial u}{\partial t} = \nabla^2 u \equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial r^2}
\]  

(157)

At \(r = 0\) the right hand side appears to contain singularities, which can be approximated as follows:
Construct a circle of radius $\delta r$ as in the figure then we name the value of the origin by $u_0$, and we write:

$$\nabla^2 u = \frac{4(u_m - u_0)}{(\delta r)^2} + O(\delta r^2) \quad \text{for} \quad u_m = \frac{u_1 + u_2 + u_3 + u_4}{4} \quad (158)$$

We may rotate the axis by $\delta \theta$ and get another prediction for $u_m$, the best mean value available is given by adding all values and dividing by their number.

When a 2D problem in cylindrical coordinates possesses circular symmetry $\frac{\partial^2 u}{\partial \theta^2} = 0$ we get the simpler form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r}. \quad (159)$$
A similar problem arises at $r = 0$ with spherical polar coordinates in which the Laplacian operator assumes the form:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{\cot \theta}{r} \frac{\partial u}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}$$  \hspace{1cm} (160)

By the same argument the previous equation can be replaced at $r = 0$ by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u^2}{\partial z^2}$$  \hspace{1cm} (161)

which can be approximated by

$$\nabla^2 u = \frac{6(u_m - u_0)}{(\delta r)^2} + O(\delta r^2)$$  \hspace{1cm} (162)

where $u_m$ is the mean of $u$ over the sphere of radius $\delta r$. 

72
If the problem is symmetrical with respect to the origin, that is independent of $\theta$ and $\phi$ we get the simpler form

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{2 \partial u}{r \partial r}. \]  

(163)

with $\frac{\partial u}{\partial r} = 0$ at $r = 0$

In the case of symmetrical heat flow problems for hollow cylinders and spheres that exclude $r = 0$ simpler equations than the above may be employed by suitable changes of variable.

- The change of variable $R = \log_r r$ transforms the cylindrical equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r}. \]

(164)

to

\[ e^{2r} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial R^2}. \]

(165)
The change of dependent variable given by $u = \frac{w}{r}$ transforms the spherical equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r}.$$  \hspace{1cm} (166)


to

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial r^2}.$$  \hspace{1cm} (167)