# Partial Differential Equations: Analytic Solutions

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# Introduction

# Introduction i

- A differential equation involving more than one independent variable is called a **partial differential equations** (PDEs)

- Many problems in applied science, physics and engineering are modeled mathematically with PDE.

- In this course will study finite-difference methods in solving numerically PDEs, which are based on formulas for approximating the 1st and the 2nd derivatives of a function.

- In this lecture we present analytical ways for the solution of the simplest PDEs in order to use them as benchmark of our numerical methods as well as understanding what type of solutions one should expect.

# Introduction ii

PDEs are classified as one of three types, with terminology borrowed from the conic sections. For a 2nd-degree polynomial in $x$ and $y$

$$Ax^2 + Bxy + Cy^2 + D = 0$$

the graph is a quadratic curve, and when

- $B^2 - 4AC < 0$ the curve is a **ellipse**,
- $B^2 - 4AC = 0$ the curve is a **parabola**
- $B^2 - 4AC > 0$ the curve is a **hyperbola**

In the same way a PDE of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right) = 0 \tag{1}$$

where $A$, $B$ and $C$ are constants, is called **quasilinear**.

# Introduction iii

There are 3 types of quasilinear equations:

- If $B^2 - 4AC < 0$, the equation is called **elliptic**,
- If $B^2 - 4AC = 0$, the equation is called **parabolic**
- If $B^2 - 4AC > 0$, the equation is called **hyperbolic**

# Introduction iv

Two classic examples of PDEs are the 2-D Laplace and Poisson eqns:

$$\nabla^2 u = 0\,, \quad \nabla^2 u = g(x,y) \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad 0 < y < 1 \tag{2}$$

with boundary conditions:

- $u(x,0) = f_1(x)$ for $y = 0$ and $0 \leq x \leq 1$
- $u(x,0) = f_2(x)$ for $y = 1$ and $0 \leq x \leq 1$
- $u(x,0) = f_3(x)$ for $x = 0$ and $0 \leq y \leq 1$
- $u(x,0) = f_4(x)$ for $x = 1$ and $0 \leq y \leq 1$

for which $B = 0$, $A = C = 1$ and thus they are **elliptic** PDEs.

# Introduction v

The **wave equation**

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \quad \text{for} \quad 0 < x < L \quad \text{and} \quad 0 < t < \infty \tag{3}$$

with a given initial position and velocity functions

- $u(x, 0) = f(x)$ for $t = 0$ and $0 \leq x \leq L$
- $u_t(x, 0) = g(x)$ for $t = 0$ and $0 \leq x \leq L$

is a classic example of **hyperbolic** PDE.

The 3D form is:

$$\frac{\partial^2 u(\vec{x}, t)}{c^2 \partial t^2} - \nabla^2 u(\vec{x}, t) = 0 \tag{4}$$

The **heat equation** which describes how the distribution of heat evolves in time. It is practically the diffusion equation for solids.

$$\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{5}$$

the initial temperature distribution at $t = 0$ is

- $u(x, 0) = f(x)$ for $t = 0$ and $0 \leq x \leq L$

and the boundary conditions at the ends of the rod are

- $u(x, t) = c_1$ for $x = 0$ and $0 \leq t \leq \infty$
- $u(L, t) = c_2$ for $x = L$ and $0 \leq t \leq \infty$

is an example of **parabolic** PDE.

The 3D form is:

$$\frac{\partial u(\vec{x}, t)}{\partial t} - \alpha^2 \nabla^2 u(\vec{x}, t) = 0 \tag{6}$$

# Parabolic PDEs

# Parabolic PDEs i

We will present a simple method in solving analytically **parabolic PDEs.**

The most important example of a parabolic PDE is the **heat equation**. For example, to model mathematically the change in temperature along a rod. Let's consider the PDE:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad 0 \leq x \leq 1 \quad \text{and for} \quad 0 \leq t < \infty \tag{7}$$

with the **boundary conditions** :

$$u(0, t) = 0, \quad u(1, t) = 0 \quad \text{for} \quad 0 \leq t < \infty \tag{8}$$

and the **initial conditions** :

$$u(x, 0) = \phi(x), \quad \text{for} \quad 0 \leq x \leq 1\,. \tag{9}$$

This is the equation of 1D heat flow, where
$u_t$ : is the rate of change with respect to time
$u_{xx}$ : is the concavity of the temperature profile $u(t, x)$ (comparing the temperature at one point to the temperature at the neighbouring points).

# Parabolic PDEs ii

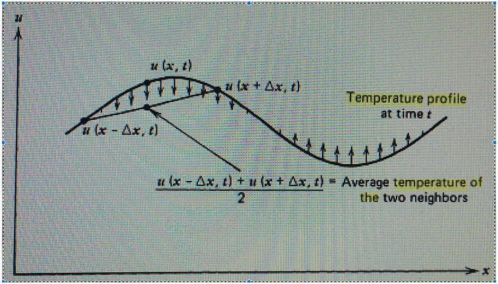



**Figure 1:** Arrows indicating change in temperature according to $u_t = \alpha^2 u_{xx}$.

# Parabolic PDEs iii

In order to find a solution we make the ansatz that $u(t, x)$ can be written as the product of two functions $T(t)$ and $X(x)$ i.e.

$$u(t, x) \equiv T(t) \cdot X(x) \tag{10}$$

Substituting in eqn (7) we get:

$$T'(t)X(x) = \alpha^2 T(t)X''(x) \tag{11}$$

This relation can be written as:

$$\frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda^2 \tag{12}$$

where $\lambda$ is a proportionality constant.

In this way we reduced the initial parabolic PDE into 2 ODEs

$$\begin{aligned} \dot{T} &- \lambda^2\alpha^2 T = 0 && (13) \\ X'' &- \lambda^2 X = 0 && (14) \end{aligned}$$

# Parabolic PDEs iv

with obvious solutions:

$$T(t) = C\, e^{-\lambda^2 \alpha^2 t} \tag{15}$$

$$X(x) = \tilde{A} \sin(\lambda x) + \tilde{B} \cos(\lambda x) \tag{16}$$

Then $u(t,x)$ gets the form:

$$u(t,x) = C e^{-\lambda^2 \alpha^2 t} \left[ \tilde{A} \sin(\lambda x) + \tilde{B} \cos(\lambda x) \right] , \tag{17}$$

by substituting $C \cdot \tilde{A} \to A$ and $C \cdot \tilde{B} \to B$ we get rid of the integration constant $C$.

The two remaining constants of integrationò $A$ and $B$ will be further constrained by the use of the boundary conditions (8).

- The boundary condition $u(0,t) = 0$ leads to $B = 0$, thus the solution becomes:

$$u(t,x) = A e^{-\lambda^2 \alpha^2 t} \sin(\lambda x) , \tag{18}$$

# Parabolic PDEs v

- The 2nd boundary condition $u(1, t) = 0$ leads to:

$$e^{-\lambda^2\alpha^2 t}\sin(\lambda) = 0 \quad \Rightarrow \quad \sin(\lambda) = 0 \tag{19}$$

with the following allowed values of the constant $\lambda$:

$$\lambda = \pm\pi, \pm2\pi, \pm3\pi, \ldots \quad \text{i.e.} \quad \lambda = \pm n\pi \quad \text{for} \quad n = \pm1, \pm2, \ldots \tag{20}$$

- Thus all functions of the form:

$$u_n(t, x) = A_n e^{-(n\pi\alpha)^2 t}\sin(n\pi x) \quad \text{for} \quad n = \pm1, \pm2, \ldots \tag{21}$$

are solutions of the PDE (7), while the constants of integration $A_n$ should be estimated by using the initial conditions provided by (9).

- The general solution will be the sum of all solutions of the form (21), i.e. :

$$u(t, x) = \sum_{n=1}^{\infty} A_n e^{-(n\pi\alpha)^2 t}\sin(n\pi x) \quad \text{for} \quad n = \pm1, \pm2, \ldots . \tag{22}$$

- The initial condition (9), $u(0,x)=\phi(x)$, by setting $t=0$ into eqn (22), can be written in the form:

$$\phi(x)=u(0,x)=\sum_{n=1}^{\infty}A_n\sin(n\pi x) \tag{23}$$

The constants $A_n$ can be found via the initial form of $\phi(x)$ by using the **orthonality conditions of trigonometric functions**, i.e.:

$$\int_0^1 \sin(n\pi x)\cdot\sin(m\pi x)dx=\begin{cases}0, & m\neq n\\ 1/2, & m=n\end{cases} \tag{24}$$

thus if we multiply (23) with $\sin(m\pi x)$ (where $m$ is any integer) and integrate from $0$ to $1$ we get:

$$\int_0^1 \phi(x)\cdot\sin(m\pi x)dx=A_m\int_0^1 \sin^2(m\pi x)dx=\frac{1}{2}A_m\,. \tag{25}$$

Thus the constants $A_n$ can be calculated by integration constants of the form:

$$A_n = 2\int_0^1 \phi(x)\sin(n\pi x)dx \tag{26}$$

and obviously they depend in a unique way on the initial form of the function $\phi(x)$.

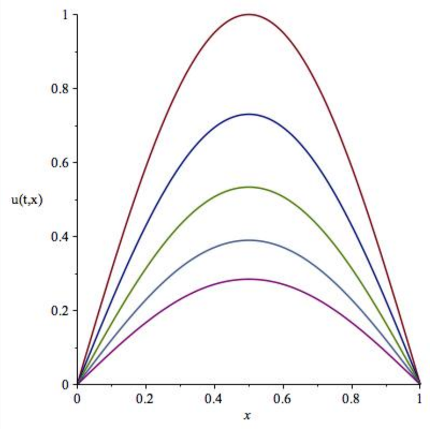
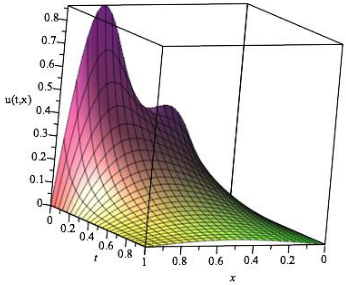



**Figure 2:** Evolution in time of $u_1(t, x)$ (for $t = 0 \ldots 0.4$) and $\phi(x) = \sin(x)$.




Evolution in time ($t = 0 \ldots 0.4$) of

$$u(t,x) = \sum_{n=1}^{4} A_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x)\,. \tag{27}$$

Here the coefficients $A_n$ get the values: $A_1 = 0.596$, $A_2 = -0.275$, $A_3 = 0.181$, $A_4 = -0.135$.

# Hyperbolic PDEs -1D

# Hyperbolic PDEs - 1D i

Here we will present an analytic method for solving 1D hyperbolic PDEs (1D wave equation)

$$u_{tt} = c^2 u_{xx} \tag{28}$$

This PDE can represent the oscillations of a string.

- The function $u(t,x)$ represents the deviation from equilibrium and the constant $c$ the propagation velocity of the waves.

- For simplicity, we will consider that the string is fixed on both ends and its length is $\ell = 1$.

In other words the **boundary conditions** will be:

$$u(t,0) = u(t,1) = 0 \quad \text{for} \quad t \geq 0 \tag{29}$$

while the **initial conditions** are:

$$u(0,x) = \phi(x) \quad \text{and} \quad u_t(0,x) = \psi(x) \quad \text{for} \quad 0 \leq x \leq 1\,. \tag{30}$$

We will apply again the ansatz used earlier for parabolic PDEs, i.e. we will write the function $u(t,x)$ as:

$$u(t,x) = T(t) \cdot X(x) \tag{31}$$

where $T$ is function of time $t$ and $X$ is function of the spatial dimension $x$.

By substitution in (28) we get:

$$\ddot{T} \cdot X = c^2 T \cdot X'' \tag{32}$$

if we separate the terms of the above equation appropriately we get:

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X} = \lambda^2 \quad \text{where} \quad \lambda = \text{constant} \tag{33}$$

Which leads to two ODEs:

$$\ddot{T} \quad - \quad \lambda^2 c^2 T = 0 \tag{34}$$

$$X'' \quad - \quad \lambda^2 X = 0\,. \tag{35}$$

The general solutions if the previous equations are:

$$T(t) = \alpha_1 \cos(c\lambda t) + \alpha_2 \sin(c\lambda t) \tag{36}$$

$$X(x) = \beta_1 \cos(\lambda x) + \beta_2 \sin(\lambda x)\,. \tag{37}$$

Due to the boundary condition $u(t,0) = u(t,1) = 0$ we will get $X(0) = X(1) = 0$ which means that:

$$X(0) = \beta_1 = 0 \tag{38}$$

$$X(1) = \beta_2 \sin(\lambda) = 0\,. \tag{39}$$

The last equation will be satisfied , when

$$\lambda = n\pi \quad \text{where} \quad n = 0, 1, 2, \ldots \tag{40}$$

# Hyperbolic PDEs - 1D iv

This leads to an infinite number of potential solutions

$$X_n(x) = \beta_2 \sin(n\pi x)\,. \tag{41}$$

Based on the previous discussion the expression for $T(t)$ will be:

$$T_n(t) = \alpha_{1,n} \cos(n\pi ct) + \alpha_{2,n} \sin(n\pi ct) \tag{42}$$

As a result of (41) and (42) the wave equation is satisfied by an infinite number of solutions, which have the form:

$$\begin{aligned} u_n(t,x) &= T_n(t)\, X_n(x) \\ &= A_n \cos(n\pi ct) \sin(n\pi x) + B_n \sin(n\pi ct) \sin(n\pi x) \end{aligned} \tag{43}$$

where $A_n = \alpha_{1,n}\beta_2$ and $B_n = \alpha_{2,n}\beta_2$.

In order to find a solution that satisfies the **initial contitions** (30) we will assume that the solution is given as a superposition of an infinite number of solutions (or a subset of them) i.e. the solutions will be of the form

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} u_n(t) = \sum_{n=1}^{\infty} T_n(t)\,X_n(x) \\ &= \sum_{n=1}^{\infty} \left[A_n \cos(n\pi ct)\sin(n\pi x) + B_n \sin(n\pi ct)\sin(n\pi x)\right] \end{aligned} \tag{44}$$

From the **1st of the initial conditions** (30) $[u(0,x) = \phi(x)]$ we get:

$$u(0,x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = \phi(x) \tag{45}$$

The constants $A_n$ can be found in terms of the function describing the initial deformation of the string i.e. the $\phi(x)$ by using the orthogonality properties of trigonometric functions i.e.

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)dx = \begin{cases} 0, & m \neq n \\ 1/2, & m = n \end{cases} \tag{46}$$

Thus if we multiply (45) with $\sin(m\pi x)$ (where $m$ is any number) and integrate from $0$ to $1$ we get:

$$\int_0^1 \phi(x)\sin(m\pi x)dx = A_m \int_0^1 \sin^2(m\pi x)dx = \frac{1}{2}A_m \tag{47}$$

Thus we conclude that the constants $A_n$ will be found from integral equations of the form:

$$A_n = 2\int_0^1 \phi(x)\sin(n\pi x)dx \tag{48}$$

and obviously depend on a unique way on the form of $\phi(x)$.

# Hyperbolic PDEs - 1D vii

The **2nd initial condition** will lead us in calculating the other unknown constant i.e. the $B_n$. Thus by taking the derivative of (44) we get:

$$u_t(t,x) = \pi c \sum_{n=1}^{\infty} n \left[ B_n \cos(n\pi ct) \sin(n\pi x) - A_n \sin(n\pi ct) \sin(n\pi x) \right] \tag{49}$$

which for $t = 0$ will give us:

$$u_t(0,x) = \pi c \sum_{n=1}^{\infty} n B_n \sin(n\pi x) = \psi(x) \tag{50}$$

And thus:

$$B_n = \frac{2}{n\pi} \int_0^1 \psi(x) \sin(n\pi x) dx \quad \text{for} \quad n = 1, 2, \ldots \tag{51}$$

Which means that the solutions has been fully determined.

# Elliptic PDEs

# Elliptic PDEs

Poisson's equation

$$\nabla^2 u(x, y, z) = f(x, y, z) \tag{52}$$

is a fundamental elliptic PDE that can be met in various branches of physics e.g. gravity, electromagnetism etc

We will study this equation based on the techniques developed earlier for the other two types of PDEs (parabolic, hyperbolic).

**NOTE**: The solutions of elliptic PDEs are completly determined by the boundary conditions and thus we call them also boundary value problems (BVP).

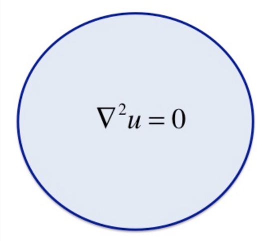
# Elliptic PDEs: Boundary Conditions

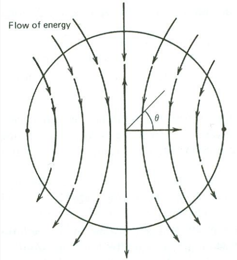
- **Dirichlet problem**: we define the values of the function $u$ on the boundaries of the domain.

*For example, consider the problem on finding the temperature at the various points of a 2D domain from the values that one can measure on the boundaries.*

- **Neumann Problem**: in this case we define the values of $\partial u/\partial n$ on the boundaries of the domain.

*For example, consider finding the continuous flow of e.g. electrons, energy etc, in the interior of domain where the flow is known only on the boundaries.*

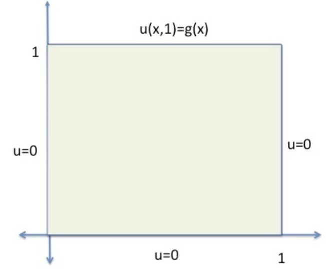


# Elliptic PDEs: 2D - Cartesian grid i

Here we will present a solution of Poisson equation in Cartesian coordinates with Dirichlet boundary conditions.

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{53}$$



**Dirichlet boundary conditions** on a square.

$$
\begin{aligned}
u(0,y) &= u(1,y) = 0 \quad 0 \le y \le 1 && (54)\\
u(x,0) &= 0 \qquad\qquad\quad 0 \le x \le 1 && (55)\\
u(x,1) &= g(x) \qquad\qquad 0 \le x \le 1 && (56)
\end{aligned}
$$

We assume that the solution can be written in the form:

$$u(x,y) = X(x) \cdot Y(y) \tag{57}$$

Then by substituting in Poisson equation (53) we get:

$$X''(x)Y(y) + X(x)Y''(y) = 0 \tag{58}$$

which leads to

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \mathrm{const} \tag{59}$$

Thus we will consider the solution of two independent boundary value problems.

# Elliptic PDEs: 2D - Cartesian grid iii

- BVP-1

$$X''(x) + \lambda^2 X(x) = 0 \quad \text{where} \quad 0 < x < 1 \quad \text{and} \quad X(0) = X(1) = 0\,, \tag{60}$$

this problems has the following eigenvalues:

$$\lambda_k = k\pi\,, \quad k = 1, 2, \ldots\,, \tag{61}$$

and the corresponding eigenfunctions:

$$X_k(x) = \sin(k\pi x)\,, \quad k = 1, 2, \ldots\,, \tag{62}$$

- BVP-2

$$Y''(y) - \beta^2 Y(y) = 0 \quad \text{where} \quad 0 < y < 1 \quad \text{and} \quad Y(0) = Y(1) = 0\,, \tag{63}$$

The general solution of (63), will be a linear combination of $e^{\beta y}$ and of $e^{-\beta y}$ and due to the boundary condition at $y = 0$ the solution will get the form:

$$Y(y) = \sinh(\beta y) \tag{64}$$

# Elliptic PDEs: 2D - Cartesian grid iv

Thus the partial solutions ($\beta^2 \equiv \lambda^2$) will have the form:

$$u_k(x, y) = \sin(k\pi x) \cdot \sinh(k\pi y), \quad k = 1, 2, \dots, \tag{65}$$

and the general solution will be a linear combination of the partial ones:

$$u(x, y) = \sum_{k=1}^{\infty} c_k \sin(k\pi x) \sinh(k\pi y) \tag{66}$$

where $c_k$ are arbitrary real constants.

If we take into account the boundary condition given by (56) [$u(x, 1) = g(x)$]

Then we get

$$g(x) = u(x, 1) = \sum_{k=1}^{\infty} g_k \sin(k\pi x) \quad \text{where} \quad g_k = c_k \sinh(k\pi) \tag{67}$$

# Elliptic PDEs: 2D - Cartesian grid v

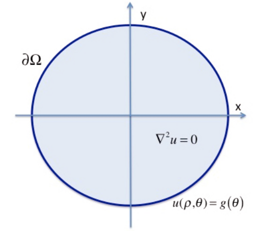
where the coefficients $g_k$, will be

$$g_k = 2\int_0^1 g(x)\sin(k\pi x)dx \tag{68}$$

and obviously

$$c_k = g_k / \sinh(k\pi) \quad \text{for} \quad k = 1, 2, \ldots \tag{69}$$

We will study the interior Dirichlet problem for disk.




The Poisson equation for the disk is:

$$\nabla^2 u \equiv u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \text{for} \quad 0 < r < \rho \tag{70}$$

With boundary condition:

$$u(\rho,\theta) = g(\theta) \quad \text{for} \quad 0 \leq \theta < 2\pi \tag{71}$$

We will solve the problem by using the method developed in earlier sections

# Elliptic PDEs - 2D - Polar ii

We will assume that the function $g(\theta)$ is periodic and thus it can be written in the form of a Fourier series

$$g(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[ a_k \cos(k\theta) + b_k \sin(k\theta) \right] , \tag{72}$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos(k\theta) d\theta \tag{73}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sin(k\theta) d\theta \tag{74}$$

We will use again the ansatz :

$$u(r,\theta) = R(r)\cdot\Theta(\theta)\,, \tag{75}$$

and then equation (70) will be written as:

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \tag{76}$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} \equiv \lambda \tag{77}$$

where $\lambda$ is a constant independent from $r$ and $\theta$.

Thus we end up with the ODEs

$$\Theta'' + \lambda\Theta = 0 \tag{78}$$

and

$$r^2R'' + rR' - \lambda R = 0 \tag{79}$$

# Elliptic PDEs - 2D - Polar iv

Obviously, the function $\Theta$ must be periodic with period $2\pi$ with respect to $\theta$. While for the ODE (78) we set the conditions:

$$\Theta(-\pi) = \Theta(\pi) \quad \text{and} \quad \Theta'(-\pi) = \Theta'(\pi) \tag{80}$$

We these conditions we get the eigenvalues:

$$\lambda_k = k^2 \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{81}$$

with eigenfunctions

$$\Theta_k(\theta) = c_1 \cos(k\theta) + c_2 \sin(k\theta) \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{82}$$

where $c_1$ and $c_2$ are arbitrary constants.

- The ODE (79) is known as **Cauchy-Euler's equation** and admits solutions of the form:

$$R(r) = r^{\beta} \tag{83}$$

which can be substituted in (79) to get:

$$\beta(\beta-1)r^{\beta}+\beta r^{\beta}-k^2 r^{\beta}=0\,, \tag{84}$$

which leads to the relation:

$$\beta=\pm k \tag{85}$$

Then for $k\geq 1$ we get a solution which will be a linear combination of $r^{-k}$ and $r^k$ i.e.

$$R(r)=c_1 r^k+c_2 r^{-k} \tag{86}$$

While for $k=\lambda=0$ the solution will be of the form:

$$R(r)=c_1\ln(r)+c_2 \tag{87}$$

Since, physically, the interior solutions cannot be divergent the solutions of the form $r^{-k}$ and $\ln(r)$ that diverge at $r=0$ are discarted. Thus the acceptable solution will be of the form:

$$R_k(r)=c_1 r^k \quad \text{for} \quad k=0,1,2,\ldots \tag{88}$$

Thus the general solution will be:

$$u(r,\theta) = \frac{\tilde{a}_0}{2} + \sum_{k=1}^{\infty} r^k \left(\tilde{a}_k \cos(k\theta) + \tilde{b}_k \sin(k\theta)\right) \tag{89}$$

where $\tilde{a}_k$ and $\tilde{b}_k$ are constants which should be extimated via the boundary condition $u(r,\theta) = g(\theta)$.

Then from the equations (72), (73) and (74) we find that:

$$\tilde{a}_k = a_k r^{-k} \quad \text{and} \quad \tilde{b}_k = b_k r^{-k} \tag{90}$$

# Problems for Elliptic PDEs

- What is the solution of the interior Dirichlet problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad 0 < r < 1$$

  for the following boundary conditions:

  1. $u(1, \theta) = 1 + \sin(\theta) + \frac{1}{2}\cos\theta$
  2. $u(1, \theta) = 2$
  3. $u(1, \theta) = \sin\theta$
  4. $u(1, \theta) = \sin 3\theta$

  Can you draw the solution?

- What is the solution of the interior Dirichlet problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad 0 < r < 2$$

  for the boundary condition $u(2, \theta) = \sin\theta$. Can you draw the solution?

- What would have been the solution o fthe previous problem if the boundary condition was $u(2, \theta) = \sin(2\theta)$. Can you draw the new solution?

# Hyperbolic PDEs - 2D

# Hyperbolic PDEs - 2D i

The PDE for the motion of a membrane is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{91}$$

where $c$ is the propagation velocity.

Eqn (91) is a 2nd order hyperbolic PDE and for the solution we need to define *initial* and the *boundary* conditions in order to find the unique solution.

- Here we will assume a *circular membrane* with fixed ends e.g. a drum:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} \right] \tag{92}$$

The **boundary condition** will be

$$u(r = R, \phi, t) = 0. \tag{93}$$

# Hyperbolic PDEs - 2D ii

We will also assume, for simplicity, that the radius of the drum is $R = 1$, and thus the boundary condition (93) will be written as:

$$u(r = 1, \phi, t) = 0. \tag{94}$$

We will assume that $u(r, \phi, t)$ can be written as the product of two functions one describing the spatial part and the other one the temporal part of the solution i.e.:

$$u(r, \phi, t) = G(r, \phi)D(t). \tag{95}$$

If we substitute (95) in (92) we can divide the spatial and the temporal part of the wave equation

$$\frac{1}{D}\frac{d^2D(t)}{dt^2} = c^2\frac{1}{G(r,\phi)}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 G}{\partial \phi^2}\right] = \text{constant} \equiv -\omega^2 \tag{96}$$

# Hyperbolic PDEs - 2D iii

- Since the left part of the equation is function **only** of the time and the right hand side **only** of the spatial coordinates, they two relations should be equal to a constant $-\omega^2$.

- Thus the problem reduced to the solution of the following two equations:

$$\frac{d^2 D}{dt^2} = -\omega^2 D \quad (97)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 G}{\partial \phi^2} = -\frac{\omega^2}{c^2}G = -k^2 G \quad (98)$$

where $k = \omega/c$ is the wavenumber.

- We will assume that $G(r,\phi)$ can be split into two functions. One of the radial coordinate and the other of the angular coordinate, i.e. $G(r,\phi) = h(r)g(\phi)$. By substitution in (98) we get

$$\frac{r^2}{h}\frac{d^2 h}{dr^2} + \frac{r}{h}\frac{dh}{dr} + k^2 r^2 = -\frac{1}{g}\frac{d^2 g}{d\phi^2} = \text{constant} = m^2 \quad (99)$$

Thus the problem has been reduced to the solution of two ODEs

$$
\begin{aligned}
\frac{d^2 g}{d\phi^2} + m^2 g &= 0 && (100) \\
\frac{d^2 h}{dr^2} + \frac{1}{r}\frac{dh}{dr} + \left(k^2 - \frac{m^2}{r^2}\right) h &= 0 && (101)
\end{aligned}
$$

The equation (100) is an ODE which describes a canonical oscillation, with respect to the angular coordinate $\phi$. The general solution is:

$$g(\phi) = A\sin(m\phi) + B\cos(m\phi) \tag{102}$$

The second equation, (101), reduced to the known Bessel equation by considering the change of variable $x = kr$

$$h'' + \frac{1}{x}h' + \left(1 - \frac{m^2}{x^2}\right) h = 0 \tag{103}$$

Then the solution of the equation (103) for the oscillating membrane is:

$$h(x) = J_m(x) = J_m(kr) \quad \text{for} \quad m = 0, 1, 2, ... \tag{104}$$

where $J_m(x)$ is the Bessel function of 1st kind which in the form of series can be given as:

$$\begin{aligned} J_m(x) &= \frac{1}{m!}\left(\frac{x}{2}\right)^m \left[1 - \frac{1}{m+1}\left(\frac{x}{2}\right)^2 + \frac{1}{(m+1)(m+2)}\,\frac{1}{2!}\left(\frac{x}{2}\right)^4 - \cdots\right] \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\;\Gamma(m+i+1)}\left(\frac{x}{2}\right)^{m+2i} \end{aligned} \tag{105}$$

⋆⋆⋆ Bessel functions obey orthogonality relations like the ones of the Legendre functions.

If $J_m(kx)$ and $J_m(lx)$ (or their derivatives) vanish at the points $a$ and $b$ then

$$\int_a^b J_m(kx)J_m(lx)xdx = 0 \quad \text{for} \quad k \neq l \tag{106}$$

while

$$\int_a^b J_m(kx)J_m(lx)xdx = \frac{x^2}{2}\left[J_m'(kx)\right]^2 \Big|_a^b = \frac{x^2}{2}\left[J_{m+1}(kx)\right]^2 \Big|_a^b \quad \text{for} \quad k = l \tag{107}$$

⋆⋆⋆ These orthogonality relations are used typically for the estimation of the coefficients in expansions of special functions here Bessel.

$$\begin{aligned}
J_0(z) &= 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 \ldots \\
J_1(z) &= \frac{1}{2}x - \frac{1}{16}x^3 + \frac{1}{384}x^5 \ldots \\
J_2(z) &= \frac{1}{8}x^2 - \frac{1}{96}x^4 + \ldots
\end{aligned}$$

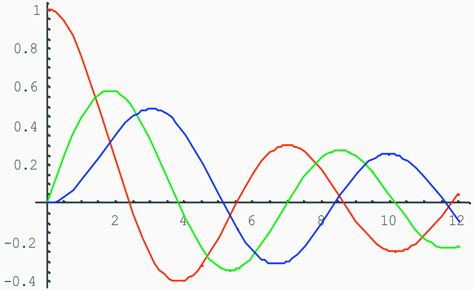



**Figure 4:** The Bessel functions $J_0(x)$, $J_1(x)$ and $J_2(x)$.

Thus the solution of (98) will be written in the form

$$G_m(r,\phi) = J_m(kr)\left[A\sin(m\phi) + B\cos(m\phi)\right] \tag{108}$$

The index $m$ in $G_m(r,\phi)$ shows that there are differnt solutions for different values of $m$.

Up to now we have not used the boundary conditions (94) or (95), which according to the splitting of the function will be written as:

$$G(r=1,\phi) = h(r=1)\cdot g(\phi) = 0 \tag{109}$$

and thus

$$J_m(k) = 0\,. \tag{110}$$

# Hyperbolic PDEs - 2D ix

I.e. the estimation of the eigenvalues $k$ has been reduced in finding the roots of the equation (110) or better the roots of the Bessel functions.

- Bessel functions have infinite number of roots and thus for every value of $m$ we have to calculate an infinite number of eigenvalues $k$ from equation (110).

- We will write $k$, as $k_{m,n}$ where the index $n$ stands for the $n$-th root of the $m$ Bessel function.

- In a similar way we will represent $\omega = ck$ i.e. as $\omega_{m,n}$.

- Some characterist values of the roots of the Bessel functions are:

$$
\begin{aligned}
J_0(k) = 0 \quad &: \quad k_{0,n} \sim 2.408,\ 5.520,\ 8.654,\ 11.791 \\
J_1(k) = 0 \quad &: \quad k_{1,n} \sim 3.832,\ 7.016,\ 10.174,\ 13.324 \\
J_2(k) = 0 \quad &: \quad k_{2,n} \sim 5.136,\ 8.417,\ 11.620,\ 14.796
\end{aligned}
$$

# Hyperbolic PDEs - 2D x

The lower frequencies and the corresponding eigenfunctions will be:

$$
\begin{aligned}
k_{0,1} &= 2.40 \qquad & \omega_{0,1} &= 2.40c \quad & G_0 &\sim J_0(2.40r) \\
k_{1,1} &= 3.83 \qquad & \omega_{1,1} &= 3.83c \quad & G_1 &\sim J_1(3.83r)\left[A\sin(\phi) + B\cos(\phi)\right] \\
k_{2,1} &= 5.14 \qquad & \omega_{2,1} &= 5.14c \quad & G_2 &\sim J_2(5.14r)\left[A\sin(2\phi) + B\cos(2\phi)\right] \\
k_{0,2} &= 5.52 \qquad & \omega_{0,2} &= 5.52c \quad & G_0 &\sim J_0(5.52r)
\end{aligned}
$$

The time dependent solution of (92) for specific values of $m$ and $n$ will be:

$$
\begin{aligned}
u_{m,n}(r,\phi,t) = J_m(k_{m,n}r) \quad &\times \quad \left[A_{m,n}\sin(m\phi) + B_{m,n}\cos(m\phi)\right] \\
&\times \quad \left[\Gamma_{m,n}\sin(\omega_{m,n}t) + \Delta_{m,n}\cos(\omega_{m,n}t)\right]
\end{aligned}
\tag{111}
$$

where $\omega_{m,n} = ck_{m,n}$.

Since equation (92) is a *linear* equation of motion, the general solution can be written as a superposition of solutions described in (112).

Thus by summing on $m$ and $n$ we get:

$$u(r,\phi,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{m,n} u_{m,n}(r,\phi,t) \tag{112}$$

or in analytic form

$$\begin{aligned} u(r,\phi,t) &= \sum_{m,n=0}^{\infty} K_{m,n} J_m(k_{m,n} r)\sin(m\phi)\left[\Gamma_{m,n}\sin(\omega_{m,n}t) + \Delta_{m,n}\cos(\omega_{m,n}t)\right] \\ &+ \sum_{m,n=0}^{\infty} L_{m,n} J_m(k_{m,n} r)\cos(m\phi)\left[\Gamma_{m,n}\sin(\omega_{m,n}t) + \Delta_{m,n}\cos(\omega_{m,n}t)\right] \end{aligned} \tag{113}$$

where $K_{m,n} = C_{m,n}A_{m,n}$ and $L_{m,n} = C_{m,n}B_{m,n}$.

The unknown quantities in (113) are the (time independent) coefficients $K_{m,n}\Gamma_{m,n}$, $K_{m,n}\Delta_{m,n}$, $L_{m,n}\Gamma_{m,n}$ and $L_{m,n}\Delta_{m,n}$.

By using properly the initial conditions we can get the coefficients from the relations:

$$K_{m,n}\Gamma_{m,n} = \frac{1}{\omega_{m,n}\pi J_{m,n}}\left[I_2^{m,n}I_3^{m,n} + I_1^{m,n}I_4^{m,n}\right] \tag{114}$$

$$L_{m,n}\Gamma_{m,n} = \frac{1}{\omega_{m,n}\pi J_{m,n}}\left[I_2^{m,n}I_5^{m,n} + I_1^{m,n}I_6^{m,n}\right] \tag{115}$$

$$K_{m,n}\Delta_{m,n} = \frac{1}{\pi J_{m,n}} I_1^{m,n}I_3^{m,n} \tag{116}$$

$$L_{m,n}\Delta_{m,n} = \frac{1}{\pi J_{m,n}} I_1^{m,n}I_5^{m,n} \tag{117}$$

Where the following integrals should be calculated:

$$J_{m,n} = \int_0^1 J_m^2(k_{m,n} r) r dr \tag{118}$$

$$I_1^{m,n} = \int_0^1 h_1(r) J_m(k_{m,n} r) r dr \tag{119}$$

$$I_2^{m,n} = \int_0^1 h_2(r) J_m(k_{m,n} r) r dr \tag{120}$$

$$I_3^{m,n} = \int_0^{2\pi} g_1(\phi) \sin(m\phi) d\phi \tag{121}$$

$$I_4^{m,n} = \int_0^{2\pi} g_2(\phi) \sin(m\phi) d\phi \tag{122}$$

$$I_5^{m,n} = \int_0^{2\pi} g_1(\phi) \cos(m\phi) d\phi \tag{123}$$

$$I_6^{m,n} = \int_0^{2\pi} g_2(\phi) \cos(m\phi) d\phi \tag{124}$$

# Vibrating circular membrane i

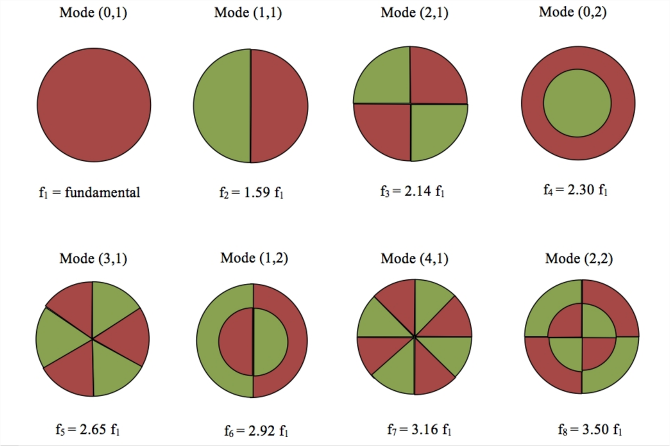



**Figure 5:** Normal modes of oscillation of a circular membrane. (Kyle Forinash)

# Vibrating circular membrane ii

Concluding the steps needed for the solution of the problem are:

1. Calculate the roots of Bessel functions.
2. Calculate the integrals $I_i^{m,n}$ .
3. Sum a large (depending on the computer) number of terms.

Animations:

http://resource.isvr.soton.ac.uk/spcg/tutorial/tutorial/Tutorial_files/Web-standing-membrane.htm

https://www.youtube.com/watch?v=Zkox6niJ1Wc

http://commons.wikimedia.org/wiki/Category:Drum_vibration_animations