# Gravitational Waves

*Theory*

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# Suggested Reading

## Books

- *Gravitational Waves: Volume I: Theory and Experiments* by Michele Maggiore, Oxford University Press (2007)
- *Gravitational Waves: Volume II: Astrophysical Sources* by Michele Maggiore, Oxford University Press (2018)
- *Gravitational-Wave Physics and Astronomy* by Jolien D.E. Greighton & Warren G. Andreson, Wiley (2011)
- *A First Course in General Relativity* B.F.Schutz, Cambridge (2009) (2nd edition)
- *Gravitation and Spacetime* by Hans C. Ohanian & Remo Ruffini, Cambridge University Press (2013)
- *Gravitational-Wave Astronomy* by Nils Andersson, Oxford University Press (2020)
- *Gravitation* by Charles W. Misner, Kip S. Thorne and John Archibald Wheeler (Sep 15, 1973) W.H. Freeman

## Review articles

- *Gravitational wave astronomy: in anticipation of first sources to be detected* L P Grishchuk, V M Lipunov, K A Postnov, M E Prokhorov, B S Sathyaprakash, Physics- Uspekhi 44 (1) 1 -51 (2001)
- *The basics of gravitational wave theory* E.E. Flanagan and S.A. Hughes, New Journal of Physics 7 (2005) 204
- *Gravitational Wave Astronomy* K.D. Kokkotas Rev. in Mod. Astroph., Vol 20, "Cosmic Matter", WILEY-VCH, (2008) arXiv:0809.1602 [astro-ph]

# Linearized Theory I

Weak gravitational fields can be represented by a slightly deformed Minkowski spacetime :

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu} + O(h_{\mu\nu})^2\,, \quad |h_{\mu\nu}| \ll 1 \tag{1}$$

here $h_{\mu\nu}$ is a small metric perturbation.

The indices will be raised and lowered by $\eta_{\mu\nu}$ i.e.

$$h^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu} \tag{2}$$

$$h = \eta^{\mu\nu}h_{\mu\nu} \tag{3}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad \text{(why?)} \tag{4}$$

and we will define the traceless ($\phi_{\mu\nu}$) tensor:

$$\phi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{5}$$

for which we get $\phi = \eta^{\mu\nu}\phi_{\mu\nu} = h - 2h = -h$ and

$$h_{\mu\nu} = \phi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\phi\,. \tag{6}$$

# Linearized Theory II

The Christoffel symbols & the Ricci tensor will become :

$$\Gamma^{\lambda}_{\mu\nu} \approx \frac{1}{2}\eta^{\lambda\rho}\left(h_{\rho\nu,\mu} + h_{\mu\rho,\nu} - h_{\mu\nu,\rho}\right) \quad (7)$$

$$R^{\sigma}{}_{\mu\nu\rho} \approx \frac{1}{2}\left(h^{\sigma}{}_{\rho,\nu\mu} + h_{\mu\nu,\rho}{}^{\sigma} - h_{\mu\rho,\nu}{}^{\sigma} - h^{\sigma}{}_{\nu,\rho\mu}\right) \quad (8)$$

$$R_{\mu\nu} \approx \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} \approx \frac{1}{2}\left(h^{\alpha}{}_{\nu,\mu\alpha} + h^{\alpha}{}_{\mu,\nu\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h^{\alpha}{}_{\alpha,\mu\nu}\right) \quad (9)$$

$$R = \eta^{\mu\nu}R_{\mu\nu} \approx h_{\alpha\beta}{}^{\alpha\beta} - h^{\alpha}_{\alpha}{}^{,\beta}_{,\beta} \quad (10)$$

Finally, Einstein tensor gets the form:

$$G^{(1)}_{\mu\nu} = \frac{1}{2}\left(h^{\alpha}{}_{\nu,\mu\alpha} + h^{\alpha}{}_{\mu,\nu\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h^{\alpha}{}_{\alpha,\mu\nu}\right) - \eta_{\mu\nu}\left(h_{\alpha\beta}{}^{,\alpha\beta} - h^{\alpha}_{\alpha}{}^{,\beta}_{,\beta}\right) \quad (11)$$

**Einstein's equations** reduce to (how?):

$$-\phi_{\mu\nu}{}_{,\alpha}^{,\alpha} - \eta_{\mu\nu}\phi_{\alpha\beta}{}^{,\alpha\beta} + \phi_{\mu\alpha}{}_{,\nu}^{,\alpha} + \phi_{\nu\alpha}{}_{,\mu}^{,\alpha} = \kappa T_{\mu\nu} \quad (12)$$

# Linearized Theory III

Then by using the so called Hilbert (or Lorenz or Harmonic or De Donder) gauge similar to Lorenz gauge ($A^{\alpha}{}_{,\alpha} = A_{\alpha}{}^{,\alpha} = 0$) in EM [1]

$$\phi^{\mu\alpha}{}_{,\alpha} = \phi_{\mu\alpha}{}^{,\alpha} = 0 \tag{13}$$

we come to the following equation:

$$\phi_{\mu\nu}{}_{,\alpha}^{,\alpha} \equiv \Box\phi_{\mu\nu} \equiv -\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi_{\mu\nu} = -\kappa T_{\mu\nu} \tag{14}$$

which is a simple wave equations describing ripples of spacetime propagating with the speed of light (why?). [2]

These ripples are called gravitational waves.

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[1] The De Donder gauge is defined in a curved background as$\partial_\mu(g^{\mu\nu}\sqrt{-g}) = 0$.

[2] Equations (13) and (14) together imply for consistency that $\partial^\nu T_{\mu\nu} = 0$, which is the conservation of energy-momentum in the linearized theory. In the full theory the partial derivative should be replaced by the covariant.

# GW: about Gauge conditions (*) i

General Relativity is invariant under the group of all possible coordinate transformations $x^\mu \to x'^\mu(x)$ where $x'^\mu$ is and arbitrary function of $x^\mu$.

Under this transformations (which should be invertible and differentiable) the metric transforms as

$$g_{\mu\nu} \to g'_{\mu\nu}(x') = \frac{\partial x^\kappa}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu} g_{\kappa\lambda}(x) \,. \tag{15}$$

This is usually referred as the gauge symmetry of GR.

When we make the assumption (1) we practically fix the coordinate system and we assume that the approximation is valid in an extended region of the space. Still a gauge symmetry remains and by careful choice of coordinates the linearized Einstein equations can be simplified.

We can fix $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ and make small changes in the coordinates that leave $\eta_{\mu\nu}$ unchanged but induce small changes in $h_{\mu\nu}$.

# GW: about Gauge conditions (*) ii

For example lets consider a change of the form:

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \tag{16}$$

where $\xi^{\mu}$ are 4 small arbitrary functions of the same order as $h^{\mu\nu}$.

Then

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}{}_{\nu} + \partial_{\nu}\xi^{\mu} \quad \text{and} \quad \frac{\partial x^{\mu}}{\partial x'^{\nu}} = \delta^{\mu}{}_{\nu} - \partial_{\nu}\xi^{\mu} \tag{17}$$

Thus, the metric transforms as:

$$\begin{aligned} g'_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma} &= \left(\delta^{\rho}_{\mu} - \partial_{\mu}\xi^{\rho}\right)\left(\delta^{\sigma}_{\nu} - \partial_{\nu}\xi^{\sigma}\right)\left(\eta_{\rho\sigma} + h_{\rho\sigma}\right) \\ &\approx \eta_{\mu\nu} + h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} = \eta_{\mu\nu} + h'_{\mu\nu} \end{aligned} \tag{18}$$

Then in the new coordinate system we get

$$h'_{\mu\nu} = h_{\mu\nu} - \left(\xi_{\mu,\nu} + \xi_{\nu,\mu}\right) \tag{19}$$

This transformation is called **gauge transformation**.

This is analogous to the gauge transformation in Electromagnetism ($A^{\mu}{}_{,\mu} = 0$).

That is, if $\tilde{A}_{\mu}$ is a solution of the EM field equations then another solution **that describes precisely the same physical situation** is given by

$$A_{\mu} = \tilde{A}_{\mu} - \psi_{,\mu} \tag{20}$$

where $\psi$ is any scalar field.

Then the gauge condition $A^{\mu}{}_{,\mu} = A_{\mu}{}^{,\mu} = 0$ means that

$$\Box\psi = \tilde{A}^{\mu}{}_{,\mu} = f(x). \tag{21}$$

where $\Box\psi = \psi_{,\mu}^{;\mu} = \psi_{,\mu}{}^{,\mu}$.

Then by solving the equation $\Box\psi = f(x)$ we get the appropriate scalar function $\psi$.

# GW: about Gauge conditions (*) iv

From (19) it is clear that if $\tilde{h}_{\mu\nu}$ is a solution to the linearised field equations then the same physical situation is also described by

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} - (\xi_{\mu,\nu} + \xi_{\nu,\mu}) \tag{22}$$

and

$$\begin{aligned} \phi^{\mu\nu} &= h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h \\ &= \tilde{h}^{\mu\nu} - (\xi^{\mu,\nu} + \xi^{\nu,\mu}) - \frac{1}{2}\eta^{\mu\nu}\left(\tilde{h} - 2\xi^{\sigma}{}_{,\sigma}\right) \\ &= \tilde{\phi}^{\mu\nu} - \left(\xi^{\mu,\nu} + \xi^{\nu,\mu} - \eta^{\mu\nu}\xi^{\sigma}{}_{,\sigma}\right) \\ &= \tilde{\phi}^{\mu\nu} - \Xi^{\mu\nu} \end{aligned} \tag{23}$$

**NOTE**

- This is a gauge transformation and not a coordinate one
- We are still working on the same set of coordinates $x^{\mu}$ and have defined a new tensor $h_{\mu\nu}$ whose components in this basis are given by (22) or (23).

# GW: about Gauge conditions (*) v

We can easily see that from (19) or (23) we can get

$$\phi^{\mu\nu}{}_{,\nu} = \tilde{\phi}^{\mu\nu}_{,\nu} - \Xi^{\mu\nu}_{,\nu} = \tilde{\phi}^{\mu\nu}_{,\nu} - \Box\xi^{\mu} \tag{24}$$

Therefore, if we assume that initially $\tilde{h}^{\mu\rho}$ obeys $\tilde{\phi}^{\mu\rho}_{,\rho} = f^{\mu}(x)$ we can choose the function $\xi^{\mu}$ so that to satisfy

$$\Box\xi^{\mu} = f^{\mu}(x) \tag{25}$$

Then we can get the Hilbert gauge

$$\phi^{\mu\nu}{}_{,\nu} = 0 \tag{26}$$

**NOTE:**

- This gauge condition is preserved by any further gauge transformation of the form (22) provided that the functions $\xi^{\mu}$ satisfy $\Box\xi^{\mu} = 0$ or equivalently $\Xi^{\mu\nu}{}_{,\nu} = 0$. [3]

# GW: about Gauge conditions (*) vi

- The choice of the Hilbert gauge $\phi^{\mu\nu}{}_{,\nu} = 0$, gives **4** conditions that reduce the **10** independent components of the symmetric tensor $h_{\mu\nu}$ to **6**!
- Eqn (23) tells us that, from the **6** independent components of $\phi_{\mu\nu}$ which satisfy $\Box\phi_{\mu\nu} = 0$, we can subtract the functions $\Xi_{\mu\nu}$, which depend on 4 independent arbitrary functions $\xi_\mu$ satisfying the same equation

$$\Box\xi^\mu = 0. \tag{29}$$

  This implies also that $\Box\Xi_{\mu\nu} = 0$ (how?) since in the flat spacetime d'Alembertian $\Box$ commutes with $\partial$.
- This means that from the **6** independent components of $\phi_{\mu\nu}$ satisfying the equation $\Box\phi_{\mu\nu} = 0$ we can subtract the functions $\Xi_{\mu\nu}$ which depend on 4 independent arbitrary functions $\xi_\mu$, and which satisfy the same equation i.e., eqn (29).
  This means that we can choose the functions $\xi_\mu$, so as to impose **4** conditions on $h_{\mu\nu}$.

- We can choose $\xi^0$ such that the trace $\phi^{\mu}{}_{\mu} = \phi = 0$ (**TRACELESS**). Note that if $\phi = 0$ then $\phi_{\mu\nu} = h_{\mu\nu}$.
- The 3 functions $\xi_i$ can be chosen so that $\phi^{0i} = 0$.
- Since $\phi_{\mu\nu} = h_{\mu\nu}$ the Hilbert condition $\phi_{\mu\nu}{}^{,\nu} = 0$ for $\mu = 0$ will be written

$$\phi_{00}{}^{,0} + \phi_{0i}{}^{,i} = 0 \quad \text{or} \quad h_{00}{}^{,0} + h_{0i}{}^{,i} = 0\,. \tag{30}$$

  But since we fixed $\phi^{0i} = 0$ we get $\phi_{00}{}^{,0} = 0$, i.e. $\phi_{00} = h_{00}$ is a constant in time.
- A **time-independent part** term $\phi_{00}$ corresponds to the **static part** of the gravitational interactions i.e. to the Newtonian potential of the source.
- The GW itself is the **time-dependent part** and therefore as far as the GW concerns $h_{00}{}^{,0} = 0$ means $h_{00} = 0$.

# GW: about Gauge conditions (*) viii

In conclusion, we set

$$h^{0\mu} = 0\,, \quad h^i{}_i = 0\,, \quad h^{ij}{}_{,i} = 0 \tag{31}$$

This defines the **transverse-traceless gauge** or TT gauge.

By imposing the Lorenz gauge, we have reduced the **10** degrees of freedom of the symmetric matrix $h_{\mu\nu}$ to **6** degrees of freedom, and the residual gauge freedom, associated with the functions $\xi^\mu$ satisfying eqn (29) has further reduced these to just **2** degrees of freedom.

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[3] **NOTE:** Equation (25) always admits a solution, because the d'Alembertian operator is invertible. If $G(x)$ is the Green's function of the d'Alembertian operator so that

$$\Box_x G(x-y) = \delta^4(x-y) \tag{27}$$

then the corresponding solution is:

$$\xi^\mu(x) = \int G(x-y) f^\mu(y) d^4x \tag{28}$$

# GW: Properties

- Equation (14) is the basis for computing the generation of GWs within the linearised theory.

- To study the propagation of GWs as well as the interaction with test masses (and therefore the GW detector) we are interested for the equations outside the source, i.e. where $T_{\mu\nu} = 0$.

- GWs are periodic changes of spacetime curvature and for weak gravitational fields far away from sources they described by a simple wave equations which admits a solution of the form:

$$\phi_{\mu\nu} = A_{\mu\nu} \cos\left(k_\alpha x^\alpha\right) \, , \tag{32}$$

where $A_{\mu\nu}$ is a symmetric tensor called polarization tensor including information of the amplitude and the polarization properties of the GWs.
$k^\alpha \equiv (k^0 = \omega/c, \vec{k})$ is the wave-vector.
For a single plane wave with given wave-vector $k^\alpha$ $(\hat{\mathbf{n}} = \mathbf{k}/|\mathbf{k}|)$, from eqn (31) we see that the non-zero components of $h_{ij}$ are in a plane transverse to $\hat{\mathbf{n}}$ since the condition $h_{ij}{}^{,j} = 0$ becomes $n^i h_{ij} = 0$.

This solution (32) satisfies Hilbert's gauge condition, that is:

$$0 = \phi_{\mu\nu}{}^{,\nu} = -A_{\mu\nu} k^{\nu} \sin(k_\alpha x^\alpha)$$

which lead to the orthogonality condition

$$A_{\mu\nu} k^{\nu} = 0\,. \tag{33}$$

explains the definition of **TRANSVERSE** gauge.

From the wave equation (14) we get

$$0 = \phi_{\mu\nu}{}^{,\alpha}{}_{,\alpha} = -A_{\mu\nu} k^{\alpha} k_{\alpha} \cos(k_\alpha x^\alpha) \quad \Rightarrow \quad k^{\alpha} k_{\alpha} = 0. \tag{34}$$

This relation suggests that the wave vector $k^\alpha$ is null i.e. gravitational waves are propagating with the speed of light. But, (34) implies that $\omega^2 = c^2|\vec{k}|^2$ i.e. both group and phase velocity of GWs are equal to the speed of light.

$$v_{\rm group} = \frac{\partial \omega}{\partial k} \quad \text{and} \quad v_{\rm phase} = \frac{\lambda}{T} = \frac{\omega}{k} \tag{35}$$

# GW: The Transverse - Traceless (TT) Gauge i

Based on the gauge freedom which allows to choose $\xi^\mu$ we derived the following relations

$$h^{0\mu} = 0\,, \quad h^i{}_i = 0\,, \quad h^{ij}{}_{,i} = 0 \tag{36}$$

which define the so-called **Transverse - Traceless (TT) Gauge**.

Then for a GW propagating in the $z$ direction i.e. it has a wave vector of the form $k_\mu = (\omega/c, 0, 0, -\omega/c)$ where $k_0 = \omega/c$ is the frequency of the wave that:

$$h^{\mu\nu} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - z/c)] \tag{37}$$

While $h_+$ and $h_\times$, are the amplitudes of the gravitational waves in the two polarizations.

# GW: The Transverse - Traceless (TT) Gauge ii

The GWs described in this specific gauge are **T**ransverse and **T**raceless, and we will use the notation $h_{\mu\nu}^{\mathrm{TT}}$.

The line element will be written as (where $\varphi = \omega(t - z/c)$):

$$\begin{aligned} ds^2 &= -c^2 dt^2 + [1 + h_+ \cos(\varphi)]\, dx^2 + [1 - h_+ \cos(\varphi)]\, dy^2 \\ &+ 2h_\times \cos(\varphi) dx dy \end{aligned} \tag{38}$$

# GW: Effects on particles

- A static or slowly moving particle has velocity vector $u^\mu \approx (1, 0, 0, 0)$ and one can assume that $\tau \approx t$. Then in linearized gravity the geodesic equation will be written as:

$$\frac{du^\mu}{dt} = -\frac{1}{2}\left(h_{\mu\alpha,\beta} + h_{\beta\mu,\alpha} - h_{\alpha\beta,\mu}\right) u^\alpha u^\beta \tag{39}$$

leading to

$$\frac{du^\mu}{dt} = -\left(h_{\mu 0,0} - \frac{1}{2} h_{00,\mu}\right) . \tag{40}$$

If we now use the T-T gauge ($h_{00} = h_{\mu 0} = 0$) we conclude that GWs do not affect isolated particles (at linear oder)!

- If instead we consider a pair of test particles on the cartesian axis $Ox$ being at distances $x_0$ and $-x_0$ from the origin and we assume a GW traveling in the $z$-direction then their distance will be given by the relation:

$$\begin{aligned} d\ell^2 &= g_{\mu\nu} dx^\mu dx^\nu = \dots \\ &= -g_{11}(dx)^2 = (1 - h_{11})(2x_0)^2 = (1 - h_+ \cos\omega t)(2x_0)^2 \end{aligned} \tag{41}$$

or approximatelly

$$\Delta\ell \approx \left(1 - \frac{1}{2} h_+ \cos\omega t\right)(2x_0) . \tag{42}$$

# GW: Effects...

In a similar way we can show for two particles on the *Oy* axis that:

$$\Delta\ell \approx \left(1 + \frac{1}{2}h_+ \cos\omega t\right)(2y_0)\,. \tag{43}$$

In other wards the coordinate distance of two particles is varying periodically with the time

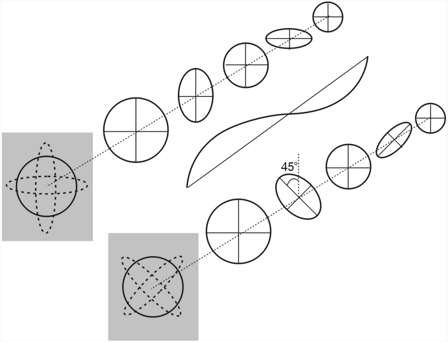


**Figure 1:** The effect of a travelling GW on a ring of particles

# GW: Effects...

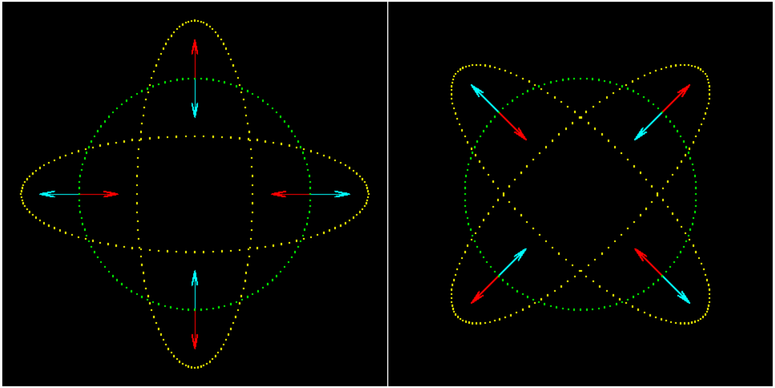

**Figure 2:** The effect of a travelling GW on a ring of particles

In a curved spacetime two geodesics that can be “parallel” initially will either converge or diverge depending on the local curvature. Consider two neighbouring geodesics $\mathcal{G}$ given by $x^\alpha(\sigma)$ and $\tilde{\mathcal{G}}$ given by $\tilde{x}^\alpha(\sigma)$ where $\sigma$ is an affine parameter. If $\xi^\alpha(\sigma)$ is a small vector connecting points of the two geodesics for the same values of $\sigma$ i.e.

$$\tilde{x}^\alpha(\sigma) = x^\alpha(\sigma) + \xi^\alpha(\sigma) \tag{44}$$

If we construct **local geodesic coordinates** about the point $P$, the Christoffel symbols will vanish but its derivatives will be non-zero there.

In this coordinate system we will get

$$\left(\frac{d^2 x^\alpha}{d\sigma^2}\right)_P = 0 \quad , \quad \left(\frac{d^2 \tilde{x}^\alpha}{d\sigma^2} + \Gamma^\alpha{}_{\mu\nu}\frac{d\tilde{x}^\mu}{d\sigma}\frac{d\tilde{x}^\nu}{d\sigma}\right)_Q = 0 \tag{45}$$

## Geodesic deviation (*) ii

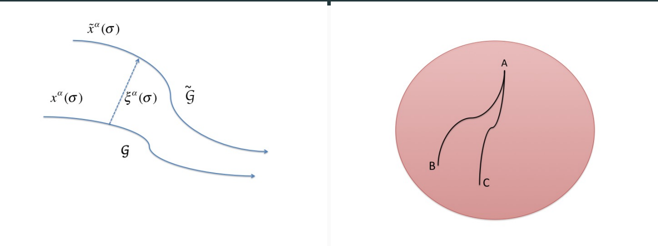


**Figure 3:** (Left) Two neighbouring geodesics. (Right) Converging geodesics on the surface of a sphere.

But since $\xi^\alpha$ is small:

$$[\Gamma^\alpha{}_{\mu\nu}]_Q = [\Gamma^\alpha{}_{\mu\nu}]_P + [\Gamma^\alpha{}_{\mu\nu,\lambda}]_P\,\xi^\lambda = [\Gamma^\alpha{}_{\mu\nu,\lambda}]_P\,\xi^\lambda \tag{46}$$

by subtracting the two equations in (45) we get (to 1st order, at $P$):

$$\frac{d^2\xi^\alpha}{d\sigma^2} + \Gamma^\alpha{}_{\mu\nu,\lambda}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\xi^\lambda = 0 \tag{47}$$

# Geodesic deviation (*) iii

However, in our geodesic coordinates the 2nd order absolute (intrinsic) derivative of $\xi^\alpha$ at $P$ is : [4]

$$\frac{D^2\xi^\alpha}{D\sigma^2} = \frac{d}{d\sigma}\left(\frac{d\xi^\alpha}{d\sigma} + \Gamma^\alpha{}_{\mu\nu}\xi^\mu x^\nu\right) = \frac{d^2\xi^\alpha}{d\sigma^2} + \Gamma^\alpha{}_{\mu\nu,\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma}\xi^\mu \tag{49}$$

where we have used the fact that $\Gamma^\alpha_{\mu\nu}(P) = 0$.

By combining the last two equations we get:

$$\frac{D^2\xi^\alpha}{D\sigma^2} + [\Gamma^\alpha{}_{\mu\lambda,\nu} - \Gamma^\alpha{}_{\mu\nu,\lambda}]\xi^\nu\frac{dx^\mu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0 \tag{50}$$

which will give

$$\frac{D^2\xi^\alpha}{D\sigma^2} + R^\alpha{}_{\mu\nu\lambda}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\xi^\nu = 0 \tag{51}$$

because the term in the square brackets is the Riemann tensor in local geodesic coordinates. This is the **equation of geodesic deviation**.

- This is a tensor relation and is hence valid in any coordinate system.

# Geodesic deviation (*) iv

- Equation (51) allows us to compute the rates of convergence or divergence of neighbouring geodesics for Riemannian spaces of arbitrary complexity.
- In a flat region $R^{\alpha}{}_{\beta\gamma\delta} = 0$ the geodesic deviation reduces to $d^2\xi^{\alpha}/d\sigma^2 = 0$ which implies that $\xi^{\alpha}(\sigma) = A^{\alpha}\sigma + B^{\alpha}$ where $A^{\alpha}$ and $B^{\alpha}$ are constants.

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[4] If we want to find the rate of change of a vector $A^{\mu}$ along a curve, we must calculate $dA^{\mu}/d\tau$. However, this is not a good measure of how much $A^{\mu}$ is really changing, since part, or maybe all , of the contribution to $dA^{\mu}/d\tau$ could be due to the curvilinear coordinates used to define the components $A^{\mu}$. Also, $dA^{\mu}/d\tau$ is not a vector (for the same reason that $\partial A^{\mu}/\partial x^{\nu}$ is not a tensor).
A better measure of the rate of change of $A^{\mu}$ along the curve is the vector quantity

$$\frac{DA^{\mu}}{D\tau} \equiv A^{\mu}{}_{;\nu}\frac{dx^{\nu}}{d\tau} = \frac{dA^{\mu}}{d\tau} + \Gamma^{\mu}{}_{\alpha\beta}A^{\alpha}\frac{dx^{\beta}}{d\tau}\,. \tag{48}$$

This is a vector, because is the product of the tensor $A^{\mu}{}_{;\nu}$ and the vector $dx^{\nu}/d\tau$. The quantity $DA^{\mu}/D\tau$ is called the derivative along the curve, or the absolute derivative.
NOTE : In local geodesic coordinates ($\Gamma^{\mu}{}_{\alpha\beta} = 0$) the derivative $DA^{\mu}/D\tau$ reduces to the ordinary derivative $dA^{\mu}/d\tau$.

# Tidal forces in Newtonian theory

Tidal forces deform the shape of bodies as they freely move in a grav. field.

Thus two nearby particles with trajectories $x^i(t)$ and $\tilde{x}^i(t)$ (in Cartesian coordinates) will be separated by a vector $\xi^i = x^i - \tilde{x}^i$

$$\frac{d^2\zeta}{dt^2} = -\left(\frac{\partial^2\Phi}{\partial x^i \partial x^j}\right)\zeta^j = -\mathcal{E}_{ij}\zeta^j \quad \text{(why?)} \tag{52}$$

where $\Phi$ is the Newtonian gravitational potential and $\mathcal{E}_{ij}$ the tidal field tensor.

The relative or tidal acceleration between two freely falling particles will be:

$$\Delta a_i = -\mathcal{E}_{ij}\zeta^j \tag{53}$$

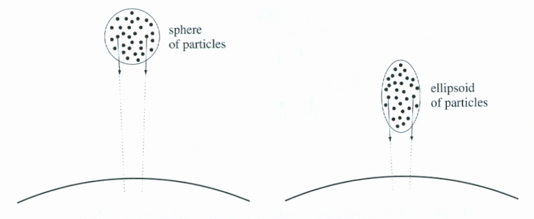


**Figure 4:** Tidal effects on a cloud of particles

# Tidal forces in a curved spacetime

Tidal effects can be also estimated in GR for two particles moving along timelike geodesics $x^\mu(\tau)$ and $\tilde{x}^\mu(\tau)$ ($\tau$ is the proper time of the 1st particle).

The separation vector between the worldlines of the 2 particles is $\xi^\mu(\tau) = \tilde{x}^\mu - x^\mu$ and thus based on (50) we get:

$$\frac{D^2\xi^\mu}{D\tau^2} = R^\mu{}_{\sigma\rho\nu} u^\sigma u^\rho \xi^\nu \equiv S^\mu{}_\nu \xi^\nu \tag{54}$$

where $S^\mu{}_\nu$ is the so called **tidal stress tensor** and $u^\sigma = d\xi^\sigma/d\tau$.

**NOTE**: This is a fully covariant tensor equation and holds in any coordinate system.

CONCLUSION: The tidal field, is a measure of the spacetime curvature.

Riemann tensor is a **"measure" of spacetime's curvature** and in linearized gravity gets the form

$$R_{\kappa\lambda\mu\nu} = \frac{1}{2}\left(\partial_{\nu\kappa} h_{\lambda\mu} + \partial_{\lambda\mu} h_{\kappa\nu} - \partial_{\kappa\mu} h_{\lambda\nu} - \partial_{\lambda\nu} h_{\kappa\mu}\right), \tag{55}$$

in the T-T gauge the Riemann tensor is considerably simplified (how?)

$$R^{\mathrm{TT}}_{j0k0} = -\frac{1}{2}\frac{\partial^2}{\partial t^2} h^{\mathrm{TT}}_{jk}, \quad \text{for} \quad j, k = 1, 2, 3. \tag{56}$$

Actually, the Newtonian limit of the Riemann tensor is (how?):

$$R^{\mathrm{TT}}_{j0k0} \approx \frac{\partial^2 \Phi}{\partial x^j \partial x^k}, \tag{57}$$

where $\Phi$ is the Newtonian potential.

In other words the Riemann tensor has also a pure physical meaning i.e. it is a measure of the tidal gravitational acceleration.

# GW: Tidal forces ii

Then the distance between two nearby particles $x^\mu(\tau)$ will $x^\mu(\tau) + \xi^\mu(\tau)$ will be described by

$$\frac{d^2\xi^k}{dt^2} \approx -R^k{}_{0j0}{}^{\mathrm{TT}}\xi^j. \tag{58}$$

The tidal force acting on a particle is (why?)

$$f^k \approx -mR^k{}_{0j0}\xi^j \approx \frac{m}{2}\frac{d^2h_{jk}^{TT}}{dt^2}\xi^j \tag{59}$$

where $m$ is particle's mass. This means that

$$f^x \approx \frac{m}{2}h_+\omega^2\cos[\omega(t-z)]\xi_0^x, \quad \text{and} \quad f^y \approx -\frac{m}{2}h_+\omega^2\cos[\omega(t-z)]\xi_0^y. \tag{60}$$

$$\nabla\vec{f} = \frac{\partial f^x}{\partial\xi_0^x} + \frac{\partial f^y}{\partial\xi_0^y} = 0\,. \tag{61}$$

Hence the divergence of the force $\vec{f}$ is zero, which tell us that the tidal force can be represented graphically by field lines.

# GW: Tidal forces iii

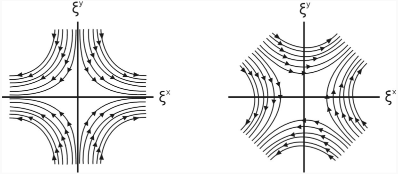


**Figure 5:** The tidal field lines of force for a gravitational wave with polarization $(+)$ (left panel) and $(\times)$ (right panel). The orientation of the field lines changes every half period producing the deformations as seen in Figure 1. Any point accelerates in the directions of the arrows, and the denser are the lines, the strongest is the acceleration. Since the acceleration is proportional to the distance from the center of mass, the force lines get denser as one moves away from the origin. For the polarization $(\times)$ the force lines undergo a $45^0$ rotation.

# GW: Properties i

- GWs, once they are generated, propagate almost unimpeded. Indeed, they are even harder to stop than neutrinos! The only significant change they suffer as they propagate is the decrease in amplitude while they travel away from their source, and the *redshift* they feel (cosmological, gravitational or Doppler).

- EM waves are fundamentally different, however, even though they share similar wave properties away from the source.

- GWs are emitted by coherent bulk motions of matter (for example, by the implosion of the core of a star during a supernova explosion) or by coherent oscillations of spacetime curvature, and thus they serve as a probe of such phenomena.

By contrast,

- Cosmic EM waves are mainly the result of incoherent radiation by individual atoms or charged particles.

# GW: Properties ii

⋆ As a consequence, from the cosmic electromagnetic radiation we mainly learn about the form of matter in various regions of the universe, especially about its temperature and density, or about the existence of magnetic fields.

• Strong GWs are emitted from regions of spacetime where gravity is very strong and the velocities of the bulk motions of matter are near the speed of light.

Since most of the time these areas are either surrounded by thick layers of matter that absorb EM radiation or they do not emit any at all (black holes), the only way to study these regions of the universe is via GWs.

# GW: The energy of GWs i

The fact that GWs carry energy and momentum is already clear from the discussion on the interaction with test masses.

To get an explicit expression of the energy-momentum tensor of GWs we can follow two different routes one more geometrical and the other more field-theoretical

**A.** According to GR, any form of energy contributes to the curvature of space-time, thus we can ask "whether GWs are themselves a source of space-time curvature".

**B.** We can treat linearised gravity as any other classical field theory and apply Noether's theorem, the standard field-theoretical tool that answers this question.

# GW: The energy of GWs ii

In order to include the contribution of the energy-momentum associated with the gravitational field itself one must modify the linearised Einstein's equations to read

$$G^{(1)}_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} + \mathcal{T}_{\mu\nu}\right) \tag{62}$$

where $G^{(1)}_{\mu\nu}$ is the linearized Einstein tensor, $T_{\mu\nu}$ is the energy-momentum tensor of any matter present and $\mathcal{T}_{\mu\nu}$ is the energy-momentum tensor of the gravitational field itself.

On the other hand Einstein's equations may expand beyond first order to obtain

$$G_{\mu\nu} \equiv G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} + \cdots = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{63}$$

This suggest that, to a good approximation, we should make the identification

$$\mathcal{T}_{\mu\nu} \equiv \frac{c^4}{8\pi G} G^{(2)}_{\mu\nu}\,. \tag{64}$$

The terms in the Einstein tensor that are 2nd-order in $h_{\mu\nu}$ are given by

$$G^{(2)}_{\mu\nu} = R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(2)} - \frac{1}{2}h_{\mu\nu}R^{(1)} + \frac{1}{2}\eta_{\mu\nu}h^{\rho\sigma}R^{(1)}_{\rho\sigma} \tag{65}$$

- It should be noted that $\mathcal{T}_{\mu\nu}$ is *covariant under global Lorentz transformations*, but *not under general coordinate transformations*. It is may be shown that it is **not** invariant under the gauge transformations given by eqn (19).
- Actually, $\mathcal{T}_{\mu\nu}$ is not a tensor but a **pseudo-tensor** and its is known as the **Landau-Lifschitz pseudotensor**.
- The stress-energy carried by GWs cannot be localized within a wavelength (since one can always transform to a free-falling frame in which the gravitational effects disappear). Instead, one can say that a certain amount of stress-energy is contained in a region of the space which extends over several wavelengths.

- This suggests that at each point in the spacetime, one should average $G^{(2)}_{\mu\nu}$ over a small region in order to probe the physical curvature of the spacetime, which gives a gauge-invariant measure of the gravitational field. Thus we may replace (64) with

$$\mathcal{T}_{\mu\nu} = \frac{c^4}{8\pi G} \left\langle G^{(2)}_{\mu\nu} \right\rangle . \tag{66}$$

The energy-momentum tensor can be written as (HOW?):

$$\mathcal{T}^{\mu\nu} = \frac{1}{4}\left[2\phi^{\alpha\beta,\mu}\phi_{\alpha\beta}{}^{,\nu} - \phi^{,\mu}\phi^{,\nu} - \eta^{\mu\nu}\left(\phi^{\alpha\beta,\sigma}\phi_{\alpha\beta,\sigma} - \frac{1}{2}\phi_{,\sigma}\phi^{,\sigma}\right)\right] \tag{67}$$

which in the TT gauge of the linearized theory becomes (HOW?)

$$\mathcal{T}^{GW}_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \left(\partial_\mu h^{TT}_{ij}\right)\left(\partial_\nu h^{TT}_{ij}\right)\right\rangle . \tag{68}$$

where the angular brackets indicate averaging over several wavelengths.

# GW: The energy of GWs v

For the special case of a plane wave propagating in the $z$ direction, the energy-momentum tensor has only 3 non-zero components, which take the simple form (HOW?)

$$\mathcal{T}_{00}^{GW} = \frac{\mathcal{T}_{zz}^{GW}}{c^2} = -\frac{\mathcal{T}_{0z}^{GW}}{c} = \frac{1}{32\pi}\frac{c^2}{G}\omega^2\left(h_+^2 + h_\times^2\right), \qquad (69)$$

where $\mathcal{T}_{00}^{GW}$ is the energy density, $\mathcal{T}_{zz}^{GW}$ is the momentum flux and $\mathcal{T}_{0z}^{GW}$ the energy flow along the $z$ direction per unit area and unit time .