The History of Radial Stellar Pulsation Theory

A short look at the long road to understand what was seen

A Historical Preludium of a Kind: Over the years, many authors writing on the history of stellar variability considered it remarkable how long it took astronomers to accept that *variable* stars do indeed exist in the sky. They concluded that it cannot be attributed to sloppy or insufficient observing techniques. It could have been the conceptual barrier of the Aristotelian totalitarism that dominated the mainstream of natural philosophy that prevented the acceptance of such a phenomenon. If this was indeed the case, is unlikely to be ever proofed beyond doubt. In any case, in Europe, the first testimonies of observations of variable stars – not being novae or supernovae – date indeed from around the early Copernican-Tychonic era.[1] For detailed historical descriptions of observing records we refer to the Hagen (1921) and to the historical introductions in Perdang (1985, 1990).

The pulsating variable star that is the prototype for one of the most influential classes of variable stars in astronomy was discovered in 1784 by Goodricke (1786). He was the first to observe the periodic and asymmetric light variation of δ Cephei[2] in the same year as he discovered the eclipsing binary star β Lyrae. This was only two years after he discovered the variability of Algol (in the historical literature Montanari is claimed to have suspected Algol to be variable already in 1672). From the very beginning, Goodricke attributed Algol's light variation to eclipses (due to revolving planets, however). For whatever reason, he did not attribute the same mechanism to δ Cephei's variability. In the absence of any other viable ideas on how to cause stellar variability, Cepheids were interpreted as binaries also for almost 130 years.

The situation changed towards the end of the 19th century when also astronomy experienced the influence of the rapidly evolving physical sciences. Before that, the study of variable stars amounted mainly to book-keeping of ephemerides and in phenomenologically classifying the rapidly growing number of variable stars. By the end of the century, however, there was an attempt to understand stars on the basis of fundamental physical laws. The various phases of the emerging discipline

[1] The large-amplitude Mira variables were the variable stars that attracted the astronomers' attention first. The Friesean astronomer Fabricius mentioned the variability of *o* Ceti (i.e. Mira) in 1596 in a letter to Tycho Brahe. Kirch discovered the variability of χ Cyg (1687), Maraldi that of R Hyd (1704), and Koch the one of R Leo (1782) (see Hagen (1921)).

[2] Pigott actually discovered the *first* Cepheid variable, η Aql, about one month before Goodricke made his observations of δ Cep (see Gilman 1978).

of stellar astrophysics can also be traced in the development of the theory of the intrinsic variability of variable stars. Hence, the story of the appearance of stellar pulsation theory is also that of understanding the laws and mechanisms that dictate the stars' lives.

This historical essay will stop at around 1962. In that year, Baker and Kippenhahn (1962) published their numerical solutions and their explanation of Cepheid-type instability. Thereafter, the use of electronic computers[3] became wide-spread. Accordingly, by the relative ease to obtain numerical solutions for the differential equations describing stellar pulsations, the literature and the diversity of the research aspects in pulsation theory grew explosively. Today, stellar pulsation theory is used as an accurate tool to probe many different aspects of stellar physics in almost all evolutionary phases. Through the seismological potential of multimode pulsations they might play an important role in a possible renaissance of stellar astronomy in the future which is clearly necesary if larger-scale astrophysical topics aim at becoming mature quantitative scientific fields.

Radial Stellar Pulsations: After the discovery of the first variable stars with asymmetric light curves (as shown for example in Fig.1 and the very early assimilation of the idea of their being binary stars, this attitude remained unchallenged to the beginning of the 20th century. As late as 1913, Brunt (1913) attributed the *proof* of the binary nature of δ Cep to Belopolski (1894) who observed spectroscopically its radial velocity variations. In the year 1899 Schwarzschild (1900) discovered that the light variability of η Aql is accompanied by a continuous change of the color and hence of the spectral type. The attempt was made to embed this behavior in the binary theory, despite the considerable problems it caused. With the improving quality of variable-star observations and an accumulating number of different viewpoints of this astronomical problem, increasingly more complicated orbital constructs of multiple star systems had to be invented to account for the asymmetries and bumps in the lightcurves of Cepheid variables. It is noteworthy that – of course – in no case ever was more than one stellar component observed in the spectrum. Brunt (1913) listed comprehensively the severe problems inherent in the binary hypothesis. He did not, though, question at all the basic assumption of two or more stars being involved in this variability phenomenon.

Early Strugglers: In the same year as Brunt's (1913) discussion of the binary nature of the "short period" variables appeared, Plummer (1913) wrote about serious problems with the interpretation of the radial velocity data of ζ Gem. Still anchored in the binary-star picture, he speculated that by tidal effects displacements of the atmosphere relative to its center might be present. In the following year Plummer (1914) suggested, without being more specific, a radial pulsation mechanism in *sin-*

[3]In the early years of stellar astrophysics human computers or analog integrating devices were the only means of obtaining numerical solutions. In terms of efficiency, they were, however, much inferior to their electronic successors. The calculations of whole stellar models and the solution of stability problems on the grand scale that proved to be necessary and experimentation with detailed micro-physics had to wait for the development of electric/electronic computing devices. Baker and Kippenhahn performed their calculations on the "home-grown" computer G2 which was one of a series of electronic computers designed and built at the Max-Planck-Institut für Physik in Göttingen.

gle stars as a possible way out of the numerous problems with "certain classes" of variable stars.[4]

In his seminal paper, Shapley (1914) analyzed systematically the observational situation of the short-period variables for which complicated elliptic orbits had to be postulated and criticized the binary-star explanation. He concluded that a radial pulsation of the atmospheres of single stars would represent the observed facts better. In particular, Shapley was able to argue that in many cases the postulated orbits were smaller than the size of the visible component of the binary system. This point questioned most convincingly the binary hypothesis. The paper by Shapley proposed no specific physical mechanisms for the maintenance of stellar pulsations. Surprisingly, Shapley did not mention the period – luminosity (PL) relation of Cepheids found two years earlier by Leavitt (1912). If he had trusted the observation that the Cepheids are not wildly scattered in the giant region on the Hertzsprung-Russell (HR) diagram (cf. Russell (1913)) or had used the period – color relation he mentioned in the paper of 1914, he could have provided a formidable physical explanation of the PL relation with the help of the period – mean-density relation and hence a strong support for the pulsation hypothesis.[5]

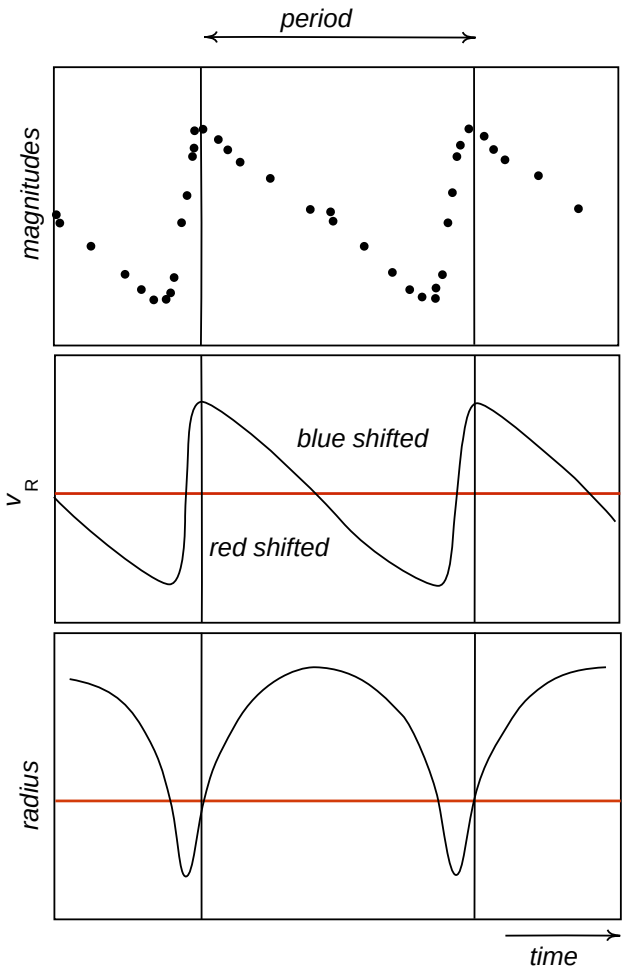

Figure 1: Typical variabilty structure of a radially pulsating star. The top panel shows the temporally discrete measurements of the pulsators brightness. The middle panel sketches the the associated radial-velocity variation as measured by an observer; the corresponding radius variation is shown in the lowest panel. Characteristic magnitudes of the fluctuations will be mentioned when discussing particular families of variables.

[4]Shapley (1914) expressed the opinion that Plummer must have had in mind the variable stars which were classified later as Cepheid-type pulsators.

[5]Hertzsprung (1913) referred to the PL relation of Leavitt (1912) and used it to estimate (although incorrectly) the distance to the Small Magellanic Cloud and characterized it as extra-galactic.

At the time of Shapley's paper (1914) astronomers were establishing the separation of stars into dwarfs and giants. The studies of Hertzsprung (1907; 1911; 1913) and Russell (1913; 1914) showed that stars populate at least two major sequences on the plane spanned by the spectral class or color index and the absolute magnitude. The short-period Cepheid stars were recognized as belonging to a group of stars with small proper motion and small parallaxes and hence of large absolute magnitude. At the same time these stars were not attributed masses that significantly exceed that of the sun (Hertzsprung 1907, 1913; Russell 1913). These views enabled Shapley (1914) to make his strongest point against the binary hypothesis: The large dimensions of the Cepheid stars compared with the postulated binary orbits.

Shapley's study of 1914 was not the ultimate death-blow for binary theories for Cepheids. During the 1920s and the 1930s research articles continued to appear, though at a decreasing rate, defending the binary hypothesis (e.g. Vogt 1921, 1927; Pannekoek 1922; Hellerich 1925 and references therein). In 1925, for example, the eminent astronomer Jeans hypothesized that Cepheids and the long-period variables are binary stars in the process of fission (Jeans, 1925a). Even later, Hoyle & Lyttleton (1943) revisited Jeans' proposition and conjectured the Cepheids to be contact systems with a common envelope that does not partake in the motion of the embedded binary system. Neither theoretical approach received wide recognition, however. Enough evidence for an intrinsic pulsational motion had already been accumulated by then. In particular, Baade (1928) devised a test for the pulsation hypothesis. The pulsations demand the stellar surface area to change periodically. This is reflected in the luminosity and temperature changes. The two effects can be separated and a plot of radius versus phase can be deduced. The radial velocity curve provides an independent check of the radius variation. Later, the method was improved by Becker (1940), van Hoof (1943), and Wesselink (1946). Today we know this approach as the Baade – Wesselink method and it plays an important role in physically calibrating classical pulsators. Schwarzschild (1938a) chose a different approach to test the pulsation hypothesis of the Cepheids. He started from the diffusion approximation of radiation to describe the luminosity and its variation at the stellar photosphere. He further assumed the pulsations to lead to a sequence of adiabatic state changes. Eventually, Schwarzschild adopted observed radial-velocity data with which he reconstructed the luminosity variation. The agreement with the observed light curve of δ Cephei which he obtained appeared convincing and clearly supported the pulsation hypothesis.

The idea that stars pulsate was not completely *terra incognita* by the time of Shapley's paper (1914): Robert Emden (1907) discussed comprehensively the theory of polytropes that came to play an important role in physically modeling the structure of stars in his famous monograph "*Gaskugeln*". In the same publication, Emden extensively reviewed and re-did the astronomically relevant parts of the analyses of Ritter. August Ritter, professor for mechanics at the polytechnical university at Aachen, published a series of 19 articles on the behavior of gaseous bodies between 1878 and 1889. In the sixth article of the series, Ritter (1879) addressed the problem of radially pulsating spheres of constant density. He showed that the pulsation period is proportional to the inverse square-root of the density: the *period – density relation*. Towards the end of the article, Ritter considered the application of his cal-

culations to stars. He concluded that since the physical origin of the variable stars was not known one might assume that they are pulsating gas spheres. The temperature change during a pulsation cycle would then cause a luminosity change. Unfortunately, these propositions were made long before the photographic work of Schwarzschild (1900) and of the realization that giant stars with mean densities much lower than that of the sun exist. Except for the review in Emden's book, Ritter's work seems not to have had any impact on the astronomical community. Indeed, none of the issues of the *Vierteljahresschrift der Astronomischen Gesellschaft* contains an obituary or a notice of Ritter's death in 1908.

Figure 2: Sir Arthur Stanley Eddington 1882 – 1944

In 1917, Eddington, who was to play a key role in establishing the theory of stellar pulsations, entered the field with a short article on stellar pulsations. He realized that pulsating stars should be considered as thermodynamical engines for which energy gain and dissipation are compensating each other. Therefore, infinitesimal perturbations that initially grow could eventually be limited by the action of viscosity, i.e. the entering of a *limit cycle* behavior was considered possible; this insight is very important since hitherto only rather violent events, such as collisions were thought to be able to initiate stellar oscillations, which thereafter die out on a long time-scale. Actually, the maintenance of stellar pulsations was already recognized by Eddington in this first paper as a major problem for the theory. Without the necessary physics at hand, and indeed stellar astrophysics was far from having developed enough at the time, Eddington imagined a *variable transparency* of the stellar matter to be a suitable mechanism to temporally modulate the energy flux. In the first extensive formal theoretical paper (Eddington, 1918), stellar pulsation theory was treated in the adiabatic approximation and periods for polytropic gas spheres were calculated. Furthermore, Eddington addressed the question of the ratios of the specific heats in stars. The particular values of the theoretically derived pulsation periods turned out to depend on the particular choice of this ratio. Very little was known then about the constitution of stellar material. Eddington assumed the value of $\gamma \equiv c_P/c_V$ to vary between 5/3 and 4/3. Some examples with $\gamma = 1.429$ were calculated. This particular choice might have been motivated by the realization that a value for $\gamma = 5/3$ results in significantly too short a period for δ Cephei so that a value closer to 4/3 was to be preferred. An ideal mono-atomic gas admits $\gamma = 5/3$ whereas a radiation pressure dominated configuration, which is only marginally stable as pointed out already by Ritter, has $\gamma = 4/3$. As the role of radiation pressure was an important stellar physical ingredient for Eddington, he also believed it to be important for Cepheid variables. He concluded, though, that his analyses did

not depend explicitly on the radiation pressure as long as the total pressure was used in the derivation of the dynamical equations. Eventually he evaluated the period – density relation, which he formulated in terms of the central density for his polytropic models. The list of observed Cepheid variables included in the paper (Eddington, 1918)were found to obey the relation well. The second formal paper of Eddington was devoted exclusively to stellar pulsation theory (Eddington, 1919) wherein he discussed the problem of dissipation of pulsation energy. He observed that the pulsation modes cannot live on energy extracted from contraction (which was thought by then to be the major energy source of the stars and the driving force behind stellar evolution); otherwise the period change due to the resulting density increase would have to be observable within a human lifetime.

Another important point which appears in the 1919 paper of Eddington was the discussion of the origin of the *phase lag*. It is observed (as shown in Fig.1) that the maximum brightness does not occur at maximum compression of the star but it is delayed to approximately maximum outward expansion velocity. If the motion of the pulsating star is expressed in a sinusoidal form, then the phase shift amounts to about 90°. In nonlinear reality, the shift is less than 1/4 of a period. Eddington attributed the phase lag to nonadiabatic effects. He argued that in an adiabatic motion, the temperature fluctuation has to be in phase with the density fluctuation and hence with the radius/velocity variation. If, by some unspecified nonadiabatic effect, the energy flux is being dammed in the outermost regions this will cause a delay in the luminosity variation relative to the dynamical motion. Due to the complexity of the equations he did not succeed, however, in quantitatively reproducing the quarter-period lag deduced from Cepheid observations. Later, Schwarzschild (1936; 1938b) hypothesized that running waves in the outermost regions of Cepheids can be regarded as the origin of the observed phase lag. With his semi-empirical approach he indeed succeeded well in achieving correspondence between observation and his computations. Later on, more realistic simulations showed that the acoustic modes in Cepheids are described rather well by standing waves, so that Schwarzschild's approach had to be dismissed. Eddington (1941) reconsidered the phase-lag problem after hydrogen had been established as a major constituent of stellar material. He attributed the phase lag to the action of the hydrogen ionization zone at an appropriate depth of the Cepheid envelopes, realizing that the hydrogen ionization zone in stars can lead to any amount of phase lag depending on the mass depth of this layer. In the case of Cepheids the location of this region is just optimal to minimize dissipation so that the phase lag is roughly 90°. Although this reasoning was not completely right (Rosseland, 1950), Castor (1968), also presented in Cox (1980), established quantitatively, in a now generally accepted model, that the very *narrow* H-ionization region is indeed the origin of the phase lag between light and radial-velocity curve.

Edgar (1933) discussed the possibility that first overtone modes are also excited in stars, not only fundamental modes.[6] He calculated fundamental and first overtone

[6]Jeans criticized the stellar pulsation theory with the argument that in principle not only the fundamental mode but also higher overtones are possible solutions to the equations. Since there was, at that time, no evidence for such modes being excited in variable stars, Jeans concluded that the pulsation hypothesis had to be dismissed.

solutions for the standard model as defined in the ICS (Eddington, 1926). Edgar performed a factorization of the temporal dependence into $\exp(i\sigma_k t)$ where σ_k denotes the oscillation frequency of the kth mode. In general, σ_k is a complex quantity, with the imaginary part measuring the damping or growth time for the particular mode. Eventually, Edgar arrived at an integral formula for this imaginary part based on what we would call today a quasi-adiabatic approximation. He used the adiabatic eigensolution to express the thermodynamic perturbation quantities. The particular integral formulation was used for a long time thereafter, and it is actually encountered even these days. Edgar did not know what destabilizes the stars, but he found that the first overtone modes were strongly damped, more so than the fundamental. In the conclusions Edgar (1933) speculated that a varying index in the power-law representation of the opacity function, which indeed varies strongly in the case of partial ionization of chemical species, for example, might provide a source to destabilize stars vibrationally. At the time of his article, the true elemental composition and the ionization profiles obtaining in a star were not known in sufficient detail.

By the end of the 1920s it became clear that nuclear transmutation can serve as a viable source of energy to explain the age of the stars and of the universe as a whole. Atkinson & Houtermans (1929) showed that hydrogen can be transformed into helium inside stars. The nuclear energy generation rate ϵ in an element of stellar matter can then be parameterized by the state variables density and temperature in the form [7] $\epsilon \propto \rho^m \cdot T^n$. Once nuclear fusion appeared to be responsible for the stars' luminosities the question of stability of such stellar models was of eminent importance. Eddington (1926) had already shown in his monograph that a rapid increase of ϵ with temperature, i.e. a large enough n, contributes to pulsational destabilization of nuclearly active layers. The outer regions with $\epsilon = 0$, on the other hand, were always damping pulsations. The global stability property is determined by the relative weights of driving and damping regions throughout the star. Hence, not only the exponent of the nuclear energy generation rate itself but also the structural details of the star, in this case those influenced by the nuclear energy generation, are important in the stability analyses.

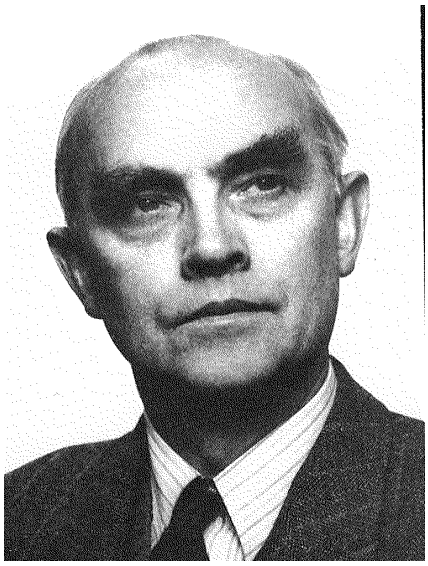

Figure 3: Svein Rosseland: 1894 – 1985

The ideas of Eddington (presented in detail in ICS 1926, after having been outlined in 1917) concerning the inner workings of the stars already contain the whole success story without, however, identifying the physical sources. He considered

[7] This particular form of parameterizing the energy generation rate was used in the theory of stellar structure even before the physical origin was known. Often, "radioactive" processes of some unknown sort were invoked, see e.g. Jeans (1925b).

the stars as thermo-mechanical engines that could, under favorable circumstances, pulsate self-excitedly. He thought of two different possibilities for this to happen. One, the classical approach, which works in any engine, is to modulate the heat input during a working cycle. In stars, this corresponds to an increased rate of energy liberation by 'sub-atomic' sources during the compression phase. The ingenious idea, which appears as the second suggestion: "...fantastic in an ordinary engine but not necessarily so in the star..." is the *valve* mechanism. In this picture the heat transfer is temporarily modulated so that even a constant heat input into the engine could maintain its working if the valve were to appropriately modify the energy flux: at maximum compression the heat leak must be minimal. For stars, this means that opacity should be largest during the compression phase.

In the 1930s, when the first stability studies of nuclear-burning stars were performed, the two aspects of energy-flux modulation were still treated separately. This was certainly also technically motivated as self-consistent calculations even of simple stellar models were still out of reach. Cowling (1934) derived a critical exponent $n \approx 7 - 8$ above which energy generating layers become convectively unstable. This result is in good agreement with present-day beliefs. Additionally, Cowling relaxed the constraint of homologous radial displacements (as used before by Jeans (1925b), where he found rather low values of the characteristic exponents to ensure pulsational stability for the stars). Indeed, the radial displacements were found to vary considerably as a function of radius, allowing a broader stability domain in terms of n than derived before. In particular, Cowling (1934) concluded that all stars that are convectively stable (convective instability is induced by nuclear burning in his models) are also pulsationally stable as long as $\gamma > 1.38$. A more consistent treatment of the convectively unstable central regions in the pulsation equations was presented by Cowling (1935). He realized that the convective instability of nuclear-burning regions did not necessarily influence the pulsational (in)stability of the star. Hence, the large temperature and density exponents of nuclear burning rates need not destabilize a star, thus making it more plausible that such burning is indeed the source of stellar radiation. In none of his models did Cowling (1934; 1935) consider radiation pressure to be important. Therefore, his limiting n for pulsationally stability turned out to be rather large. Ledoux (1941) included the effects of radiation pressure not only in the pulsation analysis but also in the structure of the stellar model. The stellar model was still not self-consistently calculated but rather composed of a point-source envelope which floated on a convective core. Assuming an exponent of $n = 16$ in the nuclear energy generation rate and a Kramers-type opacity law throughout the star, Ledoux derived an upper mass limit of about $100 M_\odot$ for pulsationally stable chemically homogeneous stars. As the stellar mass increases, the central parts with nuclear energy generation gain weight in the work integral which sums up positive and negative (damping) contributions throughout the star. Eventually, the growing amplitude of the relative radial displacement in the centermost regions due to the influence of radiation pressure on the stratification of the star overcomes the radiative damping in the envelope. Analyses after Ledoux's (1941) study reduced the critical mass somewhat when implementing more realistic constitutional physics; the general picture did not change, however. Even now, the onset of pulsational instability above a critical stellar mass is frequently considered to be the mechanism limiting the mass

function at its upper end.

An interesting sidelight which illustrates the potential power of stellar pulsation theory concerns the pulsating white-dwarf stars that were discovered only by the middle of the 1970s. Before then, their absence led Ledoux & Sauvenier-Goffin (1950) to indirectly (and correctly) conclude that white dwarfs must extract the energy for their luminosity from sources different from nuclear burning. Otherwise, in particular due to their compact structure, the effect of nuclear reactions would pulsationally destabilize such stars. Although pulsating white dwarfs are known today, their pulsational instabilities are attributed to mechanisms acting in their envelopes very similar to those in Cepheids, possibly on different chemical elements.

Figure 4: Paul Ledoux 1914 – 1988

The discussion of the upper end of the mass function of main-sequence stars and its relation to stellar stability was the first partial success of excitation physics acting in pulsating stars to explain observed astronomical facts. Nonetheless, the agent maintaining Cepheid pulsations – the original goal – continued to remain in the dark.

Exciting Cepheids: At present we are beginning to appreciate that many disjoint areas of the HR diagram are populated by classes of pulsating stars that are destabilized by different mechanisms. It seems that most of the major phases of stellar evolution host some kinds of stellar pulsation, so that the class of the Cepheids is only one of a number of different families of pulsating variable stars on the HR diagram. Derived from their influence in many areas of modern astronomy they remain, however, the most prominent representatives of regular variable stars. The Cepheids are known to be confined along a line (Eddington 1941, Rosseland 1949) or better a narrow strip which is referred to now as the *classical instability strip* (Sandage 1958)[8] on the HR diagram. The distinct locus they trace out on the HR diagram was a key element that was to be explained by any successful theory of pulsations of Cepheids. Pulsational variability seems hence to require very specific envelope conditions which can be fulfilled only within a narrow, almost vertical strip on the HR diagram.

In the paper of 1941, Eddington came back to his earlier hypothesis that a variable transmissivity for photons might influence pulsational instability in at least some variable stars. As it was broadly accepted by then that the stellar material

[8]Sandage deduced the finite extension in color of the instability strip for Cepheids by analogy with the RR Lyrae instability gap on the horizontal branches of globular clusters. He supported his hypothesis by referring to (by then) unpublished data from Small Magellanic Cepheids analyzed by Arp which indicated a spread of Δ(B-V) = 0.2 at comparable periods.

consisted mostly of hydrogen, Eddington (1941) suggested that the partial ionization zone of hydrogen can minimize dissipation in the non-adiabatic outer layers of stars in such a way that the *nuclear* driving of the central regions could overcome the otherwise overwhelming dissipation in the envelopes. Despite his deep insight into the essential influence of the envelope on stellar pulsations and the ingenious concepts concerning the workings of thermo-mechanical engines elaborated on in the ICS monograph, Eddington (1941) did not abandon the idea that nuclear energy sources were exclusively responsible for maintaining the oscillatory motion of pulsating stars. He positively rejected the valve-mechanism as the *source* of stellar pulsations.

Figure 5: John Paul Cox 1926 – 1984

One reason that nuclear driving of the pulsations remained attractive for such a long time was the insufficient understanding of the structure of giant stars. As complete stellar models were not calculated before the 1950s it was not appreciated that these stars show a very strong central mass concentration. Its usual measure, the ratio of the central to the mean stellar density: $\rho_C/\langle\rho\rangle$ exceeds 10^5 in giant models which suitably describe Cepheid variables. Sen (1948) pointed out explicitly that centrally concentrated models seem to be required to explain the asymmetric velocity curves and the amplitudes in Cepheids. He suggested a polytropic index of 4 to be appropriate. The resulting value of $\rho_C/\langle\rho\rangle = 632$ is still much lower than that used by Epstein (1950) in the first pulsation analysis ($2.1 \times 10^5 - 2.4 \times 10^7$) using the most realistic stellar structure integrations of the time (calculated around M. Schwarzschild on the the ENIAC in Princeton). Before Epstein's paper, the ratio of the central displacement amplitude to that of the surface was many orders of magnitudes too large. Hence, the influence of the nuclear driving was heavily overestimated. Epstein realized that the displacement in the centermost regions (i.e. in the nuclear active regions) was about 10^{-6} of that at the surface. Epstein's main concern in his seminal paper was focused on the pulsation periods of his stellar models rather than on their excitation. By approximating the eigenfrequencies for the stellar models with the help of a variational principle he introduced the notion of the *weight function*. With this concept at hand he was able to quantify the importance of those regions in the stellar interior that determined the magnitude of the oscillation period. The weight function clearly proved that the pulsation periods were mostly influenced by the outer envelope regions of the giant stars and that the regions close to the center have a only marginal effect.

John Cox, for his PhD thesis at Indiana University, used Epstein's models to study their pulsational stability (Cox, 1955). He included the nonadiabatic effects in the pulsation treatment. The models, however, did not include ionization zones of hy-

drogen and helium, so the outer regions only damped the pulsations. On the other hand, Cox included the exciting contribution ("ϵ-mechanism") from the CNO-cycle of hydrogen burning in the centermost regions.[9] The very small amplitudes of the radial pulsations close to the stellar center led to an accordingly small driving effect in the work integral which was overwhelmed by the radiative damping in the nonadiabatic part of the envelope. Cox found that pulsations in his models were dying out on a time-scale of 10 days. This value is very much shorter than the thousands of years which were adopted hitherto (cf. Rosseland 1949). Eventually, Cox had to conclude that the improved stellar modeling of Cepheids led to more pronounced difficulties than ever before to account for the observed pulsations of these stars. Despite his failure to locate the driving agent he could, nevertheless, restrict considerably the choice of possible mechanisms that remained to be considered for the explanation of stellar pulsations. The sought-for source had to be located within the outermost 15 % of the stellar radius – nuclear driving was at last positively out of discussion. In a later paper, Cox (1958) tried to more closely identify the driving mechanism by studying the nonadiabatic flux perturbation in schematic models. By studying the induced phase lags, which should be in accordance with observed pulsating variables, he derived conditions about the spatial variation of the energy flux perturbation. Cox concluded that the ionization of a predominant element would be a natural source. This conclusion was based on a result of Schatzman (1958) who had found clear drops in the flux perturbations in regions of partial ionization of abundant chemical elements. The paper of Cox (1958) mentioned the work of Zhevakin (to be discussed below) but did not adopt Zhevakin's finding of He^+ ionization to be the driving agent in Cepheids. In the same year, Cox & Whitney (1958) inferred, after developing a chain of semi-theoretical arguments to derive a P-L relation, that only partial ionization of He^+ provides *quantitative* agreement with the empirical P-L relation. It is noteworthy that the identification of the driving mechanism happened without actually computing corresponding work integrals. The reference to the observed phase lags, which only much later were found to be due to the hydrogen ionization zone, led Cox & Whitney (1958) to identify the He^+ partial ionization zone as the driving agent. This approach stimulated rather harsh critique by Zhevakin (1961; 1963a) who considered the Cox & Whitney (1958) work to rely on fallacious assumptions. In 1960, having access to a IBM 704 computer, Cox eventually evaluated work integrals for model systems including He^+ partial ionization. He indeed encountered strong excitation in the appropriate layers. Still, Cox was not yet able to reach definite conclusions about the pulsational stability properties of the global systems due to significant damping even close to the inner boundary of the employed models. Furthermore, Cox's models did not include hydrogen and first helium ionization in the outermost regions so that their expected positive contributions could not be evaluated.

A different approach to solving the excitation problem of stellar pulsations was chosen by Zhevakin in the Soviet Union. In 1948 he submitted his candidate's the-

[9]Note that in 1955 the evolutionary status of Cepheids had not yet been clarified. Cepheids are now generally accepted to be helium core-burning stars performing blueward loops on the HR diagram during which they cross the instability strip several times.

sis on "Cepheids as Thermodynamic Auto-Oscillation Systems" in Leningrad. In 1952, he introduced a one-zone approximation of stellar models to study their pulsation properties (Zhevakin, 1952). Therein, he arrived at the conclusion that nuclear processes were insufficient to maintain stellar pulsations. In a quasi-adiabatic analysis of stellar envelopes including a zone of partial He^+ ionization, Zhevakin (1953) found significant driving that led him to postulate this zone as essential to overcoming the usually dominating dissipation in the stellar envelopes.

In nonadiabatic calculations on a four-zone "discrete stellar model" Zhevakin (Zhevakin, 1954) reiterated his previous conclusion that the He^+ ionization zone plays the decisive role in driving Cepheid-type pulsations. Pulsational instabilities were, however, encountered for rather high helium abundances only. Assuming standard population I heavy-element concentration, instabilities were confined to helium mass fractions $Y > 0.43$. This unusually high value might be connected with the inadequacies of the quasi-adiabatic analysis. Later studies confirming this result were based on discrete models which were tuned to optimally reproduce quasi-adiabatic pulsations of the corresponding continuous model. A considerable number of papers was published by Zhevakin in subsequent years attempting to encompass the different phenomenological properties and variabilities in other types of pulsating stars with his approach (see e.g. Zhevakin (1963a) for the extensive list of his references). All of Zhevakin's studies were based essentially on the discrete model approach. At the beginning, his work does not seem to have had much influence in the west – which can be attributed to the fact that his publications appeared in Russian only and that the cold war was in a very cold phase. Furthermore, his discrete models involved rather elaborate tuning of the free parameters when making the transition to the discrete approximation of the pulsation equations. This might have caused researchers who had failed to explain the Cepheids to attribute little weight to results based on such a description. Zhevakin received eventually considerable international recognition when he was invited contribute his understanding of stellar pulsation theory to the first volume of the *Annual Review of Astronomy and Astrophysics* (Zhevakin, 1963b).

Figure 6: Sergei Aleksandrovich Zhevakin 1916 – 2001

The approach of the discrete stellar model differed conceptually from the previously used "Woltjer method". In the latter, the nonadiabatic differential equations describing the pulsation problem were treated as an iteratively solved perturbation problem to the adiabatic problem. Usually only one iteration step was performed where the adiabatic solutions were used in the nonadiabatic flux perturbation part; this approach is also known as the quasi-adiabatic approximation. The spatial

structure of the underlying stellar model is treated as continuously as possible. Of course, if the the differential equations were solved numerically then some sort of discretization was still used, but the spatial mesh could be thought of as arbitrarily dense, as required by the numerical analysis tool. In contrast, the discrete model was an *a priori* slicing of the stellar model into a small number of discrete shells. The differential operators were translated into difference quotients so that a small set of coupled (linear or nonlinear) equations resulted. To study the method and the accuracy of discrete approaches Whitney & Ledoux (1957) appear to have been the first to numerically calculate periods for linear adiabatic stellar pulsation problems with finite difference schemes in the way it is done today. Since Ledoux & Whitney had access to computers they set out, from the very beginning, to formulate the problem for n shells where n can be arbitrarily chosen. Their test computations on simple stellar models usually used 25 shells. Since they distributed the gridpoints equally spaced in radius, $5 - 10$ % deviation of the lowest order eigenfrequencies were found when compared with the then customary Woltjer method. In this respect it is noteworthy that Zhevakin solved the system of equations he obtained in his discretization *manually*. Only in the mid-sixties did Aleshin publish results from numerical realizations of Zhevakin's method to electronic computers.

The detailed computational studies of Baker & Kippenhahn (1962) brought the final recognition for partial He^+ ionization as the driving agent of Cepheid-type variables. They performed, on the G2 computer of the Max-Planck-Institut für Physik, self-consistent numerical computations of stellar envelopes with detailed spatial ionization structure. Relying on a shooting approach, Baker & Kippenhahn obtained low-order solutions of the radial pulsation equations. Based on observations, they assumed a pulsation period (i.e. a real part of the oscillation frequency) and modeled the stellar envelopes. The integration of the pulsation equations led to perturbed physical quantities which were assumed to be close to eigenfunctions so that the integral work performed by the particular oscillation mode could be computed. The sign of the work integral tells if a mode is excited or damped in time. A number of models studied by Baker & Kippenhahn with choices of luminosity, effective temperature, and mass assumed to be appropriate for δ Cephei were found to have pulsationally unstable fundamental modes. Confirmation of the appropriateness of these parameters by stellar evolution computations still had to wait a few years.

The frequent reference to particular models of electronic computers throughout this text are deliberate and should emphasize the strong influence numerical calculations always had in the subject of stellar astrophysics. The first time we encountered in the literature (concerning our particular topic here) results from the use of automatic machine computations was in a paper of Schwarzschild (1941),[10] where he computed the first 4 radial overtone modes in $n = 3$ polytropes (results that were frequently referred to for a long time; see e.g. Ledoux & Walraven (1958), p. 472). A great proliferation of numerical papers occurred in the 1960s when the

[10]Schwarzschild, then at Rutherford Observatory of Columbia University, refers to the punched-card machine of the Thomas J. Watson Astronomical Computing Bureau.

first mainframe computing machines became available in large research institutions. Thereafter, detailed stellar stability studies became increasingly a matter of routine.

Nonlinear Thoughts: Even a superficial look at the light or radial-velocity curves of pulsating stars convinces us that linear pulsation theory can at best provide partial explanations. The temporal asymmetry (see Fig. 1) of the observables can not be accounted for by the sinusoidal solutions resulting from the linear approximation. Linear theory is appropriate to explain under which conditions arbitrary small perturbations can grow but it fails to describe the finite amplitude state of well developed oscillations. For many years it remained even unclear what effect nonlinearities have on the periods. Rosseland (1943) argued by comparing the mathematical with the physical pendulum that the asymmetry of the velocity curve lets us infer the lengthening of the nonlinear period relative to the one derived from linear theory. To support his pulsation theory of stars, Eddington (1919) attempted to model the asymmetries of the velocity curves of Cepheids by extending the perturbation equations to second order terms in the disturbances, a study which was extended to more detail by Kluyver (1935a). In his George Darwin Lecture, Rosseland (1943) elaborated on the nonlinear stellar pulsation problem by using the Hamiltonian approach which had been introduced to pulsation theory a few years before by Woltjer (1935). In the second-order approximation, the form of the velocity curve depends on two parameters only, viz. the period and the ratio of the radius amplitude of the pulsation to the stellar radius. Rosseland (1943, Appendix) could reproduce asymmetries in the velocity curves which were comparable with observations only for much too large displacement amplitudes. He expected that higher-order coupling terms might change the results considerably. The problem was, however, actually associated with the constant-density models which were called upon to describe the interior structure of stars. Schwarzschild & Savedoff (1949) indeed pointed out that in their standard model ($n = 3$ polytrope) the nonlinear pulsation period was essentially the same as in the linear approximation.

When studying arbitrary pulsational displacements ξ in a star one can attempt to separate the temporal (t) and the spatial contributions (parameterized by the mass at radius r: M_r) in the form

$$\xi(M_r, t) = \sum_i f_i(M_r) \cdot q_i(t) \,. \tag{1}$$

Eddington (1919) assumed the time dependence to be describable by suitable mathematical functions. The displacement was expanded in a Fourier series which was, to keep the problem tractable, terminated at the second order terms. He eventually ended up with differential equations for the spatial coefficients. Woltjer (1935), on the other hand, favored the approach of adopting a complete set of spatial functions $f_i(M_r)$ and solving for the unknown time coefficients. In the adiabatic approximation, the resulting equations can be transformed into canonical Hamiltonian form. As a complete spatial basis set of functions f_i, Woltjer chose the eigenfunctions of the linear, adiabatic boundary – eigenvalue problem. His approach can be considered as a starting point to present-day's amplitude equation formalisms which are fashionable in mostly formal analyses of nonlinear pulsation problems.

Woltjer (1935) applied his method of incorporating nonlinearities to the study of the effects of close resonances between eigenmodes of the stellar models. The particular dealing with resonances in the equations is reminiscent of methods which are encountered in celestial mechanics. Some of the RR Lyrae variables show modulations of their light-curves on a time-scale of weeks to months, a phenomenon known as the Blazhko effect. Kluyver (1935b) attempted to explain the Blazhko phenomenon as the beat period arising from the nonlinear coupling of two radial pulsation modes. She emphasized this hypothesis with the results from the standard model and a ratio of the specific heats of $\gamma = 20/13$ for which the fundamental and the second overtone modes are close to commensurability. A persistent problem with the close-resonance picture was that for its applicability two simultaneously – and not only one – excited pulsation modes must be present. At that time, though, no excitation mechanism at all was known for the stellar pulsations. Later, Woltjer returned to this nagging problem (Woltjer, 1943a; Woltjer, 1943b; Woltjer, 1946) and expanded the formalism without however achieving satisfactory solutions.

As seen above, the most accessible nonlinear problem is the study of the asymmetry of the pulsational velocity curve; this can be addressed with a nonlinear *adiabatic* description. The reproduction of lightcurves, on the other hand, needs the treatment of the full nonadiabatic system of equations. Hence, it took much longer before significant progress emerged. Schwarzschild (1938a) adopted a semi-empirical description of the photospheric density variation. His arbitrary adjustable coefficients allowed the luminosity change to be in phase with velocity rather than with radius variation. Despite a satisfactory functional form for the lightcurve, the physical reason for the particular phase relation between luminosity and velocity remained obscure.

Long before the true excitation mechanism of stellar pulsations was uncovered considerable research efforts were directed towards the amplitude limitation mechanisms, assuming linearly unstable modes to be present; much of Woltjer's work (1937; 1943a; 1943b; 1946) concentrated on this aspect. In his nonlinear mode-coupling formalism, modes with nearly commensurable frequencies can exchange energy. If one of them only is pulsationally excited, some of its energy can be drained in the resonantly coupled companion so that eventually the amplitude levels off after initial growth. But more than the saturation of the temporal evolution of the amplitudes was attacked with nonlinear mode coupling. The same approach was adopted for decades when trying to arrive at some understanding of the origin of bumps and other irregularities encountered in the observed velocity- and light-curves of pulsating variable stars.

Most physically relevant nonlinear pulsation configurations are not accessible to an analytical or even semi-analytical treatment. Therefore, projects attempting to understand or at least reproduce in detail observed properties of pulsating variables emerged only after the advent of electronic computers. Even today, research continues in attempting to satisfactorily simulate the lightcurves of observed pulsating stars to understand their features and to eventually link them with the relevant stellar physical processes.

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