The evolution of the f-mode instability and gravitational wave detection prospectives

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Abstract. We study the dynamical evolution of the gravitational-wave driven instability of the f-mode in rapidly rotating relativistic stars with a polytropic equation of state. We use linear perturbation theory to describe the evolution of the mode amplitude and follow the trajectory of a newborn neutron star through its instability window. An unstable f-mode with a saturation energy of about $10^{-6} M_{\odot} c^2$ may generate a gravitational-wave signal which can be detected by the Einstein Telescope detector from the Virgo cluster. The effects of the magnetic field on the evolution and the detectability of the gravitational radiation are relevant when its strength is higher than 10^{12} G, while an unstable r-mode becomes dominant it reaches the maximum saturation value allowed by non-linear mode couplings. From the thermal evolution we find also that the heat generated by shear viscosity during the saturation phase completely balances the neutrinos' cooling and prevents the star from entering the regime of mutual friction. The evolution time of the instability is therefore longer and the star loses significantly larger amounts of angular momentum via gravitational waves.

1. Introduction

Rapidly rotating neutron stars born in a core collapse may develop non-axisymmetric instabilities and radiate a significant amount of gravitational waves [1]. A dissipative process, as gravitational radiation, may actually drive unstable non-axisymmetric modes via the well-known Chandrasekhar-Friedman-Schutz (CFS) mechanism [2, 3, 4]. This instability occurs when a mode which is counter-rotating with respect to the star is seen co-rotating by an inertial observer. The angular momentum radiated by gravitational waves induces an increasingly negative angular momentum of the mode which thus becomes unstable and grows up to a finite amplitude where it is saturated by some dissipative processes. In realistic neutron stars the f-mode instability may be limited by bulk and shear viscosity, non-linear interaction with other modes, superfluid effects, etc. The parameter space of the instability is therefore reduced with respect to an inviscid system and is typically described by an instability window in a temperature/star's rotation plane.

The gravitational wave signal generated during the instability can be potentially observed by the current and next generation of Earth based laser interferometers, and the identification of the oscillation modes from the spectrum will help us to unveil the properties of the dense matter at super-nuclear densities and clarify the neutron star physics by using Astereoseismology [5, 6].

In this contribution we outline our work on the evolution of the gravitational-wave driven f-mode instability [7], in which we use perturbation theory to determine a system of ordinary differential equations to describe the mode's amplitude, the star's rotation and the thermal evolution. In this formalism we incorporate the effects of viscosity and magnetic field, and the impact of an unstable r-mode on the f-mode instability.

2. Evolution of the f-mode instability

In a realistic star, an oscillation mode approximately evolves as $e^{i\omega t - t/\tau}$, where ω is the mode's frequency and τ the damping/growth timescale. A way to determine τ is to calculate the variation in time of the mode's energy given in the rotating frame E by [8]

$$\frac{dE}{dt} = -\frac{2E}{\tau} \,, \tag{1}$$

where the global timescale τ of equation (1) results from the individual dissipative mechanisms that act on the mode. For stars in which only gravitational waves, shear and bulk viscosity dissipate energy, we have [8]:

$$\frac{1}{\tau} = \frac{1}{\tau_{\rm gw}} + \frac{1}{\tau_{\rm s}} + \frac{1}{\tau_{\rm b}} \,, \tag{2}$$

where $\tau_{\rm gw}$ is the gravitational radiation time scale, while $\tau_{\rm s}$ and $\tau_{\rm b}$ are, respectively, the damping time due to shear and bulk viscosity. These quantities can be determined by using appropriate volume integrals where we need to insert the f-mode frequency and eigenfunctions [9, 10]. We determine the mode's properties by studying the time evolution of the linearised equations of rapidly rotating and relativistic stellar models. In particular, we consider uniformly rotating stars with a polytropic equations of state (EoS).

To study the evolution of the f-mode instability we can derive a system of equations for the mode amplitude and stellar rotation by using equation (1) and the evolution of the angular momentum:

$$\frac{dJ}{dt} = \frac{dJ_{gw}}{dt} + \frac{dJ_{mag}}{dt} \,, \tag{3}$$

where $J = J_s + J_c$ is the total angular momentum, which consists of the star's angular momentum J_s and the canonical angular momentum of the mode J_c. The latter one can be related to the mode energy by the well known equation $J_c = -mE/\omega$ [11], where m is the azimuthal number and ω the mode frequency measured in the rotating frame. The dissipative terms in equation (3) are the gravitational radiation torque,

$$\frac{dJ_{gw}}{dt} = -\frac{2J_c}{\tau_{\rm gw}} \tag{4}$$

and the magnetic torque dJ_{mag}/dt. This quantity for a standard orthogonal rotator model in vacuum with a dipolar magnetic field is given by [12]

$$\frac{dJ_{mag}}{dt} = -\frac{R^6}{6c^3} B_p^2 \Omega^3 \,, \tag{5}$$

where Ω and R are, respectively, the star's angular velocity and radius, and $B_{\rm p}$ is the magnetic field at the magnetic pole. This formula provides a magnetic spin-down comparable to that of a standard pulsar model with magnetosphere [12, 13].

The thermal evolution of a neutron star can be studied with a global energy balance between the relevant radiative and viscous processes [12]:

$$C_v \frac{dT}{dt} = -L_\nu + H_s \,, \tag{6}$$

where C_v is the total heat capacity at constant volume, L_ν is the neutrino's luminosity produced by the modified Urca processes that operate in the background star, see [7], and H_s is the heating rate generated by shear viscosity:

$$H_s = \frac{2E}{\tau_{\rm s}} \,. \tag{7}$$

Now, we may regard the mode energy as a function of the mode amplitude α and the stellar rotation rate Ω, i.e. we can write it as $E = \alpha \tilde{E}\,(\Omega)$. From equations (1) and (3) we can therefore derive a system of ordinary differential equations:

$$\frac{d\alpha}{dt} = -\frac{2\alpha}{\tau_{\rm gw}} - \frac{2\alpha}{\tau_{\rm v}}\frac{1+\alpha Q}{D} + \frac{2P}{D}\frac{\alpha}{\tau_{\rm mag}} \,, \tag{8}$$

$$\frac{d\Omega}{dt} = \frac{2F}{D}\left(\frac{\alpha}{\tau_{\rm v}} - \frac{1}{\tau_{\rm mag}}\right) \,, \tag{9}$$

where the total viscous damping time is defined as $1/\tau_{\rm v} = 1/\tau_{\rm s} + 1/\tau_{\rm b}$, $\tau_{\rm mag}$ is the timescale due to the magnetic torque, while D, F, P and Q are functions of the background stars (see [7] for their definitions).

In the non-linear study of the secular bar-mode instability [14, 15, 16, 17], it is reasonable to discern two dynamical phases of the f-mode instability, namely the mode's exponential growth and its non-linear saturation [9]. As the star cools down and enters the instability window, the mode grows exponentially while the star slowly spins down on viscous timescales. This behaviour is well described by equations (8)-(9). This initial growth phase ends either when non-linear dynamics or dissipative processes saturate the mode amplitude at a finite value. After this point, the mode amplitude is nearly constant while the star loses angular momentum via gravitational radiation [16, 17]. We can therefore approximate this evolution phase by assuming that $d\alpha/dt = 0$, which allows us to recast equations (8)-(9) into a single relation,

$$\frac{d\Omega}{dt} = -\frac{2F}{1+\alpha Q}\left(\frac{\alpha}{\tau_{\rm gw}} + \frac{1}{\tau_{\rm mag}}\right) \,. \tag{10}$$

The star is then spun down by gravitational radiation and magnetic torque. Eventually, the star will leave the instability window and the f-mode will be damped by gravitational radiation.

We normalise the mode amplitude as $E = \alpha E_{\rm rot}$, where $E_{\rm rot}$ is the stellar rotational energy. For our polytropic models an $\alpha = 1$ corresponds to $E \sim 10^{-2}\, M_\odot c^2$. Note that with this definition the fluid variables and the gravitational-wave strain scale as $\alpha^{1/2}$.

3. Stellar Models

We model the compact objects as relativistic neutron stars which obey a polytropic EoS,

$$p = K\rho^{1+1/N} \tag{11}$$

where p is the fluid pressure, K is the polytropic constant and N is the polytropic index. The rest-mass density ρ is related to the fluid energy density ε via the relation $\varepsilon = \rho + Np$.

We consider two sequences of uniformly rotating stars up to the mass shedding limit $\Omega_{\rm K}$. The first model represents a "standard" neutron star with gravitational mass $M = 1.4\, M_\odot$ and polytropic index $N = 1$. The non-rotating member of this sequence has a circumferential radius of $R = 14.15$ km, while the maximum rotation rate is $\nu_{\rm K} = \Omega_{\rm K}/2\pi = 673.1$ Hz. The second model is a polytrope with $N = 2/3$, and baryonic mass $M_b = 1.6\, M_\odot$ and a Kepler limit of $\nu_{\rm K} = \Omega_{\rm K}/2\pi = 1783$ Hz. This model is used to widen the parameter space of the instability, as it describes a supramassive star, i.e. an object that does not have a stable configuration in the static limit [18, 19, 20].

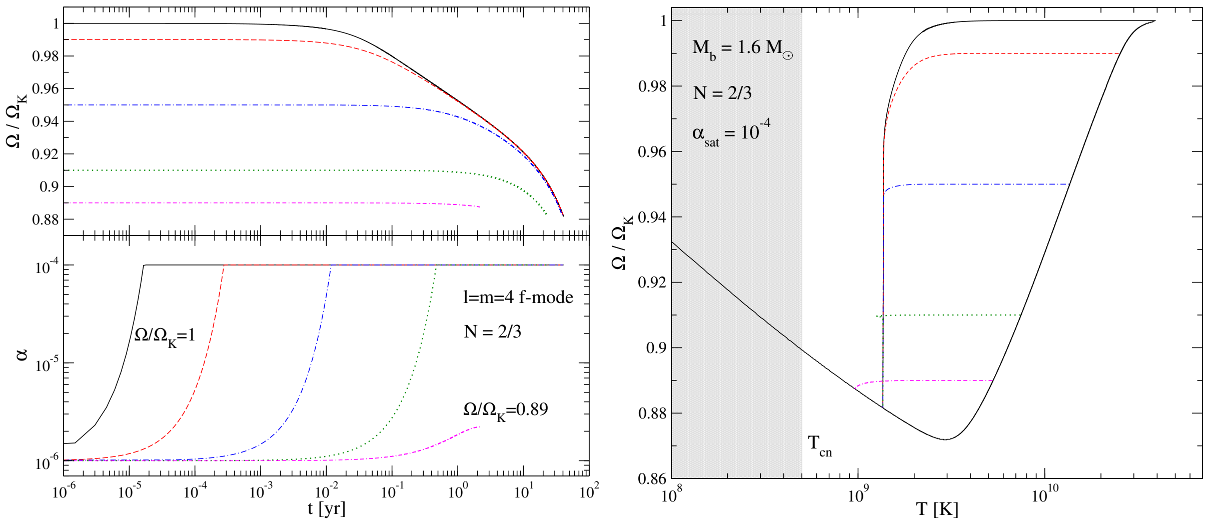

Figure 1. Evolution of the $l = m = 4$ f-mode instability for a relativistic polytrope with $N = 2/3$, baryonic mass of $M_b = 1.6\,M_\odot$ and $B_\mathrm{p} = 10^{11}$ G. The star enters the instability window at different spins *(see right panel)* and correspondingly the left panel depicts the time-evolution of the stellar rotation rate *(top left panel)* and mode amplitude *(bottom left panel)*, where the initial amplitude $\alpha = 10^{-6}$ saturates at $\alpha_{sat} = 10^{-4}$. The shaded region in the right panel represents the temperature range where the neutrons and protons of the core are expected to be in a superfluid/superconducting state in accordance with recent models for the observed cooling of Cassiopeia A [21, 22].

4. Results

We study now the dynamical evolution of the f-mode and consider how the magnetic field and a simultaneous presence of an unstable r-mode affect the instability.

The ideal conditions for the f-mode instability to work are high temperatures and rotation rates. At birth, the temperature of a neutron star is around $T \simeq 10^{11}$ K and drops down rapidly in the following few days. We therefore expect that as a rapidly rotating star cools down, it enters the f-mode instability window from the high temperature side (see figure 1). We study the evolution of the star from the onset of the gravitational-wave driven instability for different initial rotation rates, and consider an initial mode amplitude of $\alpha = 10^{-6}$. This corresponds to an energy of $E \sim 10^{-8}\,M_\odot c^2$ which is a typical value for the total energy loss in gravitational waves due to quadrupole deformations shortly after a gravitational core collapse [23, 24].

In [7], we study the $l = m = 3$ and 4 f-mode for the $N = 1$ and $N = 2/3$ models. Here we report only the most promising cases. We first consider the evolution of the $l = m = 4$ f-mode instability in a low magnetised star ($B_p \leq 10^{11}$ G), and show in figure 1 the star's spin, the mode's amplitude and the star's trajectory through the instability window for the $N = 2/3$ model. At the mass shedding limit the gravitational growth time is only $\tau_\mathrm{gw} \simeq 250\,\mathrm{s}$ in contrast to the $\tau_\mathrm{gw} \simeq 10^4\,\mathrm{s}$ for the $N = 1$ model, and the total evolution lasts for about 40 years. The exponential growth of the f-mode takes about 5 minutes (14 days) for a star with an initial $\Omega = \Omega_\mathrm{K}$ ($\Omega = 0.93\,\Omega_\mathrm{K}$). After the mode saturates, figure 1 (right panel) shows that the star spins down at nearly constant temperature ($T \simeq 10^9$ K) as a result of the heat generated by shear viscosity which completely balances the neutrino cooling. *This effect prevents the star from entering the superfluid transition zone and increases the duration of the instability.* In fact, for $T \leq T_\mathrm{cn}$ the neutrons of the core become superfluid and mutual friction damps the f-mode very efficiently on short timescales [25]. For the superfluid critical temperature we chose a value of

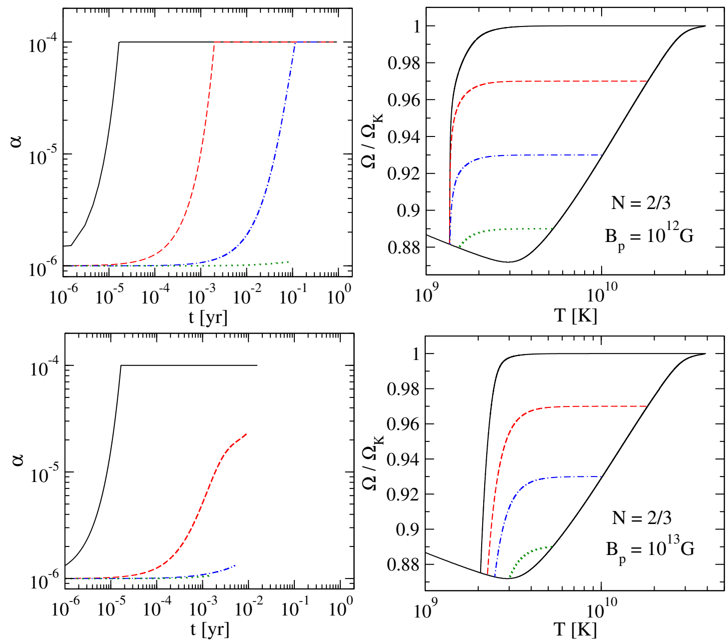

Figure 2. The impact of the magnetic torque on the evolution of the $l = m = 4$ f-mode instability for the $N = 2/3$ model. The two top panels display the mode amplitude (left panel) and the star's evolution through the instability region (right panel) for $B_{\rm p} = 10^{12}$G. The same quantities are depicted in the two lower panels for $B_{\rm p} = 10^{13}$G.

$T_{\rm cn} \simeq 5\times10^8$ K which has recently been determined from the cooling curves of Cassiopeia [21, 22]. Furthermore, figure 1 shows that when a star becomes unstable at lower rotation rates near the minimum of the instability window, e.g. $\Omega \simeq 0.89\,\Omega_{\rm K}$, the mode amplitude remains small and the shear viscosity heating does not balance the neutron star's cooling.

A strong magnetic field can accelerate the star's spin-down and limit the growth of the f-mode. In figure 2, we show the effects on the $l = m = 4$ f-mode for the $N = 2/3$ model and two dipolar magnetic field configurations with, respectively, $B_{\rm p} = 10^{12}$G and $B_{\rm p} = 10^{13}$G. For a magnetic field of $B_{\rm p} = 10^{12}$ G the total duration of the instability is shorter than a factor of about 50 with respect to the $B_{\rm p} = 10^{11}$G model. In fact, the evolution lasts about 1 yr (10 yr) for the $N = 2/3$ ($N = 1$) model. In slower rotating models, the magnetic torque clearly dominates the evolution even during the initial phase of the instability and limits considerably the growth of the mode amplitude. This behaviour is for instance evident for a $N = 2/3$ model with an initial rotation rate of $\Omega = 0.89\,\Omega_{\rm K}$. The magnetic torque dominates completely the evolution when $B_{\rm p} = 10^{13}$ G. The unstable star is quickly spun down and the mode amplitude is strongly limited even in the more massive model.

In rapidly rotating neutron stars several modes can be driven unstable by gravitational radiation at the same time. The most important ones are definitely the f- and r-modes, as they have a comparatively short growth time and therefore can generate a significant gravitational wave signal. A relevant difference between these two classes of modes is that the f-mode only gets unstable in very rapidly rotating stars while the r-mode is CFS unstable at any rotation rate. In addition, the growth time is typically shorter for the r-mode which consequently has a larger instability window. It is then reasonable to think that the r-mode should dominate the evolution of the gravitational wave driven instability. However before drawing any secure conclusion, it is necessary to know the maximum amplitude that each mode can reach during the instability. Non-linear perturbation calculations show that the r-mode may transfer energy to other inertial modes through non-linear mode coupling and has a small saturation amplitude.

As the star cools down, it is expected that the r-mode gets unstable before the f-mode as its instability window extends towards higher temperatures. The r-mode then quickly reaches its non-liner saturation value α_r^{sat} and spins down the star. However if the r-mode growth is limited by non-linear mode coupling, the star evolves towards the critical curve of the f-mode

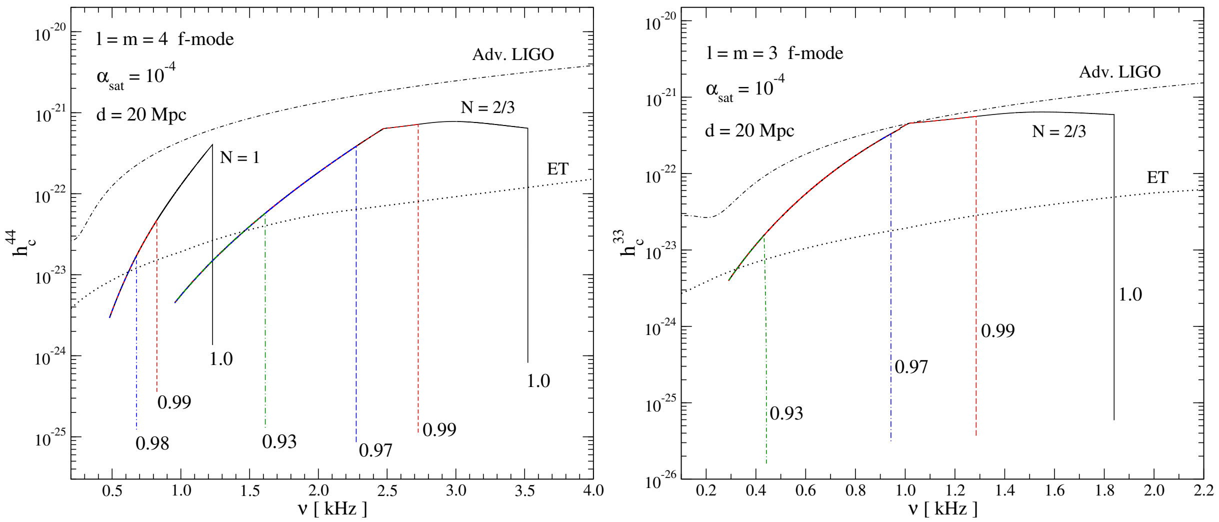

Figure 3. Characteristic strain generated by the f-mode instability for a polytrope with $B_{\rm p} = 10^{11}$G. The source is located at 20 Mpc and the saturation amplitude of the f-mode is set to $\alpha_{sat} = 10^{-4}$. *Left panel:* The gravitational-wave signal emitted by the $l = m = 4$ f-mode for the two stellar models with $N = 1$ and $N = 2/3$. The number near the vertical lines denotes the initial rotation rate $\Omega/\Omega_{\rm K}$. *Right panel:* The signal emitted the by $l = m = 3$ f-mode for the more massive model with $N = 2/3$. The sensitivity curves of Advanced LIGO and the Einstein Telescope (ET) are shown in both panels. The gravitational-wave signal generated during the f-mode instability may be detected by ET for the most part of the instability window of the $N = 2/3$ model.

and eventually also this mode is driven unstable by gravitational radiation.

We have studied several simulations with different relative saturation amplitude between the $l = m = 4$ f- and $l = m = 2$ r-modes, and found that if the r-mode reaches the maximum saturation amplitude predicted by the non-linear coupling calculations, it dominate the evolution of the instability [7]. However, it is therefore crucial to know more accurately the relative saturation amplitude between these two modes, in particular for the f-mode. This is an interesting aspect that must be clarified in a future work.

5. Gravitational Waves

The gravitational-wave signal generated during the instability of the f-mode may be potentially observed by the current and next generation of Earth-based laser interferometers. To evaluate the detection we determine the characteristic strain of the gravitational waves, which is a quantity that takes into account also the statistical amplification of the signal due to the number of oscillations accumulated in a given frequency bandwidth. It is defined by

$$h_c^{lm} \equiv \langle \left| h^{lm} {}_{-2}Y^{lm} \right| \rangle \sqrt{N_{cyc}} \qquad (12)$$

where h^{lm} is the strain emitted by the source, ${}_{-2}Y^{lm}$ is a spin-weighted spherical harmonic and $\langle \ldots \rangle$ denotes an averaging over the angles (θ, ϕ). The number of oscillation cycles can be expressed as $N_{cyc} = \nu \, \tau_{ev}$, which is written in terms of the mode frequency $\nu = (\omega - m\Omega)/2\pi$ and the time spent near a given frequency τ_{ev}. Considering the signal integration time allowed by the detector technology, we assume that a detector may integrate the signal at most for 1 year, and consequently calculate h_c^{lm} for $\tau_{ev} \leq 1$yr. This means that in N_{cyc} we use the evolution time of our simulations whenever $\tau_{ev} < 1$yr, otherwise we set $\tau_{ev} = 1$yr.

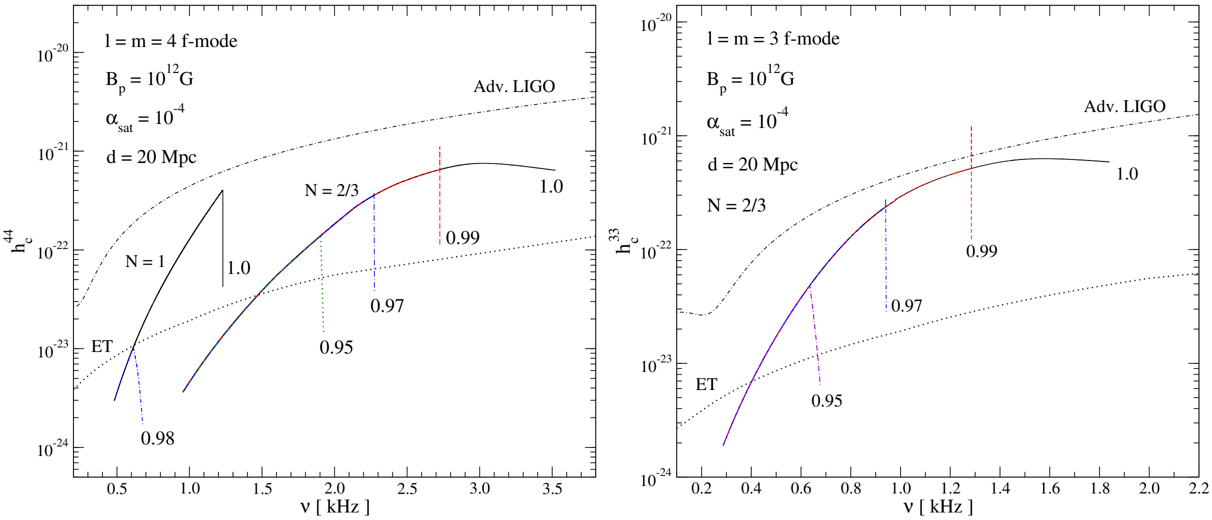

Figure 4. The characteristic strain generated by the f-mode instability for a polytropic star with $B_{\rm p} = 10^{12}$ G. The quantities and notation used in this figure are the same as in figure 3.

For the two polytropic models of this work, we calculate h_c^{lm} for several instability evolutionary paths with different initial stellar spin. We consider first a star with a weak magnetic fields, $B_{\rm p} \leq 10^{11}$ G, and focus on the $l = m = 3$ and 4 f-mode. For a saturation amplitude of $\alpha_{sat} = 10^{-4}$, figure 3 shows that the gravitational-wave signal can be detected by ET for the most part of the instability evolution for a $N = 2/3$ star located in the Virgo Cluster at a distance of 20 Mpc. For the standard $N = 1$ model an unstable $l = m = 4$ f-mode may generate a detectable signal only if the star rotates near the Kepler limit $\Omega \geq 0.98\,\Omega_{\rm K}$. From figure 3 we see that the gravitational-wave signal is initially monochromatic during the growth phase of the mode and evolves successively to lower frequencies as the star spins down.

A higher magnetic field may affect significantly the characteristic strain of the f-mode, as the number of accumulated cycles N_{cyc} decreases. The results for the $B_{\rm p} = 10^{12}$ G case are shown in figure 4 for both the $N = 1$ and $N = 2/3$ models. Although the gravitational-wave signal is slightly weaker than in the previous $B_{\rm p} = 10^{11}$ G case, it is still detectable by the Einstein Telescope (ET).

6. Conclusions

We have studied the evolution of the f-mode instability of rapidly rotating relativistic stars and have considered the effects of viscosity, magnetic fields and unstable r-modes. We have modelled the star with polytropic equations of state to explore the instability parameter space. The evolution of the star through the instability window may be accelerated by the presence of strong magnetic fields and a simultaneous presence of an unstable r-mode. Our results show that a magnetic field with $B_p \geq 10^{12}$ G affects the f-mode instability, and its influence is more relevant in neutron star models with smaller compactness (M/R). For the r-mode we find that it must reach the maximum value expected from non-linear mode coupling studies in order to affect considerably the f-mode evolution. However, a definitive answer to this issue cannot be given as the maximum f-mode amplitude is still unknown.

Another important result of our simulations is that the heat generated by shear viscosity during the f-mode saturation phase prevents the star from entering the regime of mutual friction. In fact, the shear viscosity re-heating balances the neutrino cooling and leads to a nearly isothermal evolution. The star therefore leaves the instability window at lower rotation rates, and even a moderate change in the superfluid transition temperature does not modify this result.

Our results show also that the gravitational-wave signal generated during the f-mode instability may be detected by ET if the source is located within the Virgo cluster and the mode saturates at $E \sim 10^{-6} M_{\odot} c^2$.

Acknowledgements

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