PPN formalism

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Introduction

Up to now, there exists no experiment purporting inconsistency of Einstein's theory. General relativity is definitely a beautiful theory of gravitation.

However, we may have alternative approaches to explain all gravitational phenomena. We have also faced on some fundamental unknowns in the Universe, such as dark energy and dark matter, which might be solved by new theory of gravitation.

The candidates as *an alternative gravitational theory* should satisfy at least three criteria for viability; (1) self-consistency, (2) completeness, and (3) agreement with past experiments.

Metric Theory

In only one significant way do metric theories of gravity differ from each other:
$\longrightarrow$ their laws for the generation of the metric.

- In GR, the metric is generated directly by the stress-energy of matter and of nongravitational fields.

- In *Dicke-Brans-Jordan theory*, matter and nongravitational fields generate a scalar field φ; then φ acts together with the matter and other fields to generate the metric, while "long-range field" φ CANNOT act back directly on matter.

(1) Despite the possible existence of long-range gravitational fields in addition to the metric in various metric theories of gravity, the postulates of those theories demand that *matter and non-gravitational fields be completely oblivious to them*.
(2) The only gravitational field that enters the equations of motion is the metric.

Thus the metric and equations of motion for matter become the primary entities for calculating observable effects.

Post-Newtonian Limit

The comparison of metric theories of gravity with each other and with experiment becomes particular simple when one take the *slow-motion, weak-field limit*. This approximation, known as the "*post-Newtonian limit*," is sufficiently accurate to encompass most solar-system tests.

In fact, the solar-system has weak gravity, the matter that generates solar-system gravity moves slowly, and has small internal energies, such as $|U| \leq 10^{-6}, v^2 \leq 10^{-7}, \Pi \leq 10^{-6}$, where $|U|$, v, and Π denote Newtonian potential, velocity relative to solar system center mass, and internal energy per unit mass.

Each metric theory has its own post-Newtonian approximation.

Despite the great differences between metric thoeries themselves, *their post-Newtonian approximations are very similar*.

- Metric predicted by nearly every metric theory of gravity has the same structure.
- It can be written as an expansion about the Minkowski metric in terms of dimensionless gravitational potentials of varying degrees of smallness, which are constructed from the matter variables.

Parametrized Post-Newtonian (PPN) formalism

"Order of smallness" is determined according to the rules $U \sim v^2 \sim \Pi \sim p/\rho \sim \epsilon$, $v^i \sim |d/dt|/|d/dx| \sim \epsilon^{1/2}$, and so on.

A consistent post-Newtonian limit requires determination of $g_{00} \sim \mathcal{O}(\epsilon^2)$, $g_{0i} \sim \mathcal{O}(\epsilon^{3/2})$, and $g_{ij} \sim \mathcal{O}(\epsilon)$.

The only way that one metric theory differs from another is in the numerical values of the coefficients that appear in front of the metric potentials.

The *parametrized post-Newtonian formalism* inserts parameters depending on the theory in place of these coefficients. In order to indicate general properties of metric theories of gravity, one needs to prepare **TEN parameters**.

By using these parameters, we can identify the metric theories of gravity.

One set of values makes the PPN formalism identical to the PN limit of GR.

Another set of values makes it the PN limit of Dicke-Brans-Jordan theory, etc...

PPN parameters

parameter	What it measures relative to GR
γ	How much space-curvature produced by unit rest mass ?
β	How much "nonlinearity" in the superposition law for gravity ?
ξ	Preferred-location effects ?
α_1,α_2,α_3	Preferred-frame effects ?
α_3,ζ_1,ζ_2,ζ_3,ζ_4	Violation of conservation of total momentum ?

- The parameters γ and β are used to describe the "classical" tests of GR, and in some sense the most important.
- The parameter ξ is non-zero in any theory of gravity that predicts preferred-location effects such as a galaxy-induced anisotropy in the local gravitational constant.

- In GR, (γ, β)=(1,1) and the other parameters are zero.
- In scalar-tensor, the only non-zero parameters are γ and β.
- Semi-conservative theories have five free PPN parameters ($\gamma, \beta, \xi, \alpha_1, \alpha_2$).
- Fully conservative theories have three PPN parameters (γ, β, ξ).

Summary of PPN formalism

Matter variables:

- ρ: density of rest mass as measured in a local free falling frame.
- $v^i = dx^i/dt$: coordinate velocity of the matter,
- w^i: coordinate velocity of the PPN coordinate system relative to the mean rest-frame of the universe,
- p: pressure as measured in a local free falling frame momentarily comoving with the matter,
- Π: internal energy per unit rest mass.

Metric:

$$g_{00} = -1+2U-2\beta U^2-2\xi\Phi_W+(2\gamma+2+\alpha_3+\zeta_1-2\xi)\Phi_1+2(3\gamma-2\beta+1+\zeta_2+\xi)\Phi_2+2(1+\zeta_3)\Phi_3$$
$$+2(3\gamma+3\zeta_4-2\xi)\Phi_4-(\zeta_1-2\xi)\mathcal{A}-(\alpha_1-\alpha_2-\alpha_3)w^2U-\alpha_2 w^i w^j U_{ij}+(2\alpha_3-\alpha_1)w^i V_i+\mathcal{O}(\epsilon^3),$$

$$g_{0i} = -(4\gamma+3+\alpha_1-\alpha_2+\zeta_1-2\xi)V_i/2-(1+\alpha_2-\zeta_1+2\xi)W_i/2-(\alpha_1-2\alpha_2)w^i U/2-\alpha_2 w^j U_{ij}+\mathcal{O}(\epsilon^{5/2}),$$

$$g_{ij} = (1+2\gamma U)\delta_{ij}+\mathcal{O}(\epsilon^2).$$

Metric potential:

$$U = \int \frac{\rho'}{|x-x'|} d^3x', \quad U_{ij} = \int \frac{\rho'(x-x')_i (x-x')_j}{|x-x'|^3} d^3x', \quad \Phi_W = \int \frac{\rho'\rho''(x-x')}{|x-x'|^3} \cdot \left(\frac{x'-x''}{|x-x''|} - \frac{x-x''}{|x'-x''|} \right) d^3x' d^3x'',$$

$$\Phi_1 = \int \frac{\rho' v'^2}{|x-x'|} d^3x', \quad \Phi_2 = \int \frac{\rho' U'}{|x-x'|} d^3x', \quad \Phi_3 = \int \frac{\rho' \Pi'}{|x-x'|} d^3x', \quad \Phi_4 = \int \frac{p'}{|x-x'|} d^3x',$$

$$\mathcal{A} = \frac{\rho' [v' \cdot (x-x')]^2}{|x-x'|^3} d^3x', \quad V_i = \int \frac{\rho' v'_i}{|x-x'|} d^3x', \quad W_i = \int \frac{\rho' [v' \cdot (x-x')] (x-x')_i}{|x-x'|} d^3x'.$$

Stress-energy tensor (perfect fluid):

$$T^{00} = \rho(1 + \Pi + v^2 + 2U),$$

$$T^{0i} = \rho v^i (1 + \Pi + v^2 + 2U + p/\rho),$$

$$T^{ij} = \rho v^i v^j (1 + \Pi + v^2 + 2U + p/\rho) + p\delta^{ij}(1 - 2\gamma U).$$

Competing theories of gravity

One of the important applications of the PPN formalism is the comparison and classification of alternative metric theories of gravity.
$\longrightarrow$ Use of the PPN framework eliminated many theories thought previously to be viable.

Here we shall focus on some metric theories of gravity;

theory	Arbitrary	cosmological	γ	β	ξ	α_1	α_2
GR	none	none	1	1	0	0	0
ST							
Brans-Dicke	ω_{BD}	ϕ_0	$\frac{1+\omega_{BD}}{2+\omega_{BD}}$	1	0	0	0
General	$A(\varphi), V(\varphi)$	φ_0	$\frac{1+\omega}{2+\omega}$	$1+\Lambda$	0	0	0
VT							
Unconstrained	$\omega, c_1, c_2, c_3, c_4$	u	γ'	β'	0	α_1'	α_2'
Einstein-Æther	c_1, c_2, c_3, c_4	none	1	1	0	α_1'	α_2'

The parameters γ', β', α_1', and α_2' are complicated functions of u and of the arbitrary constants.

Tests of the parameter γ

Deflection of light

A light ray with passes the Sun at a distance d is deflected by an angle

$$\delta\theta = \frac{1+\gamma}{2}\frac{4M_\odot}{d}\frac{1+\cos\Phi}{2},$$

where Φ is the angle between the Earth-Sun line and the incoming direction of the photon.

- 1995 VLBI: $(1+\gamma)/2 = 0.9996 \pm 0.0017$.
- 2004 VLBI: $(1+\gamma)/2 = 0.99992 \pm 0.00023$.

$\longrightarrow$ equivalently $\gamma - 1 = (-1.7 \pm 4.5) \times 10^{-4}$.

Time delay of light

- 1976 Viking: in a 0.1% measurement.
- 2003 Cassini: $\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$.

$\longrightarrow$ $(1+\gamma)/2$ must be within at most 0.0012% of unity.

$\longrightarrow$ Scalar-tensor theories must have $\omega > 40000$ to be compatible with this constraint.

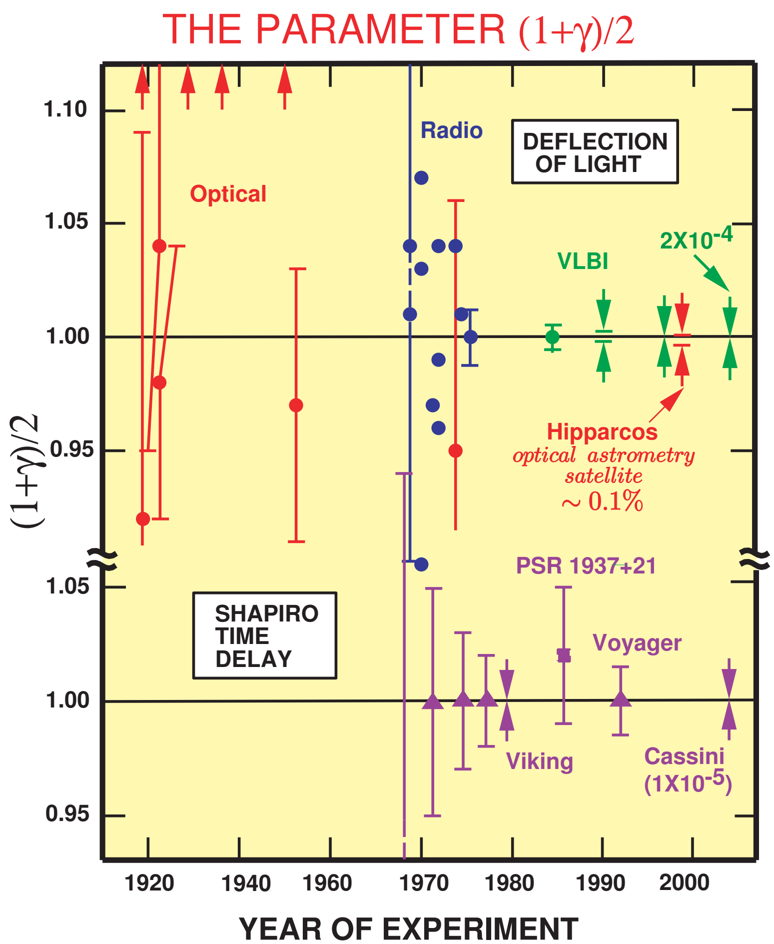

Figure 1: Estimations of the coefficient $(1+\gamma)/2$ from light deflection and time delay measurements [4].

Perihelion shift of Mercury

The predicted advance per orbit $\Delta\tilde{\omega}$, including both PPN and the Newtonian contributions resulting from a possible solar quadrupole moment, is given by

$$\Delta\tilde{\omega} = \frac{6\pi m}{p}\left[\frac{1}{3}(2+2\gamma-\beta) + \frac{1}{6}(2\alpha_1 - \alpha_2 + \alpha_3 + 2\zeta_2)\frac{\mu}{m} + \frac{J_2 R^2}{2mp}\right],$$

where m and μ are the total mass and reduced mass of the two-body system; $p \equiv a(1-e^2)$ is the semi-lotus rectum of the orbit, with the semi-major axis a and the eccentricity e; R is the mean radius of the oblate body; and J_2 is a dimensionless measure of its quadrupole moment.

- 1st term is the classical relativistic perihelion shift.
- 2nd term could be dropped. ($\mu/m \simeq 2\times 10^{-7}$ or in any fully conservative theory this term should be 0.)
- In 3rd term, J_2 is determined from the data of helioseismology; $J_2 = (2.2 \pm 0.1)\times 10^{-7}$.

Then we obtain the rate of perihelion shift $\dot{\tilde{\omega}}$ in arcseconds per century as

$$\dot{\tilde{\omega}} = 42.''98\left[\frac{1}{3}(2+2\gamma-\beta) + 3\times 10^{-4}\frac{J_2}{10^{-7}}\right].$$

$\longrightarrow$ we obtain the PPN bound $|2\gamma - \beta - 1| < 3\times 10^{-3}$.

Tests of the strong equivalence principle

Next class of solar-system experiments that test relativistic gravitational effects may be called tests of the *strong equivalence principle (SEP)*.

Among the testable violations of SEP are a violation of the weak equivalent principle (WEP) for gravitating bodies that leads to
- perturbations in the Earth-Moon orbit (Nordtvedt effect),
- preferred-location and preferred-frame effects in the locally measured G,
- possible variations in the gravitational constant over cosmological timescales.

Nordtvedt effect
Many metric theories predict that massive bodies violate WEP, that is, fall with different accelerations depending on their gravitational self-energy.
$\longrightarrow$ Such an effect would occur in theories with a spatially varying G, such as ST.

This violation of the massive-body equivalence principle is known as the "*Nordtvedt effect*."
- In GR this effect is absent, but this is present in ST.

For a spherically symmetric body, the acceleration from rest in an external U has the form

$$a = \left(1 - \eta_N \frac{E_g}{m}\right) \nabla U, \quad \eta_N = 4\beta - \gamma - 3 - \frac{10}{3}\xi - \alpha_1 + \frac{2}{3}\alpha_2 - \frac{2}{3}\zeta_1 - \frac{1}{3}\zeta_2,$$

where $E_g > 0$ is the negative of the gravitational self-energy of the body.

For astronomical bodies, E_g/m may be significant.
$\longrightarrow$ 3.6×10^{-6} for Sun; 10^{-8} for Jupiter; 4.6×10^{-10} for Earth; 0.2×10^{-10} for Moon.

If this effect is present, Earth should fall toward Sun with a slightly different acceleration than Moon.
$\longrightarrow$ This perturbation in the Earth-Moon orbit leads to a polarization of the orbit.
$\longrightarrow$ This polarization represents a perturbation in the Earth-Moon distance.
With the data Lunar Laser Ranging (LLR), $|\eta_N| = (4.4 \pm 4.5) \times 10^{-4}$.

Preferred-location and preferred-frame effects
Some theories of gravity violate SEP by predicting that the outcomes of local gravitational experiments may depend on the velocity of the laboratory relative to the mean rest frame of the universe (*preferred-frame effects*) or on the location of the laboratory relative to a nearby gravitating body (*preferred-location effects*).

In the post-Newtonian limit, preferred-frame effects are governed by α_1, α_2, and α_3, and some preferred-location effects are governed by ξ.

The most important such effects are variations and anisotropies in the locally-measured value of the gravitational constant, which lead to

- anomalous Earth tides
- variations in the Earth's rotation rate,
- anomalous contributions to the orbital dynamics of planets and Moon,
- self-accelerations of pulsars,
- anomalous torques on Sun.

$\xi < 1 \times 10^{-3}$ from data for anomalous tidal effects on Earth.
$\alpha_1 < 1 \times 10^{-4}$ by using LLR data.
$\alpha_1 < 2 \times 10^{-4}$ by using observations of the circular binary orbit of PSR J2317+1439.
$\alpha_2 < 4 \times 10^{-7}$ from the precession of the solar spin.
$\alpha_3 < 4 \times 10^{-20}$ from Period derivatives of 21 millisecond pulsars.

Tests of post-Newtonian conservation laws

With respect to the parameter ζ_3;

Alternative statement of Newton's 3rd law for gravitating systems is that the "active gravitational mass ($m_{\rm A}$)," that is the mass that determines the gravitational potential exhibited by a body, should equal the "passive gravitational mass ($m_{\rm P}$)," the mass that determines the force on a body in a gravitational field.

Remarkable planetary test of Newton's 3rd law was done for the observation of Moon. Improved knowledge of the lunar orbit through LLR and a better understanding of tidal effects in the Earth-Moon system provide the resulting limit

$$\left|\frac{(m_{\rm A}/m_{\rm P})_{\rm Al}-(m_{\rm A}/m_{\rm P})_{\rm Fe}}{(m_{\rm A}/m_{\rm P})_{\rm Al}}\right| < 4\times 10^{-12}.$$

According to the PPN formalism, the electrostatic binding energy E_e of an atomic nucleus could make a contribution to the ratio of active to passive mass of the form

$$m_{\rm A} = m_{\rm P} + \zeta_3 E_e/2.$$

$\longrightarrow$ Resulting limit on ζ_3 from the lunar experiment is $\zeta_3 < 10^{-8}$.

With respect to the parameter ζ_2;

Another consequence of a violation of conservation of momentum is *a self-acceleration of the center of mass of a binary stellar system*. A consequence of this acceleration would be non-vanishing values for $\ddot{P}$, where P denotes the period of any intrinsic process in the system.

$\longrightarrow$ Observed upper limit on $\ddot{P}_{\rm P}$ of the binary pulsar PSR 1913+16 places a strong constraint, such as $|\alpha_3 + \zeta_2| < 4 \times 10^{-5}$.

$\longrightarrow$ With the known constraint on α_3, we obtain a strong solitary bound on $\zeta_2 < 4 \times 10^{-5}$.

With respect to the parameters ζ_1 and ζ_4;

- Using the Nordtvedt effect parameter, we could make a constraint on the value of ζ_1.

$$\eta_N = 4\beta - \gamma - 3 - \frac{10}{3}\xi - \alpha_1 + \frac{2}{3}\alpha_2 - \frac{2}{3}\zeta_1 - \frac{1}{3}\zeta_2.$$

- The condition that the point-mass PPN formalism agrees with the standard perfect-fluid PPN formalism for arbitrary system;

$$6\zeta_4 = 3\alpha_3 + 2\zeta_1 - \zeta_3.$$

Current limits on the PPN parameters

parameter	Effect	Limit	Remarks
$\gamma - 1$	time delay	2.3×10^{-5}	Cassini tracking
	light deflection	4×10^{-4}	VLBI
$\beta - 1$	perihelion shift	3×10^{-3}	$J_2 = 10^{-7}$ from helioseismology
	Nordtvedt effect	2.3×10^{-4}	$\eta_N = 4\beta - \gamma - 3$ assumed
ξ	Earth tides	10^{-3}	gravimeter data
α_1	orbital polarization	10^{-4}	Lunar laser ranging
		2×10^{-4}	PSR J2317+1439
α_2	spin precession	4×10^{-7}	solar alignment with ecliptic
α_3	pulsar acceleration	4×10^{-20}	pulsar $\dot{P}$ statistics
η_N	Nordtvedt effect	9×10^{-4}	Lunar laser ranging
ζ_1	—	2×10^{-2}	combined PPN bounds
ζ_2	binary acceleration	4×10^{-5}	$\ddot{P}_{\rm p}$ for PSR 1913+16
ζ_3	Newton's 3rd law	10^{-8}	lunar acceleration
ζ_4	—	6×10^{-3}	$6\zeta_4 = 3\alpha_3 + 2\zeta_1 - 3\zeta_3$ assumed

Conclusion

- In slow-motion and weak-field limit, each metric theory has its own post-Newtonian approximation.
- Since the difference among the corresponding post-Newtonian approximations to the metric theories is in the numerical values of their coefficients, by introducing new 10 parameters we can indicate general properties of metric theories (PPN).
- Many experimental evidences provide the constraints on the PPN parameters.
- At least with the experiments in the solar system, the deviation from GR is quite little.
- Future experiments will be able to give severer constraints on the PPN parameters and more alternative theories will be screened.

References

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