Particle Trajectories & The Classical Tests

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$$
\begin{aligned}
\ddot{r} - \frac{1}{2}\nu' \dot{r}^2 - re^{\nu}\dot{\theta}^2 - re^{\nu}\sin^2\theta\dot{\phi}^2 + \frac{1}{2}e^{2\nu}\nu'\dot{t}^2 &= 0 && (1)\\
\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 &= 0 && (2)\\
\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\frac{\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi} &= 0 && (3)\\
\ddot{t} + \nu'\dot{r}\dot{t} &= 0 && (4)
\end{aligned}
$$

(how?) and from $g_{\lambda\mu}\dot{x}^{\lambda}\dot{x}^{\mu} = 1$ (or $= c^2$) (here we assume a massive particle) we get :

$$-e^{\nu}\dot{t}^2 + e^{\lambda}\dot{r}^2 + r^2\dot{\theta} + r^2\sin^2\theta\dot{\phi}^2 = 1. \tag{5}$$

If a geodesic is passing through a point P in the equatorial plane ($\theta = \pi/2$) and has a tangent at P situated also in this plane ($\dot{\theta} = 0$ at P) then from (2) we get $\ddot{\theta} = 0$ at P and all higher derivatives are also vanishing at P. That is, **the geodesic lies entirely in the plane** defined by P, the tangent at P and the center of symmetry of the space.

Schwarzschild Solution: Geodesics ii

Since the symmetry planes are equivalent to each other, it will be sufficient to discuss the geodesics lying on one of these planes e.g. the equatorial plane $\theta = \pi/2$.

The geodesics on the equatorial plane are:

$$
\begin{aligned}
\ddot{r} - \frac{1}{2}\nu'\dot{r}^2 - re^{\nu}\dot{\phi}^2 + \frac{1}{2}e^{2\nu}\nu'\dot{t}^2 &= 0 && (6)\\
\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} &= 0 && (7)\\
\ddot{t} + \nu'\dot{r}\dot{t} &= 0 && (8)\\
-e^{\nu}\dot{t}^2 + e^{\lambda}\dot{r}^2 + r^2\dot{\phi}^2 &= 1 && (9)
\end{aligned}
$$

Then from equations (7) and (8) we can easily prove (how?) that:

$$
\begin{aligned}
&\frac{d}{d\tau}\left(r^2\dot{\phi}\right) = 0 \quad \Rightarrow \quad r^2\dot{\phi} = h = \text{const} : \text{ (Angular Momentum)} && (10)\\
&\frac{d}{d\tau}\left(e^{\nu}\dot{t}\right) = 0 \quad \Rightarrow \quad e^{\nu}\dot{t} = k = \text{const} : \text{ (Energy)} && (11)
\end{aligned}
$$

Schwarzschild Solution: Geodesics iii

- For a massive particle with unit rest mass, , m_0, by assuming that τ is the affine parameter for its motion we get $p^\mu = (E/c, \vec{p}) = \dot{x}^\mu$ and $p_\mu = g_{\mu\nu}\dot{x}^\nu$. [1] Thus

$$
\begin{aligned}
p_0 &= g_{00}\dot{t} = e^\nu \dot{t} = kc^2 \quad \text{and} && (12)\\
p_\phi &= g_{\phi\phi}\dot{\phi} = -r^2\dot{\phi} = -h && (13)
\end{aligned}
$$

- An observer with 4-velocity U^μ will find that the energy of a particle with 4-momentum p^μ is:

$$E = p_\mu U^\mu$$

An observer at rest at infinity, will have $U^\mu = (1, 0, 0, 0)$ leading to $E = p_0 = kc^2$ which is conserved along the particle geodesic.

Actually, for a particle with rest mass m_0 we get $k = E/(m_0c^2)$ i.e. it is the energy per unit rest-mass.

For a particle at infinity ($e^\nu \to 1$ and thus $k \to 1$) thus $E = m_0c^2$.

- The constant h (later we may use the letter L) it is equal to the specific **angular momentum**of the particle and its is $-p_\phi$.

Finally, by substituting eqns (11) and (10) into (9) we get the "**energy equation**" for the r-coordinate (from now on we assume $c = 1$):

$$\dot{r}^2 + \frac{e^\nu}{r^2} L^2 + e^\nu = E^2 \tag{14}$$

which suggests that at $r \to \infty$ (and $\dot{r} = 0$) we get that $E = 1$.

By combining the 2 integrals of motion and eqn (9) we can eliminate the proper time τ to derive an equation for a 3D path of the particle (how?)

$$\left(\frac{d}{d\phi} u\right)^2 + u^2 = \frac{E^2 - 1}{L^2} + \frac{2Mu}{L^2} + 2Mu^3 \tag{15}$$

where we have used $u = 1/r$ and

$$\dot{r} = \frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{L}{r^2}\frac{dr}{d\phi}$$

Schwarzschild Solution: Geodesics v

The previous equation can be written in a form similar to Kepler's equation of Newtonian mechanics (how?) i.e.

$$\frac{d^2}{d\phi^2} u + u = \frac{M}{L^2} + 3Mu^2 \tag{16}$$

The term $3Mu^2$ is the relativistic correction to the Newtonian equation.

This term for the trajectories of planets in the solar system, where $M = M_\odot = 1.47664$ km, and for orbital radius $r \approx 4.6 \times 10^7 km$ (Mercury) and $r \approx 1.5 \times 10^8 km$ (Earth) gets very small values $\approx 3 \times 10^{-8}$ and $\approx 10^{-8}$ correspondingly.

[1] Remember the STR relation $p^\mu p_\mu = E^2/c^2 - p^2 = m_0 c^2$, where $p^2 = \vec{p} \cdot \vec{p}$.

Radial motion of massive particles i

For the radial motion ϕ is constant, which implies that $L = 0$ and eqn (14) reduces to

$$\dot{r}^2 = E^2 - e^{\nu}\,. \tag{17}$$

By differentiating we get an equation which reminds the equivalent one of Newtonian gravity i.e.

$$\ddot{r} = \frac{M}{r^2}\,. \tag{18}$$

- If a particle is dropped **from the rest** at $r = R$ ($\dot{r} = 0$) we get that $E^2 = e^{\nu(R)} = 1 - 2M/R$ and (17) will be written

$$\dot{r}^2 = 2M\left(\frac{1}{r} - \frac{1}{R}\right) \tag{19}$$

which is again similar to the Newtonian formula for the gain of kinetic energy due to the loss in gravitational potential energy for a particle (of unit mass) falling from rest at $r = R$.

Radial motion of massive particles ii

- For a particle dropped from the **rest at infinity** $E = 1$ the geodesic equations are simplified (how?):

$$\frac{dt}{d\tau} = e^{-\nu} \quad \text{and} \quad \frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}} \tag{20}$$

The component of the 4-velocity will be:

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} = \left(e^{-\nu}, -\sqrt{\frac{2M}{r}}, 0, 0 \right) \tag{21}$$

Then by integrating the second of (20) and by assuming that at $\tau = \tau_0 = 0$ that $r = r_0$ we get

$$\tau = \frac{2}{3}\sqrt{\frac{r_0^3}{2M}} - \frac{2}{3}\sqrt{\frac{r^3}{2M}} \tag{22}$$

which suggests that for $r = 0$ we get $\tau \to \frac{2}{3}\sqrt{\frac{r_0^3}{2M}}$ i.e. the particle takes **finite** proper time to reach $r = 0$.

Radial motion of massive particles iii

If we want to map the trajectory of the particle in the (r, t) coordinates we need to solve the equation

$$\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} = -e^{\nu}\sqrt{\frac{2M}{r}} \tag{23}$$

The integration leads to the relation

$$\begin{aligned} t &= \frac{2}{3}\left(\sqrt{\frac{r_0^3}{2M}} - \sqrt{\frac{r^3}{2M}}\right) \\ &+ 4M\left(\sqrt{\frac{r_0}{2M}} - \sqrt{\frac{r}{2M}}\right) + 2M\ln\left|\left(\frac{\sqrt{r/2M}+1}{\sqrt{r/2M}-1}\right)\left(\frac{\sqrt{r_0/2M}-1}{\sqrt{r_0/2M}+1}\right)\right| \end{aligned} \tag{24}$$

where we have chosen that for $t = 0$ to set $r = r_0$.

Notice that when $r \to 2M$ then $t \to \infty$. In other words, for an observer at infinity, it takes **infinite time** for a particle to reach $r = 2M$.

QUESTION : Can you find what will be the velocity of the radially falling particle for a stationary observer at coordinate radius $\tilde{r} = 2M$?

Radial motion of massive particles iv

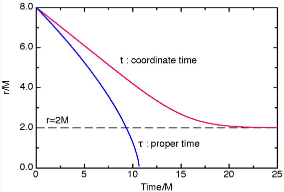


Figure 1: Radial fall from rest towards a Schwarzschild BH as described by a **comoving observer** (proper time τ) and by a **distant observer** (Schwarzschild coordinate time t)

$$\tau = \frac{2}{3}\sqrt{\frac{r_0^3}{2M}} - \frac{2}{3}\sqrt{\frac{r^3}{2M}}$$

$$t = \frac{2}{3}\left(\sqrt{\frac{r_0^3}{2M}} - \sqrt{\frac{r^3}{2M}}\right) + 4M\left(\sqrt{\frac{r_0}{2M}} - \sqrt{\frac{r}{2M}}\right) + 2M \ln \left| \left(\frac{\sqrt{r/2M} + 1}{\sqrt{r/2M} - 1}\right) \left(\frac{\sqrt{r_0/2M} - 1}{\sqrt{r_0/2M} + 1}\right) \right|$$

Circular motion of massive particles i

The motion of massive particles in the equatorial plane is described by eqn (16)

$$\frac{d^2}{d\phi^2}u + u = \frac{M}{L^2} + 3Mu^2 \tag{25}$$

For circular motions $r = 1/u = \text{const}$ and $\dot{r} = \ddot{r} = 0$. Thus we get

$$L^2 = \frac{r^2 M}{r - 3M} \tag{26}$$

if we also put $\dot{r} = 0$ in eqn (14) we get :

$$k = \frac{1 - 2M/r}{\sqrt{1 - 3M/r}} \tag{27}$$

The energy of a particle with rest-mass m_0 in a circular radius r is then given by $E = k \cdot (m_0 c^2)$.

Circular motion of massive particles ii

For the circular orbits to be **bound** we require $E < m_0 c^2$, so the limits on r for an orbit to be bound is given by $k = 1$ which leads to

$$(1 - 2M/r)^2 = 1 - 3M/r \quad \text{true when} \quad r = 4M \text{ or } r = \infty \tag{28}$$

Thus over the range $4M < r < \infty$ **circular orbits are bound**.

Circular motion of massive particles iii

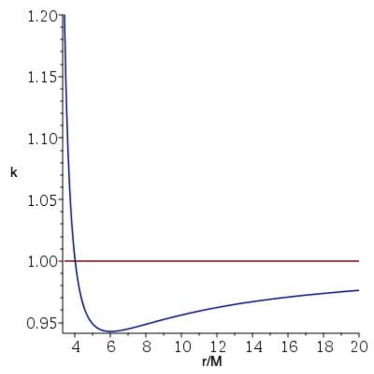


Figure 2: The variation of $k = E/(m_0 c^2)$ as a function of r/M for a circular orbit of a massive particle in the Schwarzchild geometry

Circular motion of massive particles iv

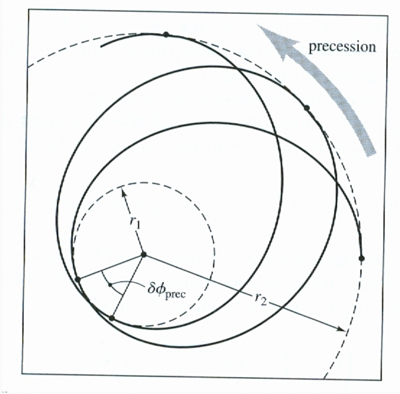


Figure 3: The shape of a bound orbit outside a spherical star or a black-hole

Circular motion of massive particles v

From the integral of motion $r^2\dot{\phi} = L$ and eqn (26) we get

$$\left(\frac{d\phi}{d\tau}\right)^2 = \frac{M}{r^2(r-3M)} \tag{29}$$

NOTICE : This equation cannot be satisfied for circular orbits with $r < 3M$. Such orbits cannot be geodesics and cannot be followed by freely falling particles. Thus according to GR a free massive particle, cannot maintain a circular orbit with $r < 3M$ around a massive body, no matter how large the angular momentum of the particle.
This is very different from Newtonian theory.

We can also calculate an expression for $\Omega = d\phi/dt$

$$\Omega^2 = \left(\frac{d\phi}{dt}\right)^2 = \left(\frac{d\phi}{d\tau}\frac{d\tau}{dt}\right)^2 = \frac{e^{2\nu}}{E^2}\left(\frac{d\phi}{d\tau}\right)^2 = \frac{M}{r^3} \tag{30}$$

which is **equivalent to Kepler's law in Newtonian gravity.**

Stability of massive particle orbits i

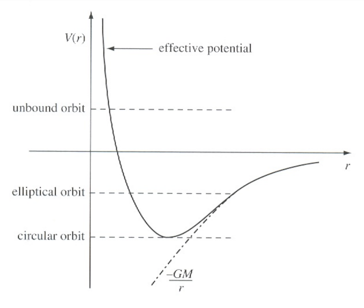
According to the previous discussion the closest bound orbit around a massive body is at $r = 4M$, however we cannot yet determine whether this orbit is stable!

In Newtonian theory the particle motion in a central potential is described by:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + V_{\text{eff}}(r) = E^2 \qquad (31)$$

where

$$V_{\text{eff}}(r) = -\frac{M}{r} + \frac{L^2}{2r^2} \qquad (32)$$



The bound orbits have two turning points while the circular orbit corresponds to the special case where the particle sits in the minimum ($dV_{\text{eff}}/dr = 0$) of the effective potential.

Stability of massive particle orbits ii

In GR the corresponding 'energy' equation (14) is

$$\dot{r}^2 + \frac{e^\nu}{r^2} L^2 + e^\nu = E^2 \tag{33}$$

which leads to an effective potential of the form

$$V_{\rm eff}(r) = \frac{e^\nu}{r^2} L^2 + e^\nu = -\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} \tag{34}$$

Circular orbits occur where $dV_{\rm eff}/dr = 0$ that is:

$$\frac{dV_{\rm eff}}{dr} = \frac{M}{r^2} - \frac{L^2}{r^3} + \frac{3ML^2}{r^4} \tag{35}$$

so the extrema are located at the solutions of the eqn $Mr^2 - Lr + 3ML^2 = 0$ which occur at

$$r = \frac{L}{2M}\left(L \pm \sqrt{L^2 - 12M^2}\right) \tag{36}$$

Note that if $L = \sqrt{12}M = 2\sqrt{3}M$ then there is only one extremum and no turning points in the orbit for lower values of L.

Stability of massive particle orbits iii

Thus the **innermost stable circular orbit** (**ISCO**) has

$$r_{\mathrm{ISCO}} = 6M \tag{37}$$

and

$$L = 2\sqrt{3}M \approx 3.46M \tag{38}$$

and it is unique in satisfying both

$$\frac{dV_{\mathrm{eff}}}{dr} = 0 \text{ and } \frac{d^2 V_{\mathrm{eff}}}{dr^2} = 0\,, \tag{39}$$

the latter is the condition for **marginal stability** of the orbit.

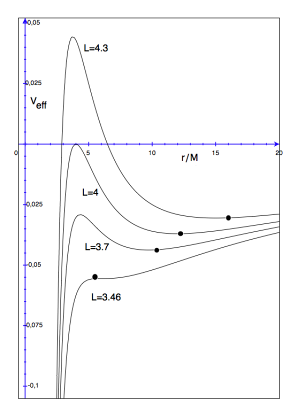


Figure 4: The dots indicate the locations of **stable circular orbits** which occur at the local minimum of the potential. The local maxima in the potential are the locations of the **unstable circular orbits**.

Stability of massive particle orbits iv

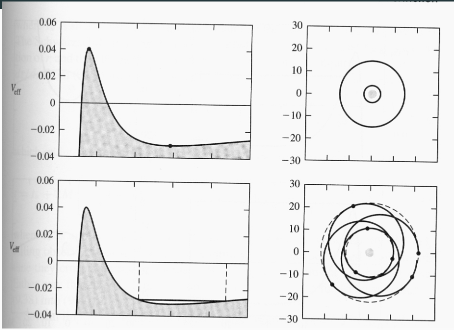


Figure 5: Orbits for $L = 4.3$ and different values of E.

- The **upper** shows **circular orbits**. A stable (outer) and an unstable (inner) one.
- The **lower** shows **bound orbits** , the particle moves between two turning points marked by dotted circles.

Stability of massive particle orbits v

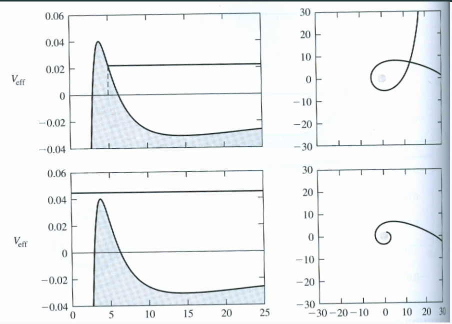


Figure 6: Orbits for $L = 4.3$ and different values of E.

- The **upper** shows **a scattering orbit** the particle comes in from infinity passes around the center of attraction and moves out to infinity again.
- The **lower** shows **plunge orbit** , in which the particle comes in from infinity.

Trajectories of photons

Photons, as any zero rest mass particle, move on null geodesics.

For photons we cannot use the proper time τ as the parameter to characterize the motion and thus we will use some affine parameter σ.

We will study photon orbits on the equatorial plane and the equation of motion will be in this case

$$
\begin{aligned}
e^{\nu}\dot{t} &= E & (40)\\
e^{\nu}\dot{t}^2 - e^{-\nu}\dot{r}^2 - r^2\dot{\phi}^2 &= 0 & (41)\\
r^2\dot{\phi} &= L & (42)
\end{aligned}
$$

The equivalent to eqn (14) for photons is

$$\dot{r}^2 + \frac{e^{\nu}}{r^2}L^2 = E^2 \tag{43}$$

while the equivalent of eqn (16) is

$$\frac{d^2}{d\phi^2}u + u = 3Mu^2 \tag{44}$$

Radial motion of photons

For radial motion $\dot{\phi} = 0$ and we get

$$e^{\nu}\dot{t}^2 - e^{-\nu}\dot{r}^2 = 0$$

from which we obtain

$$\frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right) \qquad (45)$$

integration leads to:

Outgoing Photon

$$t = r + 2M\ln\left|\frac{r}{2M} - 1\right| + \text{const}$$

Incoming Photon

$$t = -r - 2M\ln\left|\frac{r}{2M} - 1\right| + \text{const}$$

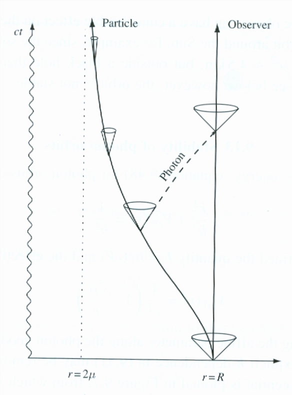


Figure 7: Radially infaling particle emitting a radially outgoing photon.

Circular motion of photons

For circular orbits we have r=constant and thus from eqn (44) we see that the only possible radius for a circular photon orbit is:

$$r = 3M \quad \left(\text{or} \quad r = \frac{3GM}{c^2}\right) \tag{46}$$

There are no such orbits around typical stars because their radius is much larger than $3M$ (in geometrical units).

Only outside black holes photons can follow such orbits.

Stability of photon orbits

We can rewrite the "energy" equation (43) as

$$\frac{\dot{r}^2}{L^2} + \frac{e^\nu}{r^2} = \frac{1}{D^2} \tag{47}$$

where $D = L/E$ and $V_{eff} = e^\nu/r$

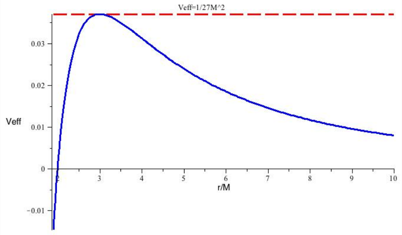


Figure 8: The effective potential for photon orbits.

We can see that V_{eff} has a single maximum at $r = 3M$ where the value of the potential is $1/(27M^2)$. Thus the circular orbit $r = 3M$ is unstable.

There are no stable circular photon orbits in the Schwarzschild geometry.

The slow-rotation limit : Dragging of inertial frames

$$ds^2 = ds^2_{\mathrm{Schw}} + \frac{4Ma}{r}\sin^2\theta dt d\phi \tag{48}$$

where $J = Ma$ is the angular momentum.

The contravariant components of the particles 4-momentum will be

$$p^\phi = g^{\phi\mu}p_\mu = g^{\phi t}p_t + g^{\phi\phi}p_\phi \quad \text{and} \quad p^t = g^{t\mu}p_\mu = g^{tt}p_t + g^{t\phi}p_\phi$$

If we assume a particle with zero angular momentum, i.e. $p_\phi = 0$ along the geodesic then the particle's trajectory is such that

$$\frac{d\phi}{dt} = \frac{p^\phi}{p^t} = \frac{g^{t\phi}}{g^{tt}} = \frac{2Ma}{r^3} = \omega(r)$$

which is the **coordinate angular velocity** of a zero-angular-momentum particle.

A particle dropped "straight in" from infinity ($p_\phi = 0$) is **dragged** just by the influence of gravity so that **acquires an angular velocity** in the same sense as that of the source of the metric.
The effect weakens with the distance and **makes the angular momentum of the source measurable** in practice.

Useful constants

Useful Constants in geometrical units

Speed of light	c =299,792.458 km/s **= 1**
Planck's constant	$\hbar = 1.05 \times 10^{-27}$ erg $\cdot s = 2.612 \times 10^{-66} \mathrm{cm}^2$
Gravitation constant	$G = 6.67 \times 10^{-8}$ cm^3/g $\cdot s^2$ **=1**
Energy	eV=1.602×10^{-12}erg $= 1.16 \times 10^4$K
	=1.7823×10^{-33}g =**1.324**$\times10^{-56}$ **km**
Distance	1 pc=3.09×10^{13}km=3.26 ly
Time	1 yr = 3.156×10^7 sec
Light year	1 ly = 9.46×10^{12}km
Astronomical unit (AU)	$1AU = 1.5 \times 10^8$ km
Earth's mass	$M_\oplus = 5.97 \times 10^{27}$ g
Earth's radius (equator)	$R_\oplus = 6378$ km
Solar Mass	$M_\odot = 1.99 \times 10^{33}$ g **=1.47664 km**
Solar Radius	$R_\odot = 6.96 \times 10^5$ km

The Classical Tests: Perihelion Advance i

For a given value of angular momentum Kepler's equation (16)

$$\frac{d^2}{d\phi^2}u + u = \frac{M}{L^2} + 3Mu^2 \qquad (16)$$

(without the GR term $3Mu^2$) admits a solution given by

$$u = \frac{M}{L^2}\left[1 + e\cos(\phi + \phi_0)\right] \qquad (49)$$

where e and ϕ_0 are integration constants.

- We can set $\phi_0 = 0$ by rotating the coordinate system by ϕ_0.
- The 2nd constant e is the orbital **eccentricity** which is an ellipse if $e < 1$.

Now since the term $3Mu^2$ is small we can use perturbation theory to get a solution of equation (16).

$$\begin{aligned}\frac{d^2}{d\phi^2}u + u &\approx \frac{M}{L^2} + \frac{3M^3}{L^4}\left[1 + e\cos(\phi)\right]^2 \\ &\approx \frac{M}{L^2} + \frac{3M^3}{L^4} + \frac{6eM^3}{L^4}\cos(\phi)\end{aligned}$$

The Classical Tests: Perihelion Advance ii

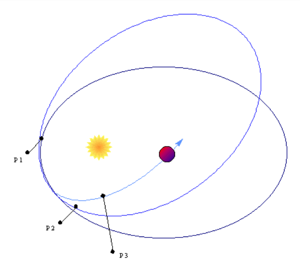
Because, $3M^3/L^4 \ll M/L^2$ and can be omitted and its corrections will be small periodic elongations of the semiaxis of the ellipse.

The term $6eM^3/L^4 \cos(\phi)$ is also small but has an accumulative effect which can be measured.

Thus the solution of the relativistic form of Kepler's equation (50) becomes ($k = 3M^2/L^2$)

$$u = \frac{M}{L^2}\left[1 + e\cos\phi + \frac{3eM^2}{L^2}\phi\sin\phi\right] \approx \frac{M}{L^2}\left\{1 + e\cos[\phi(1-k)]\right\} \tag{50}$$

The Classical Tests: Perihelion Advance iii



The perihelion of the orbit can be found be maximizing u that is when $\cos[\phi(1-k)] = 1$ or better if $\phi_n(1-k) = 2n\pi$. This mean that after each rotation around the Sun the angle of the perihelion will increase by

$$\phi_n = \frac{2n\pi}{1-k} \quad \Rightarrow \phi_{n+1} - \phi_n = \frac{2\pi}{1-k} \approx 2\pi(1+k) = 2\pi + \frac{6\pi M^2}{L^2} \qquad (51)$$

i.e. in the relativistic orbit the perihelion is no longer a fixed point, as it was in the Newtonian elliptic orbit but it moves in the direction of the motion of the planet, advancing by the angle

$$\delta\phi \approx \frac{6\pi M^2}{L^2} \tag{52}$$

By using the well known relation from Newtonian Celestial Mechanics $L^2 = Mr_0(1-e^2)$ connecting the perihelion distance r_0 with L we derive a more convenient form for the perihelion advance

$$\delta\phi \approx \frac{6\pi M}{r_0(1-e^2)} \tag{53}$$

The Classical Tests: Perihelion Advance v

Solar system measurements

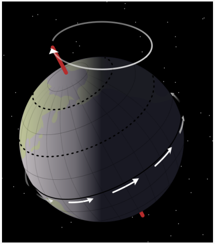
Mercury	$(43.11 \pm 0.45)''$	$43.03''$
Venus	$(8.4 \pm 4.8)''$	$8.6''$
Earth	$(5.0 \pm 1.2)''$	$3.8''$

In binary pulsars separated by 10^6km the perihelion advance is extremely important and it can be up to $\sim 2^o$/year (about 1000 orbital rotations)

The Classical Tests: Perihelion Advance vi

Sources of the precession of perihelion for Mercury

Amount (arcsec/century)	cause
5025.6	Coordinate (precession of the equinoxes)[2]
531.4	Gravitational tugs of the other planets
0.0254	Oblateness of the Sun
42 ± 0.04	General relativity
5600.0	Total
5599.7	Observed



[2]Precession refers to a change in the direction of the axis of a rotating object. There are two types of precession, torque-free and torque-induced.

The Classical Tests : Deflection of Light Rays i

Deflection of light rays due to presence of a gravitational field is a prediction of Einstein dated even before the GR.

This prediction has been verified in 1919 by Eddington during a total solar eclipse.

Photons follow null geodesics, this means that $ds = 0$ and the integrals of motion (E and L) are divergent but not their ratio L/E.

Thus the photon's equation of motion on the equatorial plane of Schwarszchild spacetime will be:

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2 \,. \tag{54}$$

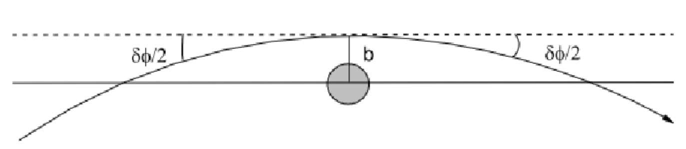
with an approximate solution (we omit at the moment the term $3Mu^2$)

$$u = \frac{1}{b}\cos(\phi + \phi_0) \tag{55}$$

describing a straight line, where b and ϕ_0 are integration constants.

The Classical Tests : Deflection of Light Rays ii

Actually, with an appropriate rotation $\phi_0 = 0$ the length b is the distance of the line from the origin.



Since the term $3Mu^2$ is very small we can substitute u with the Newtonian solution (55) and we need to solve the non-homogeneous ODE

$$\frac{d^2u}{d\phi^2} + u = \frac{3M}{b^2}\cos^2\phi\,. \tag{56}$$

admitting a solution of the form

$$u = \frac{\cos\phi}{b} + \frac{M}{b^2}\left(1 + \sin^2\phi\right) \tag{57}$$

for distant observers ($r \to \infty$) we get $u \to 0$,

The Classical Tests : Deflection of Light Rays iii

Thus for $r \to \infty$ we get a relation between ϕ, M and the parameter b

$$\frac{\cos\phi}{b} + \frac{M}{b^2}\left(1 + \sin^2\phi\right) = 0 \tag{58}$$

and since $r \to \infty$, this means that $\phi \to \pi/2 + \epsilon$ leading to $\cos\phi \to 0 + \epsilon$ and $\sin\phi \to 1 - \epsilon$ we get:

$$\epsilon \approx -\frac{2M}{b} \tag{59}$$

Since, $\phi \to \pi/2 + 2M/b$ for $r \to \infty$ on the one side and $\phi \to 3\pi/2 - 2M/b$ on the other side the total deviation will be the sum of the two i.e.

$$\delta\phi = \frac{4M}{b}\,. \tag{60}$$

For a light ray tracing the surface of the Sun gives a deflection of $\sim 1.75''$.

The Classical Tests : Deflection of Light Rays iv

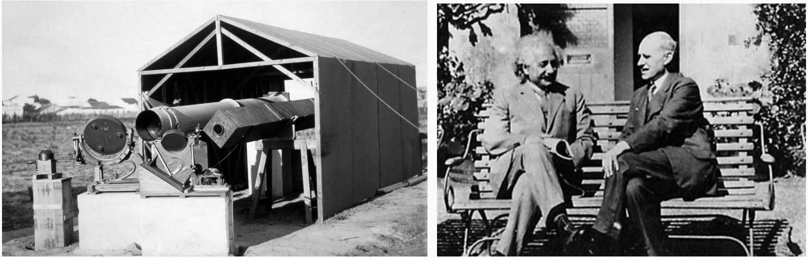


Figure 9: (Left): Instruments used to observe the 1919 total solar eclipse, Sobral, Brazil. Photograph: Science & Society Picture Library/SSPL via Getty Images. (Right): Einstein and Eddington, Photographed at the University of Cambridge Observatory, UK, in 1930.

Instruments used to observe the 1919 total solar eclipse, Sobral, Brazil. Photograph: Science & Society Picture Library/SSPL via Getty Images

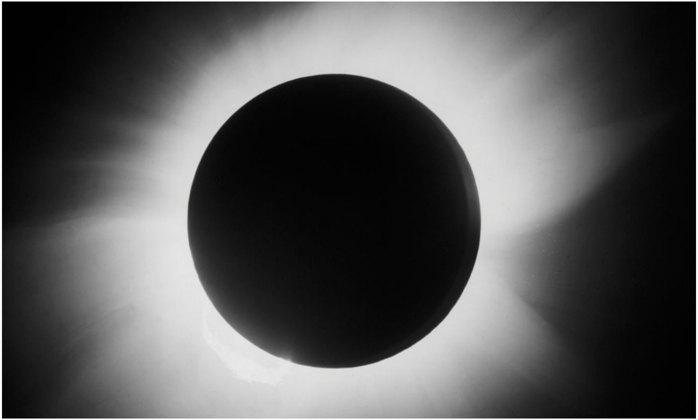


Figure 10: Total solar eclipse, 29 May 1919. Glass positive photograph of the corona, taken at Sobral in Brazil, with a telescope of 4in in aperture and 19ft focal length. Photograph: Science & Society Picture Library/SSPL via Getty Images.

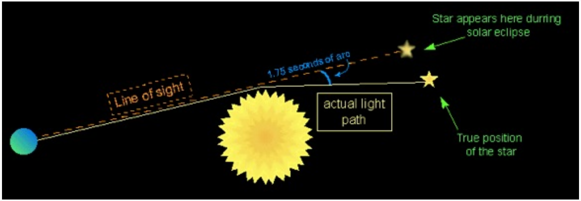
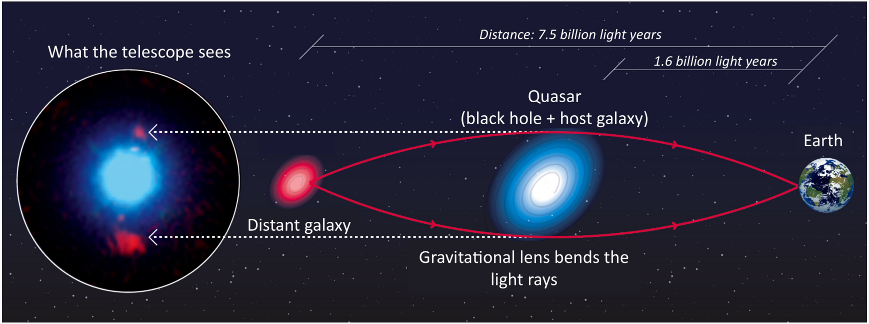


Figure 11: A schematic presendation of the light bending during solar eclipse.

EINSTEIN: "In that case, I would have to feel sorry for God, because the theory is correct."

Gravitational Lensing i

The deflection of light rays is a quite common phenomenon in Astronomy and has many applications. We typically observe "crosses" or "rings"



Gravitational Lensing ii

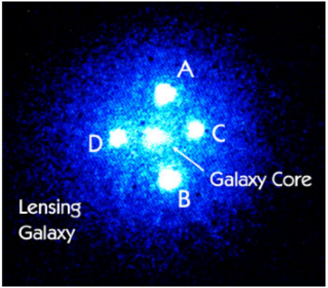


Figure 12: **Einstein Cross** (G2237+030) is the most characteristic case of gravitational lens where a galaxy at a distance 5×10^8lys focuses the light from a quasar who is behind it in a distance of 8×10^9 lys. The focusing creates 4 symmetric images of the same quasar. The system has been discovered by John Huchra.

Gravitational Lensing iii

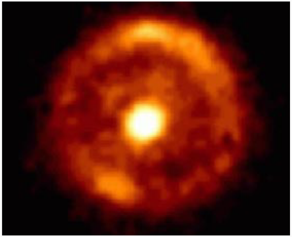
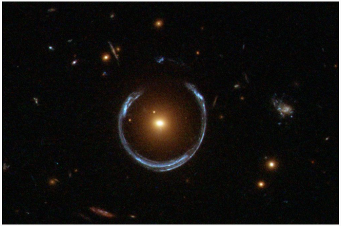
 

Figure 13: **Einstein rings** are observed when the source, the focusing body and Earth are on the same line of sight. This ring has been discovered by Hubble space telescope.

Gravitational Lensing iv

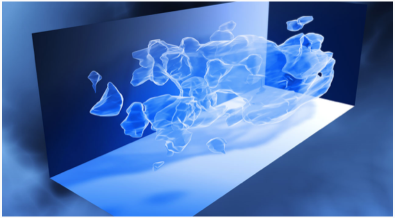


Figure 14: Mapping the dark matter in the Universe

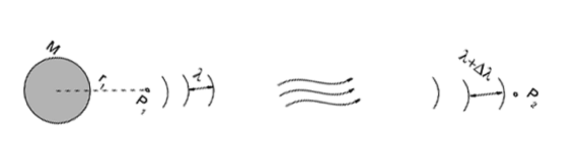


Figure 15: Let's assume 3 static observers on a Schwarszchild spacetime, one very close to the source of the field the other in a medium distance from the source and the third at infinity.

The clocks of the 3 observers ticking with different rates. The clocks of the two closer to the source are ticking slower than the clock of the observer at infinity who measures the so called ' coordinate time" i.e.
$d\tau_1 = \left(1 - \frac{2M}{r_1}\right)^{1/2} dt$ and $d\tau_2 = \left(1 - \frac{2M}{r_2}\right)^{1/2} dt$ this means that

$$\frac{d\tau_2}{d\tau_1} = \frac{\left(1 - \frac{2M}{r_2}\right)^{1/2}}{\left(1 - \frac{2M}{r_1}\right)^{1/2}} \approx 1 + \frac{M}{r_1} - \frac{M}{r_2} \,. \tag{61}$$

The Classical Tests : Gravitational Redshift ii

If the 1st observer sends light signals on a specific wavelegth λ_1 from $c = \lambda/\tau$ we get a relation between the wavelength of the emitted and received signals

$$\frac{\lambda_2}{\lambda_1} \approx 1 + \frac{M}{r_1} - \frac{M}{r_2} \quad \Rightarrow \quad \frac{\lambda_2 - \lambda_1}{\lambda_1} = \frac{\Delta\lambda}{\lambda_1} = \frac{M}{r_1} - \frac{M}{r_2} . \tag{62}$$

A similar relation can be found for the frequency of the emitted signal:

$$\frac{\nu_1}{\nu_2} = \frac{\left(1 - \frac{2M}{r_2}\right)^{1/2}}{\left(1 - \frac{2M}{r_1}\right)^{1/2}} . \tag{63}$$

While the photon **redshift** z is defined by

$$1 + z = \frac{\nu_1}{\nu_2} \tag{64}$$

QUESTION : What will be the redshift for signals emitted from the surface of the Sun, a neutron star and a black hole?

The Classical Tests : Gravitational Redshift iii

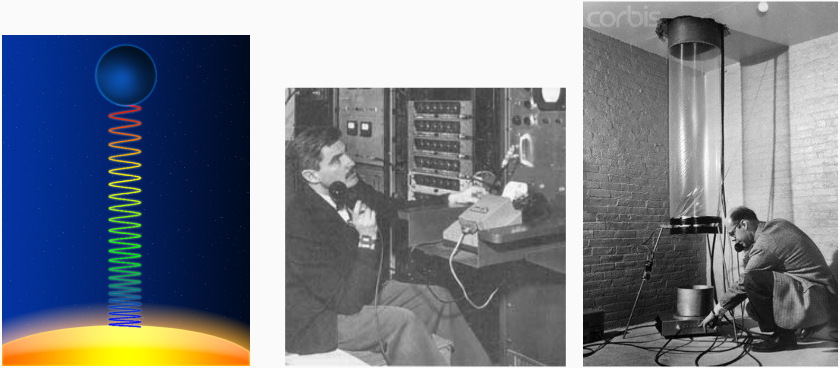
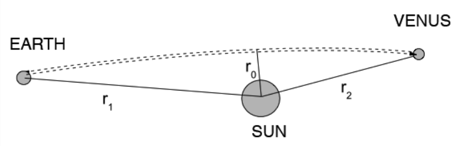


Figure 16: Gravitational Redshift : Verification by Pound-Rebka 1959-65

A more recent test (late 60s) where the delay of the radar signals caused by the gravitational field of Sun was measured. This experiment suggested and performed by I.I. Shapiro and his collaborators.



The line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ for the light rays i.e. for $ds = 0$ on the equatorial plane has the form

$$0 = \left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{dr}{dt}\right)^2 - r^2 \left(\frac{d\phi}{dt}\right)^2 \tag{65}$$

For the study of the radial motion we should substitute the term $d\phi/dt$ from the integrals of motion (10) and (11) i.e. we can create the quantity:

$$D = \frac{L}{E} = r^2 \left(1 - \frac{2M}{r}\right)^{-1} \frac{d\phi}{dt} \tag{66}$$

Then eqn (65) becomes

$$\left(1-\frac{2M}{r}\right)-\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^{2}-\frac{D^{2}}{r^{2}}\left(1-\frac{2M}{r}\right)^{2}=0\,. \tag{67}$$

At the point of the closest approach to the Sun, r_0, there should be $dr/dt=0$ and thus we get the value of $D^2=\frac{r_0^2}{1-2M/r_0}$. Leading to an equation for the radial motion:

$$\frac{dr}{dt}=\left(1-\frac{2M}{r}\right)\left[1-\left(\frac{r_0}{r}\right)^{2}\frac{1-2M/r}{1-2M/r_0}\right]^{1/2} \tag{68}$$

and by integration we get:

$$\begin{aligned} t_1 &= \int_{r_0}^{r_1}\frac{dr}{\left(1-\frac{2M}{r}\right)\sqrt{1-\left(\frac{r_0}{r}\right)^{2}\frac{1-2M/r}{1-2M/r_0}}} \\ &= \sqrt{r_1^2-r_0^2}+2M\ln\left[\frac{r_1\sqrt{r_1^2-r_0^2}}{r_0}\right]+M\sqrt{\frac{r_1-r_0}{r_1+r_0}} \\ &\approx r_1+2M\ln\left(\frac{2r_1}{r_0}\right)+M \end{aligned}$$

- For flat space we have $t_1 = r_1$ (remember $c = 1$)i.e. the term $\tilde{t}_1 = 2M \ln(2r_1/r_0) + M$ is the relativistic correction for the first part of the orbit in same way we get a similar contribution as the signal returns to Earth.

- Thus the total "extra time" is:

$$\begin{aligned} \Delta T &= 2\left(\tilde{t}_1 + \tilde{t}_2\right) \\ &= 4M\left[1 + \ln\left(\frac{4r_1 r_2}{r_0^2}\right)\right] \end{aligned} \tag{69}$$

Figure 17: I.I. Shapiro

Figure 18: Cassini Spacecraft on its way to Saturn in 2003; Deviations from GR 0.002% – the most stringent test of the theory so far.

The Classical Tests : Radar Delay iv

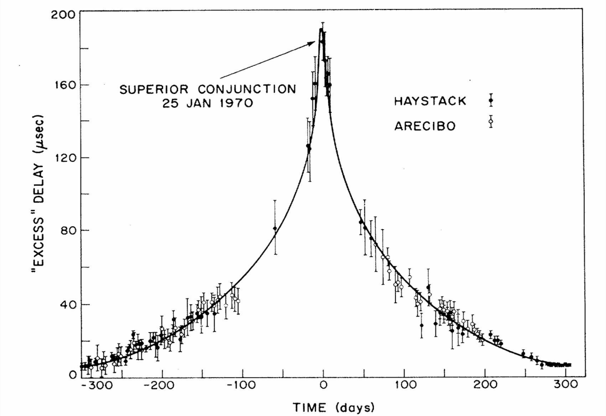
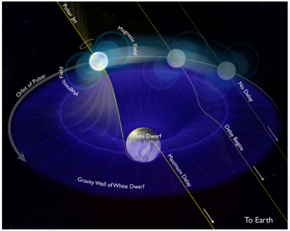


Figure 19: Comparison of the experimental results with the prediction of the theory. The results are from I.I. Shapiro's experiment (1970) using Venus as reflector.

EXAMPLE: weighting a pulsar



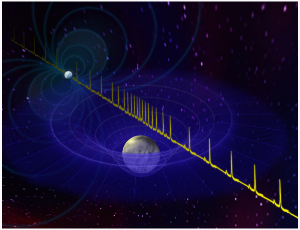


Figure 20: As a pulsar passes behind its heavy companion star, its pulses are delayed by the mass of the companion. (Credit B. Saxton/NRAO/AUI)

- If a pulsar is in orbit around a companion WD, its pulses of light will follow the space curve caused by that star. When the companion WD star is in front of the pulsar, the pulses take a little longer to reach us than when the WD is clear of the pulsar. The amount of delay tells you the amount of mass of the star causing the delay. [3]

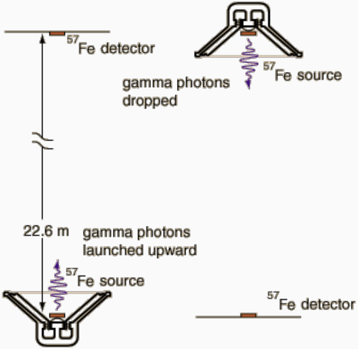
- Astronomers using the NSF's Green Bank Telescope (GBT) have discovered the most massive neutron star ($2M_\odot$) yet found, a discovery with strong and wide-ranging impacts across several fields of physics and astrophysics.

[3] NRAO:http://www.nrao.edu/index.php/learn/science/weighing-pulsars

Appendix

The Pound-Rebka-Snider experiment

- The first successful, high-precision redshift measurement was the series of Pound-Rebka-Snider experiments of 1960-65 that measured the frequency shift of gamma-ray photons (14keV) from ^{57}Fe as they ascended or descended the JPL tower at Harvard University.
- The high accuracy achieved - one percent - was obtained by making use of the Mössbauer effect to produce a narrow resonance line whose shift could be accurately determined.



- Other experiments since 1960 measured the shift of spectral lines in the Sun's gravitational field and the change in rate of atomic clocks transported aloft on aircraft, rockets and satellites.