Solutions of Einstein's Equations & Black Holes

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January 22, 2020

Embedding diagrams i

In order to get some feeling for the global geometry of the Schwarzchild black hole we can try to represent aspects of it by embeddings in 3-space.

Let us look at the space of constant time and also suppress one of the angular coordinates, say $\theta = \pi/2$. The metric becomes:

$$dl^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2 \tag{1}$$

The metric of the Euclidean embedding space is:

$$dl^2 = dz^2 + dr^2 + r^2 d\phi^2 \tag{2}$$

which on $z = z(r)$ becomes

$$dl^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\phi^2 \, . \tag{3}$$

Embedding diagrams ii

These are the same if

$$1 + \left(\frac{dz}{dr}\right)^2 = \left(1 - \frac{2M}{r}\right)^{-1} \quad \rightarrow \quad z = 2\sqrt{2M(r-2M)} \qquad (4)$$

- ⋆ We cannot follow the geometry for $r < 2M$.

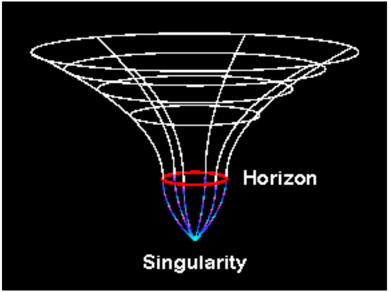


Figure 1: Embedding Diagram for Schwarzschild spacetime

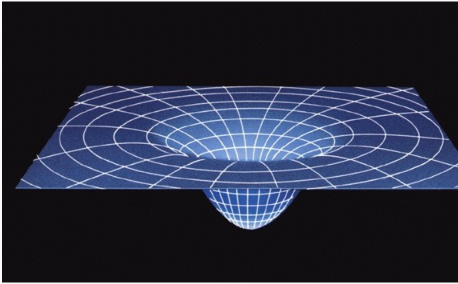


Figure 2: An embedding diagram depicting the curvature of spacetime around a static spherical star

Embedding diagrams iii

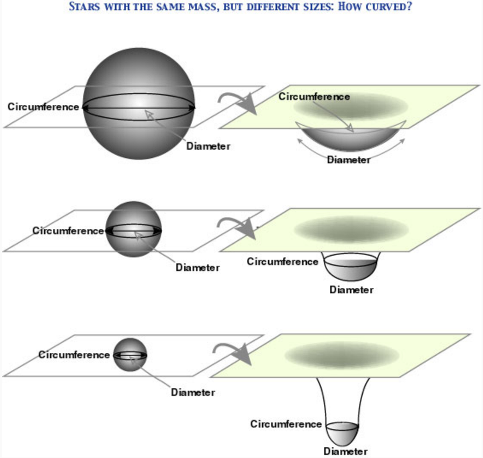


Figure 3: The figure adapted from K.S. Thorne "Black Holes and Time Warps" (1994), Fig. 3.4 on page 132.

The isotropic coordinates i

One may try to write the well known form of Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad (5)$$

as close as possible into a form where the spacelike slices are as close as possible to Euclidean i.e.

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right) dt^2 + B(r)\left(dx^2 + dy^2 + dz^2\right) \\ &\equiv -\left(1 - \frac{2M}{r}\right) dt^2 + B(r)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \qquad (6) \end{aligned}$$

we can assume a new set of coordinates $(t, \tilde{r}, \theta, \phi)$ where $\tilde{r} = \tilde{r}(r)$ in which the above metric will be written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \lambda(\tilde{r})^2 \left(d\tilde{r}^2 + \tilde{r}^2 d\theta^2 + \tilde{r}^2 \sin^2\theta d\phi^2\right) \qquad (7)$$

where $\lambda = \lambda(\tilde{r})$ is a conformal factor.

The isotropic coordinates ii

Comparing (5) with (7) we get

$$\left(1-\frac{2M}{r}\right)^{-1} dr^2 = \lambda(\tilde{r})^2 d\tilde{r}^2 \quad \text{and} \quad r^2 = \lambda(\tilde{r})^2 \tilde{r}^2 \tag{8}$$

From the last two equations we get:

$$\frac{dr}{\sqrt{r^2 - 2Mr}} = \pm \frac{d\tilde{r}}{\tilde{r}} \tag{9}$$

since we want $\tilde{r} \to \infty$ when $r \to \infty$ we choose the sign $(+)$ and by integration we get

$$r = \tilde{r}\left(1+\frac{M}{2\tilde{r}}\right)^2 \tag{10}$$

Then $r^2 = \lambda^2 \tilde{r}^2$ implies:

$$\lambda^2 = \left(1+\frac{M}{2\tilde{r}}\right)^4 \tag{11}$$

The isotropic coordinates iii

$$ds^2 = -\left(\frac{1 - M/2\tilde{r}}{1 + M/2\tilde{r}}\right)^2 dt^2 + \left(1 + \frac{M}{2\tilde{r}}\right)^4 \left[d\tilde{r}^2 + \tilde{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \quad (12)$$

⋆ Isotropic coordinates used by experimental gravitation because one can expand on a more natural way the gravitational quantities and to compare directly with Newtonian flat space gravity as well as with alternative theories of gravity.

- Metric (12) in the PPN formalism it becomes:

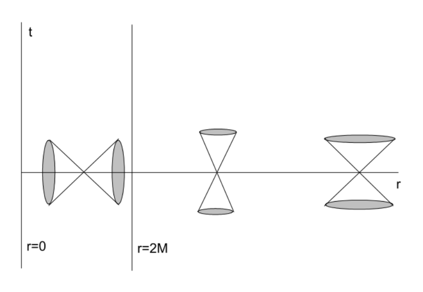
$$ds^2 \approx -\left[1 - \frac{2M}{\tilde{r}} + 2\beta\left(\frac{M}{\tilde{r}}\right)^2\right] dt^2 + \left[1 + 2\gamma\frac{M}{\tilde{r}}\right]\left[d\tilde{r}^2 + \tilde{r}^2 d\Omega^2\right] \quad (13)$$

Schwarzschild Solution: Black Holes I

The light cone in a Schwarzschild spacetime

$$ds^2 \equiv 0 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \quad \Rightarrow \quad \frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right).$$

$r = 2M$ is a null surface and is called horizon or infinite redshift surface.



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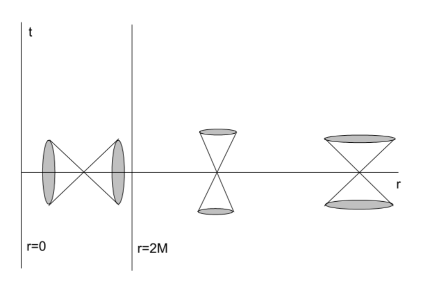


Eddington-Finkelnstein coordinates i

The Schwarzschild solution in a new set of coordinates (t', r', θ', ϕ')

$$\begin{aligned} t &= t' \pm \ln\left(\frac{r'}{2M} - 1\right), \quad r > 2M \quad \text{(outgoing)} \\ t &= t' \pm \ln\left(1 - \frac{r'}{2M}\right), \quad r < 2M \quad \text{(ingoing)} \\ r &= r', \quad \theta = \theta', \quad \phi = \phi' \end{aligned}$$

can be written as

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt'^2 - \left(1 + \frac{2M}{r}\right) dr^2 \pm \frac{4M}{r} dr dt' - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

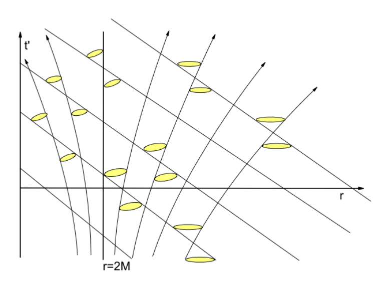
While for **null surfaces**

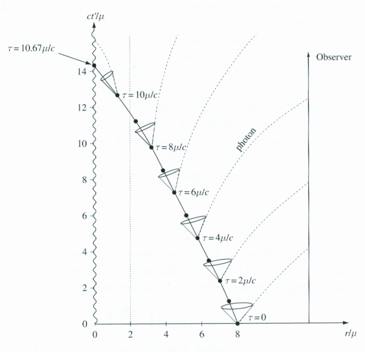
$$ds^2 \equiv 0 = \left(1 - \frac{2M}{r}\right) dt'^2 - \left(1 + \frac{2M}{r}\right) dr^2 \pm \frac{4M}{r} dr dt'$$

Eddington-Finkelnstein coordinates ii

$$\left(dt' + dr\right)\left[\left(1 - \frac{2M}{r}\right)dt' - \left(1 + \frac{2M}{r}\right)dr\right] = 0$$

$$\frac{dr}{dt'} = -1 \quad \text{and} \quad \frac{dr}{dt'} = \frac{r - 2M}{r + 2M}$$





Eddington-Finkelnstein coordinates iii

We can define the so called **tortoise coordinate** as:

$$r_* = r + 2M \ln \left| \frac{r}{2M} - 1 \right| \tag{14}$$

satisfying

$$\frac{dr_*}{dr} = \left(1 - \frac{2M}{r}\right)^{-1} \tag{15}$$

notice that $r^* \to -\infty$ as $r \to 2M$.

By using the definitions $v = t + r_*$ (ingoing) and $u = t - r_*$ (outgoing) we can get the

Ingoing form of the metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \tag{16}$$

Outgoing form of the metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2 d\Omega^2 \tag{17}$$

Eddington-Finkelnstein coordinates iv

A manifold is said to be geodesically *complete* if all geodesics are of infinite length. This means that in a *complete specetime manifold*, every particle has been in the spacetime forever and remains in it forever.

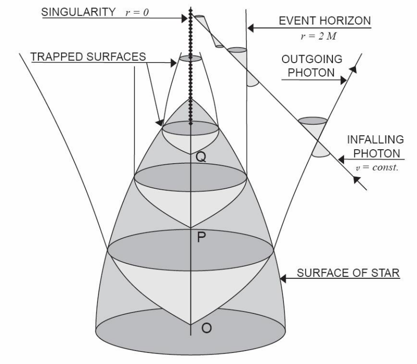


Figure 4: The spacetime of a collapsing star.

Kruskal - Szekeres Coordinates : Maximal Extension

$$
\begin{aligned}
v &= \left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right) \\
u &= \left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right)
\end{aligned}
$$

—

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$$\frac{t}{4M} = \tanh^{-1}\left(\frac{v}{u}\right)$$

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—

$$ds^2 = \frac{32M^3}{r} e^{-r/2M}\left(dv^2 - du^2\right) - r^2 d\Omega^2$$

$$du^2 - dv^2 = 0 \quad \Rightarrow \quad \frac{du}{dv} = \pm 1$$

In a maximal spacetime, particles can appear or disappear, but only at singularities.

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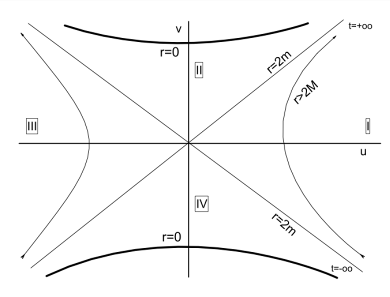
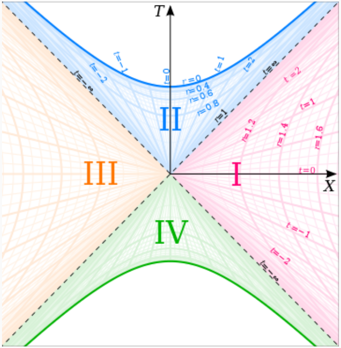


Kruskal - Szekeres Coordinates : Maximal Extension II

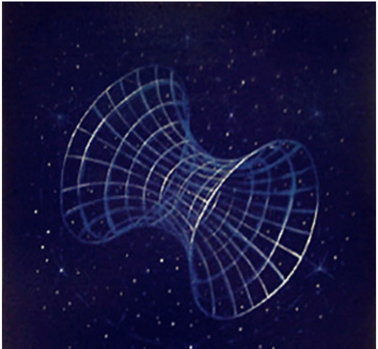


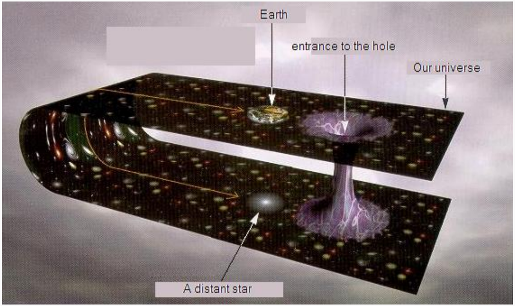
Kruskal-Szekeres diagram, illustrated for $2GM = 1$.

- The quadrants are the black hole interior (II), the white hole interior (IV) and the two exterior regions (I and III).
- The dotted 45^o lines, which separate these four regions, are the event horizons.
- The darker hyperbolas which bound the top and bottom of the diagram are the physical singularities.
- The paler hyperbolas represent contours of the Schwarzschild r coordinate, and the straight lines through the origin represent contours of the Schwarzschild t coordinate. [From Wikipedia]

Wormholes

A hypothetical "tunnel" (Einstein-Rosen bridge) connecting two different points in spacetime in such a way that a trip through the wormhole could take much less time than a journey between the same starting and ending points in normal space. The ends of a wormhole could, in theory, be intra-universe (i.e. both exist in the same universe) or inter-universe (exist in different universes, and thus serve as a connecting passage between the two).





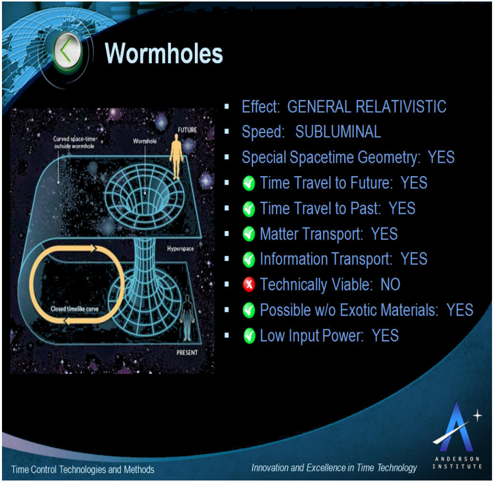


Figure 5: The Einstein-Rosen wormholes would be useless for travel because they collapse quickly. Alternatively, a wormhole containing "exotic" matter could stay open and unchanging for longer periods of time.

Exotic matter, contains negative energy density and a large negative pressure. Such matter has only been seen in the behaviour of certain vacuum states as part of quantum field theory.

If a wormhole contained sufficient exotic matter, it could theoretically be used as a method of sending information or travelers through space.

It has been conjectured that if one mouth of a wormhole is moved in a specific manner, it could allow for time travel.

Tübingen Wormholes !!!

Figure 6: Image of a simulated traversable wormhole that connects the square in front of the physical institutes of Tübingen University with the sand dunes near Boulogne sur Mer in the north of France. The image is calculated with 4D raytracing in a Morris-Thorne wormhole metric, but the gravitational effects on the wavelength of light have not been simulated

`http://www.spacetimetravel.org/wurmlochflug/wurmlochflug.html`

Reissner-Nordström Solution (1916-18) i

- In contrast to the Schwarzschild (and Kerr) solutions, which are vacuum solutions of the Einstein's equations, **the Reissner-Nordström solution is not a vacuum solution**.
- The presence of the electric field gives rise to a nonzero energy -momentum tensor throughout space.
- Then we have to solve the static Einstein-Maxwell equations with a potential

$$A_\mu = \left(\frac{Q}{r}, 0, 0, 0\right) \tag{18}$$

where Q is the total charge measured by a distant observer.

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= \kappa T_{\mu\nu} & (19) \\ F^{\mu\nu}{}_{;\nu} &= 0 & (20) \\ F^{\mu\nu;\alpha} + F^{\nu\alpha;\mu} + F^{\alpha\mu;\nu} &= 0 & (21) \end{aligned}$$

where $T_{\mu\nu}$ is the energy momentum tensor of the electromagnetic field defined in Chapter 2.

Reissner-Nordström Solution (1916-18) ii

The solution contains two constants of integration, M and Q which can be interpreted as the **mass** and the **electric charge**[1]

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad (22)$$

The horizon is for $g_{00} = 0$

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \qquad (23)$$

two horizons r_- and r_+.

[1]Where

$$M = \frac{G}{c^2} M \quad \text{and} \quad Q^2 = \frac{1}{4\pi\epsilon_o}\frac{G}{c^4} Q^2$$

with $1/4\pi\epsilon_o$ the Coulomb's force constant

The slow-rotation limit : Solution

- Stars are not spherically symmetric, but instead they are approximately axisymmetric.

- For slowly rotating bodies with mass M and angular momentum J the Einstein equations outside the body are solved approximately and the metric will have the form:

$$\begin{aligned} ds^2 &\approx \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \frac{4GJ}{rc^2} \sin^2\theta dt\, d\phi \\ &- \left(1 + \frac{2GM}{rc^2}\right) dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \end{aligned} \quad (24)$$

to order $O(1/r)$. Note, that for $J \to 0$ we get the Scwharzschild solution.

- This solution is approximately correct far from a rotating black-hole, but as we go closer to the horizon the deviations are increasing and the above solution fails to describe the spacetime, since even terms of order $O(1/r^2)$ cannot be neglected.

- Thus this solution can be a good approximation for the exterior specetime of a rotating star but not of a spinning black-hole.

The slow-rotation limit : Dragging of inertial frames

$$ds^2 = ds^2_{\rm Schw} + \frac{4J}{r}\sin^2\theta dt d\phi \quad (25)$$

where $J = Mac$ is the angular momentum.

The contravariant components of the particles 4-momentum will be

$$p^\phi = g^{\phi\mu}p_\mu = g^{\phi t}p_t + g^{\phi\phi}p_\phi \quad \text{and} \quad p^t = g^{t\mu}p_\mu = g^{tt}p_t + g^{t\phi}p_\phi$$

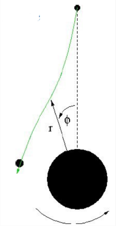
If we assume a particle with zero angular momentum, i.e. $p_\phi = 0$ along the geodesic then the particle's trajectory is such that

$$\frac{d\phi}{dt} = \frac{p^\phi}{p^t} = \frac{g^{t\phi}}{g^{tt}} \approx \frac{2J}{r^3} = \omega(r)$$

$\omega(r)$ is the **coordinate angular velocity** of a zero-angular-momentum particle.

A particle dropped "straight in" from infinity ($p_\phi = 0$) is **dragged** just by the influence of gravity so that **acquires an angular velocity** in the same sense as that of the source of the metric.

The effect weakens with the distance and **makes the angular momentum of the source measurable** in practice.



Kerr Solution (1963) i

Kerr solution is astrophysically the most important solution of Einstein's equations.

$$ds^2 = -\frac{\Delta}{\rho^2}\left[dt - a\sin^2\theta d\phi\right]^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2 + a^2)d\phi - adt\right]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

where [2]

$$\Delta = r^2 - 2Mr + a^2 \quad \text{and} \quad \rho^2 = r^2 + a^2\cos^2\theta \tag{26}$$

here $a = J/M = GJ/Mc^3$ is the angular momentum per unit mass.
For our Sun $J = 1.6 \times 10^{48} g\, cm^2/s$ which corresponds to $a = 0.185$.

- Obviously for $a = 0$ Kerr metric reduces to Schwarszchild.
- The Kerr solution is stationary but not static

Kerr Solution (1963) ii

- The coordinates are known as Boyer-Lindquist coordinates and they are related to Cartesian coordinates

$$
\begin{aligned}
x &= (r^2 + a^2)^{1/2} \sin\theta \cos\phi \\
y &= (r^2 + a^2)^{1/2} \sin\theta \sin\phi \\
z &= r \cos\theta
\end{aligned}
$$

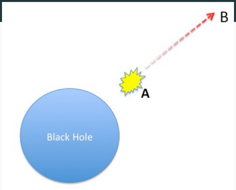
- Another writing of the solution is the following:

$$
\begin{aligned}
ds^2 &= \frac{\left(\Delta - a^2 \sin^2\theta\right)}{\rho^2} dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 - \frac{A \sin^2\theta}{\rho^2} d\phi^2 \\
&+ \frac{4Ma}{\rho^2} r \sin^2\theta \, d\phi \, dt
\end{aligned} \tag{27}
$$

where $A = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta$.

[2] If $\Delta = r^2 - 2Mr + a^2 + Q^2$ the we get the so called Kerr-Newman solution which describes a stationary, axially symmetric and charged spacetime.

Kerr Solution: Infinite redshift surface



We know that :

$$1 + z = \frac{\nu_A}{\nu_B} = \frac{g_{00}(B)}{g_{00}(A)} .$$

When

$$g_{00} = 1 - \frac{2Mr}{\rho^2} \to 0 \quad \Rightarrow r^2 + a^2 \cos^2\theta - 2Mr = 0$$

which leads to

$$R_{\pm} = M \pm \sqrt{M^2 - a^2 \cos^2\theta}$$

- This means that there are **two distinct infinite-redshift surfaces**. As in the Schwarzschild case these surfaces do not correspond to any physical singularity.
- The vanishing of the g_{00} merely tells us that a particle cannot be at rest (with $dr = d\theta = d\phi = 0$) at theses surfaces; only a light signal emitted in the radial direction can be at rest.
- **NOTE:** In the Kerr geometry, the infinite-redshift surface does not coincide with event horizons.

Kerr Solution: The Light Cone

The shape of the light cone on the equatorial plane $\theta = \pi/2$ can be derived from:

$$ds^2 \equiv 0 = g_{00}dt^2 + g_{rr}dr^2 + g_{\phi\phi}d\phi^2 + 2g_{0\phi}d\phi dt \quad (28)$$

For fixed r (i.e., $dr = 0$) we get

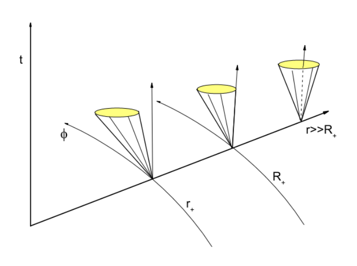
$$0 = g_{00} + g_{\phi\phi}\left(\frac{d\phi}{dt}\right)^2 + 2g_{0\phi}\left(\frac{d\phi}{dt}\right)$$

then at $g_{00} = 0$ (infinite redshift)

$$\frac{d\phi}{dt} = 0 \quad (29)$$

and

$$\frac{d\phi}{dt} = -\frac{g_{\phi\phi}}{2g_{0\phi}} \quad (30)$$



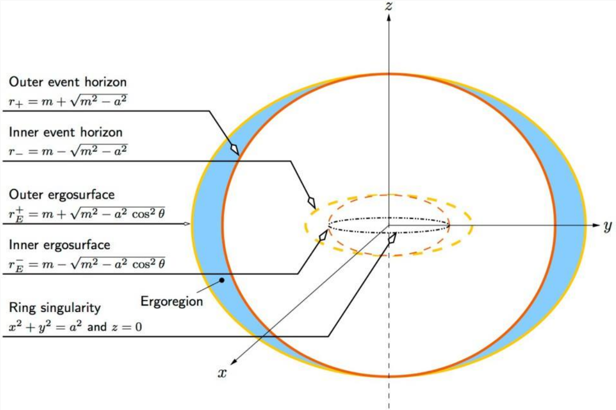
Kerr Black Hole: Horizon

The light cone in the **radial direction**:

$$0 = g_{00} dt^2 + g_{rr} dr^2 \quad \Rightarrow \left(\frac{dr}{dt}\right)^2 = -\frac{g_{00}}{g_{rr}} = \frac{\Delta(\rho^2 - 2Mr)}{\rho^4} \qquad (31)$$

thus the horizon(s) will be the surfaces:

$$\Delta = r^2 - 2Mr + a^2 = 0 \quad \Rightarrow \quad r_\pm = M \pm \sqrt{M^2 - a^2} \qquad (32)$$



We will consider test particle motion in the equatorial plane. If we set $\theta = \pi/2$ in the equation (26) we can get the Lagrangian

$$2\mathcal{L} = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 - \frac{4aM}{r}\dot{t}\dot{\phi} + \frac{r^2}{\Delta}\dot{r}^2 + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\dot{\phi}^2 \tag{33}$$

Corresponding to the ignorable coordinates t and ϕ we obtain two first integrals (where $\dot{t} = dt/d\lambda$):

$$p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = \text{constant} = -E \tag{34}$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \text{constant} = L \tag{35}$$

From (33) we can obtain (how?):

$$\dot{t} = \frac{(r^3 + a^2 r + 2Ma^2)E - 2aML}{r\Delta} \tag{36}$$

$$\dot{\phi} = \frac{2aME + (r - 2M)L}{r\Delta} \tag{37}$$

A 3rd integral of motion can be derived by $g_{\mu\nu}p^\mu p^\nu = -m^2$ which (by using (36) & (37) is:

$$r^3\dot{r}^2 = \mathcal{R}(E, L, r) \tag{38}$$

where

$$\mathcal{R} = E^2(r^3 + a^2 r + 2Ma^2) - 4aMEL - (r - 2M)L^2 - m^2 r\Delta \tag{39}$$

we can regard $\mathcal{R}$ as an "effective potential" for radial motion in the equatorial plane.

This equation can be written as:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{\mathcal{R}}{r^3} \equiv E^2 - V_{\text{eff}} \tag{40}$$

where

$$V_{\text{eff}} = \left(1 - \frac{2M}{r}\right) + \frac{\left[L^2 - a^2(E^2 - 1)\right]}{r^2} - \frac{2M(L - aE)^2}{r^3} \tag{41}$$

Circular orbits occur where $\dot{r} = 0$. This requires:

$$\mathcal{R} = 0, \quad \frac{\partial \mathcal{R}}{\partial r} = 0 \tag{42}$$

These equations can be solved for E and L to give

$$E/m = \tilde{E} = \frac{r^2 - 2Mr \pm a\sqrt{Mr}}{r\left(r^2 - 3Mr \pm 2a\sqrt{Mr}\right)^{1/2}} \tag{43}$$

$$L/m = \tilde{L} = \pm\frac{\sqrt{Mr}\left(r^2 \mp 2a\sqrt{Mr} + a^2\right)}{r\left(r^2 - 3Mr \pm 2a\sqrt{Mr}\right)^{1/2}}\,. \tag{44}$$

Here the upper sign corresponds to **corotating** orbits and the lower sign to **counterrotating** orbits.

• **Kepler's 3rd law** : for circular equatorial orbits, by using (36) & (37) we get (how?):

$$\Omega = \frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}} = \pm\frac{M^{1/2}}{r^{3/2} \pm aM^{1/2}} \tag{45}$$

Kerr Black Hole : Circular Orbits ii

Circular orbits exist from $r = \infty$ all the way down to the limiting circular photon orbit, when the denominator of Eq. (43) vanishes i.e.

$$r^2 - 3Mr \pm 2a\sqrt{Mr} = 0$$

Solving for the resulting equation we find for the photon orbit

$$r_{\rm ph} = 2M\left\{1 + \cos\left[\frac{2}{3}\cos^{-1}\left(\mp\frac{a}{M}\right)\right]\right\}. \qquad (46)$$

NOTE :

For $a = 0$, $r_{\rm ph} = 3M$

For $a = M$, $r_{\rm ph} = M$ (direct) or $r_{\rm ph} = 4M$ (retrograde).

$\star$ For $r > r_{\rm ph}$, not all circular orbits are bound.

An **unbound circular orbit** has $E/m > 1$ and for an infinitesimal outward perturbation, a particle in such an orbit will escape to infinity on an asymptotically hyperbolic orbit.

Kerr Black Hole : Circular Orbits iii

Bound orbits exist for $r > r_{\rm mb}$, where $r_{\rm mb}$ is the radius of the marginally bound circular orbit with $E/m = 1$:

$$r_{\rm mb} = 2M \mp a + 2M^{1/2}(M \mp a)^{1/2} \tag{47}$$

$r_{\rm mb}$ is the minimum periastron of all parabolic $E/m = 1$ orbits.

⋆ In astrophysical problems, particle infall from infinity is very nearby parabolic, since $u_\infty \ll c$.

⋆ Any parabolic orbit that penetrates to $r < r_{\rm mb}$ must plunge directly into the BH.

⋆ For $a = 0$, $r_{\rm mb} = 4M$, while
for $a = M$, $r_{\rm mb} = M$ (direct) or $r_{\rm mb} = 5.83M$ (retrograde)

Kerr Black Hole : Stable Orbits

Even the bound orbits are not all stable.

Stability requires that

$$\frac{\partial^2 \mathcal{R}}{\partial r^2} \leq 0 \tag{48}$$

From eqn (43) we get

$$1 - \left(\frac{E}{m}\right)^2 \geq \frac{2}{3}\frac{M}{r} \tag{49}$$

The solution for $r_{\rm ms}$, the "radius of marginally stable circular orbit" is

$$r_{\rm ms} = M\left[3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2}\right] \tag{50}$$

$$Z_1 = 1 + \left(1 - \frac{a^2}{M^2}\right)^{1/3}\left[\left(1 + \frac{a}{M}\right)^{1/3} + \left(1 - \frac{a}{M}\right)^{1/3}\right]$$

$$Z_2 = \left(3\frac{a^2}{M^2} + Z_1^2\right)^{1/2}$$

For $a = 0$, $r_{\rm ms} = 6M$, while
for $a = M$, $r_{\rm ms} = M$ (direct) or $r_{\rm ms} = 9M$ (retrograde)

Kerr Black Hole : Binding Energy i

Schwarzchild case : Stable circular orbit in the Schwarzchild case exist down to $r = 3M$ i.e. when the denominator of

$$\tilde{E}^2 = \frac{(r - 2M)^2}{r(r - 3M)} \tag{51}$$

goes to zero ($\tilde{E} = E/m \to \infty$), this happens for photons i.e. $r_{\rm ph} = 3M$.

- Circular orbits are stable if $\partial^2 V_{\rm eff}/\partial^2 r > 0$ and unstable if $\partial^2 V_{\rm eff}/\partial^2 r \leq 0$. This limit is $r_{\rm ISCO} = 6M$ then $\tilde{E}_{\rm ISCO} = \sqrt{8/9}$.

- The **binding energy** per unit mass of a particle in the last stable circular orbit ($r = 6M$) is, from (51) we get

$$\tilde{E}_{\rm bind} = 1 - \left(\frac{8}{9}\right)^{1/2} \approx 5.72\% \tag{52}$$

This is the fraction of rest-mass energy released when a particle originally at rest at infinity spirals towards the BH to the innermost stable circular orbit, and then plunges into the BH.

Kerr Black Hole : Binding Energy ii

Thus the conversion of rest mass to other forms of energy is potentially much more efficient for accretion onto a black hole than for nuclear burning which releases maximum of only 0.9% of the rest mass ($H \to Fe$).

- **Kerr case** :

The binding energy of the marginally stable circular orbits can be taken if we eliminate r from Eq (43) and using Eq (50) we find

$$\frac{a}{M} = \mp \frac{4\sqrt{2}(1-\tilde{E}^2)^{1/2} - 2\tilde{E}}{3\sqrt{3}(1-\tilde{E}^2)} \qquad (53)$$

The quantity $\tilde{E}$ decreases from $\sqrt{8/9}$ ($a = 0$) to $\sqrt{1/3}$ ($a = M$) for direct orbits and
increases from $\sqrt{8/9}$ ($a = 0$) to $\sqrt{25/27}$ ($a = M$) for retrograde orbits.

- The maximum binding energy $1 - \tilde{E}$ for a maximally rotating BH is $1 - 1/\sqrt{3}$ or 42.3% of the rest-mass energy.

Kerr Black Hole : Binding Energy iii

• This is the amount of energy that is released by matter spiralling in toward the black hole through a succession of almost circular equatorial orbits. Negligible energy will be released during the final plunge from r_{ms} into the black hole.

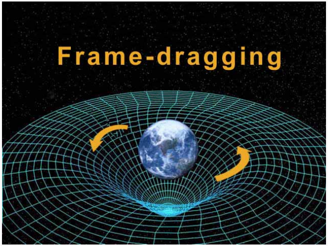
• Note that the Boyer-Lindquist coordinate system collapses r_{ms}, r_{mb}, r_{ph} and r_+ into $r = M$ as $a \to M$.

Frame draging

A particle with initial $L = 0$ at infinity released from the rest will follow a spiral towards the black-hole along a conical surface of constant θ.
It acquires an angular velocity (HOW?)

$$\frac{d\phi}{dt} = \omega(r, \theta) = \frac{2aMr}{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta} \tag{54}$$

as viewed from infinity.
Observers at fixed r and θ, with zero angular velocity, corotate with the geometry with angular momentum $\omega(r, \theta)$. Such observers define the so-called "locally non-rotating frame" (LNRF) according to such observers, the released particle described above appears to move radially locally.



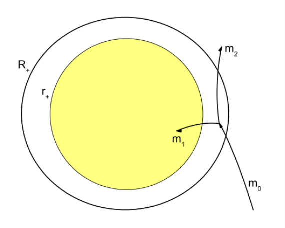
Particles inside the ergosphere may have negative energy trajectories

$$E = p_0 = m g_{0\mu} u^\mu = m \left(g_{00} u^0 + g_{0\phi} u^\phi \right) \tag{55}$$

Inside the ergosphere $g_{00} < 0$ while $g_{0\phi} > 0$ thus with an appropriate choice of u^0 and u^ϕ it is possible to have particles with negative energy.

This can lead into energy extraction from a Kerr BH. Actually, we can extract rotational energy till the BH reduces to a Schwarzschild one. The maximal energy that can be extracted corresponds to 20.7% of the initial black hole mass. Larger efficiencies are possible for charged-rotating black holes.

Starting from outside the ergosphere particle m_0 enters into the ergosphere and splits into two parts. One of them follows a trajectory of negative energy and falls into the BH, the second escapes at infinity carrying more energy than the initial particle!



Penrose Process (1969) & Superadiance (Zel'dovich 1971) ii

Superadiance (Zel'dovich 1971) Energy amplification can occur when electromagnetic or gravitational waves of suitable frequency scattered by a rotating black-hole.

In this case a part of the wave is absorbed while the scattered part, under the right conditions, can have more energy than the initial incident wave.

Penrose Process (1969) & Superadiance (Zel'dovich 1971) iii

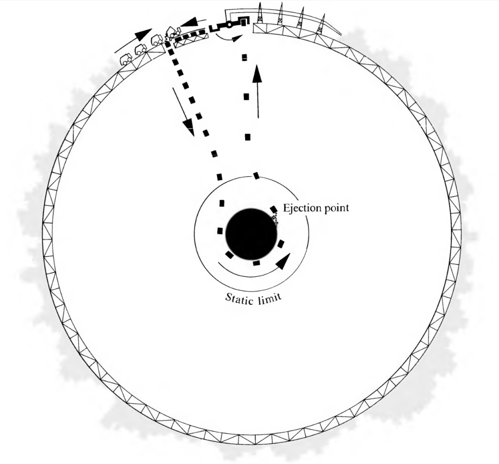


Figure 7: A city covering its energy needs from a black hole via the Penrose process. Figure adapted from Misner-Thorne-Wheeler "Gravitation" (1973)

Black-Hole: Mechanics

There is a lower limit on the mass that can remain after a continuous extraction of rotational energy via Penrose process.
Christodoulou (1969) named it **irreducible mass** M_{ir}

$$4M_{ir}^2 = r_+^2 + a^2 \equiv \left(M + \sqrt{M^2 - a^2}\right)^2 + a^2 = 2Mr_+ \tag{56}$$

while Hawking ('70) suggested that **the area of the BH horizon cannot be reduced via the Penrose process**.

The two statements are equivalent!

EXAMPLE : Calculate the area of the horizon

$$ds^2 = g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 = \left(r_+^2 + a^2\cos^2\theta\right)d\theta^2 + \frac{(r_+^2 + a^2)^2\sin^2\theta}{r_+^2 + a^2\cos^2\theta}d\phi^2$$

thus the horizon area is

$$\begin{aligned} A &= \int_0^{2\pi}\int_0^{\pi}\sqrt{g}\,d\theta d\phi = \int_0^{2\pi}\int_0^{\pi}\sqrt{g_{\phi\phi}}\sqrt{g_{\theta\theta}}\,d\theta d\phi \\ &= \int_0^{2\pi}\int_0^{\pi}(r_+^2 + a^2)\sin\theta\, d\theta d\phi = 4\pi(r_+^2 + a^2) = 8\pi M r_+ = 16\pi M_{ir}^2 \end{aligned}$$

Black Hole Thermodynamics i

Hawking's area theorem opened up a new avenue in the study of BH physics.

• The **surface area** of a BH seems to be analogous to **entropy** in thermodynamics.

• In order to associate them one needs :

- ⋆ to define precisely what is meant by the "entropy" of a BH
- ⋆ to associate the concept of **temperature** with a BH

• **Entropy** is defined as the measure of the **disorder of a system** or of the **lack of information of its precise state**. As the entropy increases the amount of information about the state of a system decreases.

• **No-hair theorem** : a BH has only 3 distinguishing features : **mass**, **angular momentum** and **electric charge** (Wheeler-Bekenstein , Carter-Hawking 1970).

Black Hole Thermodynamics ii

Bekenstein argued that the entropy of a BH could be described in terms of the number of internal states which correspond to the same external appearance. I.e. the more massive is a BH the greater the number of possible configurations that went into its formation, and the greater is the loss of information (which has been lost during its formation).

• The area of the BH's event horizon is proportional to M^2.

• The entropy of the BH can be regarded as proportional to the area of the horizon. Bekenstein has actually written

$$S_{\mathrm{bh}} = \frac{ck_B}{\hbar}\frac{A}{4} = \frac{ck_B}{\hbar} 4\pi M^2 \tag{57}$$

where A is the surface area, $\hbar$ Planck's constant divided by 2π and k_B Boltzmann's constant.

• The introduction of **BH entropy**, calls for the definition of the **BH temperature**!

Black Hole Thermodynamics: 0th Law

The temperature of a body is uniform at thermodynamical equilibrium (0th Law)

- The surface gravity of the event horizon of a BH is inversely proportional to its mass

$$\kappa_{\rm bh} = \left(\frac{GM}{r^2}\right)_{r \to r_+} = \frac{c^4}{G}\frac{1}{4M} \tag{58}$$

The surface gravity of the BH (spherical or axisymmetric) can play the role of the temperature,

And in analogy the BH temperature should be inversely proportional to its mass.

- **The less massive is a BH the hotter it would be !**

Black Hole Thermodynamics: 1st Law i

The area of the event horizon is:

$$A_h = 8\pi M r_+ = 8\pi M\left[M + \left(M^2 - a^2\right)^{1/2}\right] \tag{59}$$

Any change in A will be followed by a change in M and a, i.e.

$$\begin{aligned}
\delta A_h &= 8\pi\left[2M\delta M + \delta M\left(M^2 - a^2\right)^{1/2} + \frac{M\left(M\delta M - a\delta a\right)}{\left(M^2 - a^2\right)^{1/2}}\right] \\
&= \frac{8\pi}{\left(M^2 - a^2\right)^{1/2}}\left[M r_+ \delta M - a^2 \delta M - aM\delta a\right] \qquad (60) \\
&= \frac{16\pi r_+ \pi}{\left(M^2 - a^2\right)^{1/2}}\left[\delta M - \Omega_+ \delta J\right] \qquad (61)
\end{aligned}$$

where $\Omega_+ = a(2Mr_+)$ is the angular velocity of the black-hole, while from the definition of the angular momentum of the black hole $J = Ma$ we get $\delta J = a\delta M + M\delta a$.

We may rewrite eqn (61) as

$$\delta M = \frac{\kappa}{8\pi G}\delta A_h + \Omega\,\delta J \quad \text{where} \quad \kappa = \frac{\left(M^2 - a^2\right)^{1/2}}{2Mr_+} \tag{62}$$

- In more general case (Kerr-Newman) if we make a small change in the mass M, angular momentum J and the electric charge Q we get [3]

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega\delta J + \Phi\delta Q \tag{64}$$

where the surface gravity and horizon area of a Kerr-Newman BH are:

$$\begin{aligned} \kappa &= 4\pi\mu/A \\ A &= 4\pi[2M(M+\mu) - Q^2] \\ \mu &= (M^2 - Q^2 - J^2/M^2)^{1/2} \end{aligned}$$

This is the **1st law of black hole mechanics**

[3] In thermodynamics

$$dU = TdS - PdV \tag{63}$$

where $U = U(S, V)$ is the internal energy and S the entropy,

Black Hole Thermodynamics : 2nd Law

Second law of black hole mechanics: the horizon can never decrease assuming Cosmic Censorship and a positive energy condition [4]

⋆ This statement is not valid once quantum mechanics comes into play, because Hawking radiation carries mass and therefore reduces the surface area of the BH and hence its entropy.

⋆ The correct statement is a generalised form of the law, that the entropy of the Universe, including that of the black hole S_h, cannot decrease.
I.e. $S_{\rm Uni} = S_h + S_{\rm ext}$ cannot decrease (where $S_{\rm ext}$ is the entropy of the world excluding the BH).

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[4] Note that during the "evaporation", M decreases and thus so does A and S this violates Hawking's area theorem. However one of the postulates is that matter obeys the "strong" energy condition, which requires that the local observer always measures positive energy densities and there are no spacelike energy fluxes, which is not the case of pair creation i.e. there is a spacelike energy flux.

Black Hole Thermodynamics : 3rd Law

Third law of black hole mechanics: the surface gravity of the horizon cannot be reduced to zero in a finite number of steps.

- ⋆ **Bekenstein-Hawking entropy formula**

$$S_H = \frac{1}{G\hbar}\frac{A}{4}$$

- ⋆ **Bekenstein-Hawking temperature formula** [5]

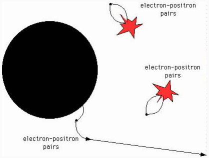
$$T = \frac{\hbar}{4\pi^2 k_B c}\kappa \approx 6.2 \times 10^{-8}\left(\frac{M_\odot}{M}\right) K \tag{65}$$

[5] NOTE: a black-hole of $1M_\odot$ will absorb far more CMB radiation than it emits. A black-hole with mass of the order of our moon will have a temperature of about 2.7K and thus will be in equilibrium with the environment.

Hawking (-Zeldovich) Radiation i

- *Classically:* Nothing can escape from the interior of a BH.
- *Quantum Mechanically:* Black holes shine like a blackbody with temperature inversely proportional to their mass (Hawking 1974).

Quantum fluctuations of the zero-energy vacuum can create particle - antiparticle pairs. These particles and antiparticles are considered to be virtual in the sense that they don't last long enough to be observed.



According to Heisenberg's principle $\Delta E \cdot \Delta t \geq \hbar$ and from Einstein's equation $E = mc^2$ we get a relation which implies an uncertainty in mass

$$\Delta m \cdot \Delta t \geq \hbar / c^2$$

This means that in a very brief interval Δt of time, we cannot be sure how much matter there is in a particular location, **even in the vacuum**.

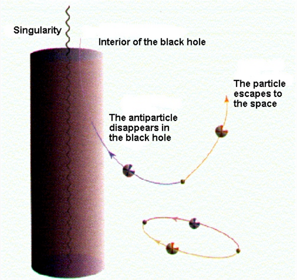
Hawking (-Zeldovich) Radiation ii

Thus particle-antiparticle pairs can be created which may last for at most, a time $\hbar/c^2\Delta m$.

Before, the time is up, they must find each other and annihilate !

In the presence of a strong field the e^- and the e^+ may become separated by a distance of Compton wavelength $\lambda = h/mc$ that is of the order of the BH radius r_+.

There is a small but finite probability for one of them to "tunnel" through the barrier of the quantum vacuum and escape the BH horizon as a real particle with positive energy, leaving the negative energy inside the horizon of the BH.



- This process is called **Hawking radiation**
- The rate of emission is as if the BH were a hot body of temperature proportional to the surface gravity.

Hawking Temperature i

Hawking calculated that the statistical effect of many such emissions leads to the emergence of particles with a thermal spectrum:

$$N(E) = \frac{8\pi}{c^3 h^3} \frac{E^2}{e^{4\pi^2 E/\kappa h} - 1} \tag{66}$$

where $N(E)$ is the number of particles of energy E per unit energy band per unit volume.

• Notice the similarity between this formula and the Planckian distribution for thermal radiation from a black body $\frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1}$.

• This similarity led Hawking to conclude that a black hole radiates as a black body at the temperature T

$$T = \frac{\hbar}{4\pi^2 k_B c} \kappa = \frac{\hbar c^3}{8\pi k_B G M} \tag{67}$$

• Note that in the classical limit $\hbar \to 0$ the temperature vanishes and the BH only absorbs photons, never emits them.

Hawking Temperature ii

The temperature expressed in terms of mass is

$$T = \frac{\hbar c^3}{8\pi k_B G M} = 6.2 \times 10^{-8} \left(\frac{M_\odot}{M} \right) K \tag{68}$$

where $M_\odot$ is the solar mass.

To estimate the power radiated, we assume that the surface of the BH radiates according to the Stefan-Boltzmann law:

$$\frac{dP}{dA} = \sigma T^4 = \frac{\pi^2}{60} \frac{k_B^4}{\hbar^3 c^2} T^4 \tag{69}$$

Substituting the temperature and noting that the power radiated corresponds to a decrease of mass M by $P = (dM/dt)c^2$, we obtain the differential equation for the mass of a radiating BH as a function of t

$$\frac{dM}{dt} = -\frac{\pi^3}{15360} \frac{\hbar\, c^4}{G^2 M^2} \tag{70}$$

By integration we get the duration of the radiation

$$t \approx \frac{Mc^2}{P} \approx \frac{5120\pi G^2}{\hbar\, c^4} M^3 \approx 8.41 \times 10^{-17} \left(\frac{M}{\mathrm{Kg}}\right)^3 \text{sec} \tag{71}$$

Hawking Radiation : Mini - BHs

⋆ A BH with a mass comparable to that of the Sun would have a temperature $T \sim 10^{-7} K$ and will emit radiation at the rate of $\sim 10^{-16} erg/s$ and will evaporate in 6.73×10^{74}secs.

⋆ A BH with mass of about 10^{14}gr is expected to evaporate over a period of about 10^{10} years (today).

⋆ Planck mass quantum black hole Hawking radiation evaporation time:

$$t_{\mathrm{ev}} = \frac{5120\pi G^2}{\hbar c^4} m_P^3 = 5120\pi t_P = 5120\pi \sqrt{\frac{\hbar G}{c^5}} = 8.671 \times 10^{-40} s\,. \qquad (72)$$

⋆ The critical mass M_0 for a BH to evaporate after time t is

$$M_0 = \frac{\pi}{8} \frac{1}{10^{1/3}} \left(\frac{\hbar c^4 t}{G^2 M^2} \right)^{1/3} \qquad (73)$$

⋆ We do not have a theory capable of explaining what happens when a BH shrinks within the Planck radius (10^{-37} cm), and the answer to this question lies in the area of speculation.