A Short Introduction to Tensor Analysis

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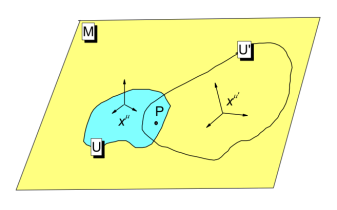
[a] This chapter based strongly on “Lectures of General Relativity” by A. Papapetrou, D. Reidel publishing company, (1974)

Scalars and Vectors i

- A n-dim manifold is a space M on every point of which we can assign n numbers $(x^1, x^2, \ldots, x^n)$ - the coordinates - in such a way that there will be an one to one correspondence between the points and the n numbers.

- Every point of the manifold has its own neighborhood which can be mapped to a n-dim Euclidean space.
- The manifold cannot be always covered by a single system of coordinates and there is not a preferable one either.



The coordinates of the point P are connected by relations of the form: $x^{\mu'} = x^{\mu'}\left(x^1, x^2, \ldots, x^n\right)$ for $\mu' = 1, \ldots, n$ and their inverse $x^{\mu} = x^{\mu}\left(x^{1'}, x^{2'}, \ldots, x^{n'}\right)$ for $\mu = 1, \ldots, n$. If there exist

$$A^{\mu'}{}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} \quad \text{and} \quad A^{\nu}{}_{\mu'} = \frac{\partial x^{\nu}}{\partial x^{\mu'}} \Rightarrow \det\left|A^{\mu'}{}_{\nu}\right| \tag{1}$$

then the manifold is called **differential**.

Scalars and Vectors ii

Any physical quantity, e.g. the velocity of a particle, is determined by a set of numerical values - its components - which depend on the coordinate system.

Tensors

Studying the way in which these values change with the coordinate system leads to the concept of **tensor**.

With the help of this concept we can express the physical laws by **tensor equations**, which have the same form in every coordinate system.

- **Scalar field** : is any physical quantity determined by a single numerical value i.e. just one component which is independent of the coordinate system (mass, charge,...)
- **Vector field (contravariant)**: an example is the infinitesimal displacement vector, leading from a point A with coordinates x^μ to a neighbouring point A' with coordinates $x^\mu + dx^\mu$. The components of such a vector are the differentials dx^μ.

Vector Transformations

From the infinitesimal vector $\vec{AA'}$ with components dx^μ we can construct a finite vector v^μ defined at A. This will be the **tangent vector** to the curve $x^\mu = f^\mu(\lambda)$ passing from the points A and A' corresponding to the values λ and $\lambda + d\lambda$ of the parameter. Then

$$v^\mu = \frac{dx^\mu}{d\lambda} \,. \tag{2}$$

Any transformation from x^μ to $\tilde{x}^\mu$ ($x^\mu \to \tilde{x}^\mu$) will be determined by n equations of the form: $\tilde{x}^\mu = f^\mu(x^\nu)$ where $\mu\,,\,\nu = 1, 2, ..., n$.

This means that :

$$d\tilde{x}^\mu = \sum_\nu \frac{\partial f^\mu}{\partial x^\nu} dx^\nu = \sum_\nu \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu \quad \text{for} \quad \nu = 1, ..., n \tag{3}$$

and

$$\tilde{v}^\mu = \frac{d\tilde{x}^\mu}{d\lambda} = \sum_\nu \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \frac{dx^\nu}{d\lambda} = \sum_\nu \frac{\partial \tilde{x}^\mu}{\partial x^\nu} v^\nu \tag{4}$$

Contravariant and Covariant Vectors i

Contravariant Vector: is a quantity with n components depending on the coordinate system in such a way that the components a^μ in the coordinate system x^μ are related to the components $\tilde{a}^\mu$ in $\tilde{x}^\mu$ by a relation of the form

$$\tilde{a}^\mu = \sum_\nu \frac{\partial \tilde{x}^\mu}{\partial x^\nu} a^\nu \tag{5}$$

This means that, if the coordinate system undergoes a transformation described by an invertible matrix $\mathbf{M}$, so that a coordinate vector $\mathbf{x}$ is transformed to $\tilde{\mathbf{x}} = \mathbf{M}\mathbf{x}$, then a contravariant vector must be similarly transformed via $\tilde{\mathbf{a}} = \mathbf{M}\mathbf{a}$.

This requirement is what distinguishes a contravariant vector from any other triple of physically meaningful quantities.

Displacement, velocity and acceleration are contravariant vectors.

Contravariant and Covariant Vectors ii

Covariant Vector: eg. b_μ, is an object with n components which depend on the coordinate system on such a way that if a^μ is **any** contravariant vector, the following sums are scalars

$$\sum_\mu b_\mu a^\mu = \sum_\mu \tilde{b}_\mu \tilde{a}^\mu = \phi \quad \text{for any} \quad x^\mu \to \tilde{x}^\mu \quad \textbf{[Scalar Product]} \tag{6}$$

The covariant vector will transform as (why?):

$$\tilde{b}_\mu = \sum_\nu \frac{\partial x^\nu}{\partial \tilde{x}^\mu} b_\nu \quad \text{or} \quad b_\mu = \sum_\nu \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \tilde{b}_\nu \tag{7}$$

The gradient vector transform is covariant

$$\nabla\phi = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^3}\right)\phi \equiv \nabla_\mu \phi \equiv \partial_\mu \phi \equiv \phi_{,\mu} \, . \tag{8}$$

What is Einstein's summation convention?

Tensors: at last

A **contravariant tensor** of **order 2** is a quantity having n^2 components $T^{\mu\nu}$ which transforms ($x^\mu \to \tilde{x}^\mu$) in such a way that, if a_μ and b_μ are arbitrary covariant vectors the following sums are scalars:

$$T^{\lambda\mu} a_\mu b_\lambda = \tilde{T}^{\lambda\mu} \tilde{a}_\lambda \tilde{b}_\mu \equiv \phi \quad \text{for any} \quad x^\mu \to \tilde{x}^\mu \tag{9}$$

Then the transformation formulae for the components of the tensors of order 2 are (why?):

$$\tilde{T}^{\alpha\beta} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} T^{\mu\nu}, \tilde{T}^\alpha_\beta = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} T^\mu{}_\nu \; \& \; \tilde{T}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} T_{\mu\nu}$$

The **Kronecker symbol**

$$\delta^\lambda{}_\mu = \begin{cases} 0 & \text{if} \quad \lambda \neq \mu \,, \\ 1 & \text{if} \quad \lambda = \mu \,. \end{cases}$$

is a mixed tensor having frame independent values for its components.

⋆ **Tensors of higher order**: $T^{\alpha\beta\gamma\ldots}{}_{\mu\nu\lambda\ldots}$

- **Tensor addition :** Tensors of the same order (p, q) can be added, their **sum** being again a tensor of the same order. For example:

$$\tilde{a}^{\nu} + \tilde{b}^{\nu} = \frac{\partial \tilde{x}^{\nu}}{\partial x^{\mu}} \left(a^{\mu} + b^{\mu} \right) \tag{10}$$

- **Tensor multiplication :** The product of two vectors is a tensor of order 2, because

$$\tilde{a}^{\alpha} \tilde{b}^{\beta} = \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \tilde{x}^{\beta}}{\partial x^{\nu}} a^{\mu} b^{\nu} \tag{11}$$

in general:

$$T^{\mu\nu} = A^{\mu} B^{\nu} \quad \text{or} \quad T^{\mu}{}_{\nu} = A^{\mu} B_{\nu} \quad \text{or} \quad T_{\mu\nu} = A_{\mu} B_{\nu} \tag{12}$$

- **Contraction**: for any mixed tensor of order (p, q) leads to a tensor of order $(p-1, q-1)$ (prove it!)

$$T^{\lambda\mu\nu}{}_{\lambda\alpha} = T^{\mu\nu}{}_{\alpha} \tag{13}$$

Tensor algebra ii

- **Trace**: of the mixed tensor $T^{\alpha}{}_{\beta}$ is called the scalar $T = T^{\alpha}{}_{\alpha}$.
- **Symmetric Tensor :** $T_{\lambda\mu} = T_{\mu\lambda}$ or $T_{(\lambda\mu)}$,
  $T_{\nu\lambda\mu} = T_{\nu\mu\lambda}$ or $T_{\nu(\lambda\mu)}$
- **Antisymmetric :** $T_{\lambda\mu} = -T_{\mu\lambda}$ or $T_{[\lambda\mu]}$,
  $T_{\nu\lambda\mu} = -T_{\nu\mu\lambda}$ or $T_{\nu[\lambda\mu]}$

Number of independent components :
Symmetric : $n(n+1)/2$,
Antisymmetric : $n(n-1)/2$

Tensors: Differentiation & Connections i

We consider a region V of the space in which some tensor, e.g. a covariant vector a_λ, is given at each point $P(x^\alpha)$ i.e.

$$a_\lambda = a_\lambda(x^\alpha)$$

We say then that we are given a **tensor field** in V and we assume that the components of the tensor are continuous and differentiable functions of x^α.

Question: *Is it possible to construct a new tensor field by differentiating the given one?*

The simplest tensor field is a scalar field $\phi = \phi(x^\alpha)$ and *its derivatives are the components of a covariant tensor*!

$$\frac{\partial \phi}{\partial \tilde{x}^\lambda} = \frac{\partial x^\alpha}{\partial \tilde{x}^\lambda}\frac{\partial \phi}{\partial x^\alpha} \quad \text{we will use:} \quad \frac{\partial \phi}{\partial x^\alpha} = \phi_{,\alpha} \equiv \partial_\alpha \phi \tag{14}$$

i.e. $\phi_{,\alpha}$ is the *gradient* of the scalar field ϕ.

The derivative of a contravariant vector field A^{μ} is :

$$\frac{\partial A^{\mu}}{\partial x^{\alpha}} \equiv A^{\mu}{}_{,\alpha} = \frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}}\tilde{A}^{\nu}\right) = \frac{\partial \tilde{x}^{\rho}}{\partial x^{\alpha}}\frac{\partial}{\partial \tilde{x}^{\rho}}\left(\frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}}\tilde{A}^{\nu}\right)$$
$$= \frac{\partial^2 x^{\mu}}{\partial \tilde{x}^{\nu}\partial \tilde{x}^{\rho}}\frac{\partial \tilde{x}^{\rho}}{\partial x^{\alpha}}\tilde{A}^{\nu} + \frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}}\frac{\partial \tilde{x}^{\rho}}{\partial x^{\alpha}}\frac{\partial \tilde{A}^{\nu}}{\partial \tilde{x}^{\rho}} \tag{15}$$

Without the first term (red) in the right hand side this equation would be the transformation formula for a mixed tensor of order 2.

The transformation $(x^{\mu} \to \tilde{x}^{\mu})$ of the derivative of a vector is:

$$A^{\mu}{}_{,\alpha} - \underbrace{\frac{\partial^2 x^{\mu}}{\partial \tilde{x}^{\nu}\partial \tilde{x}^{\rho}}\frac{\partial \tilde{x}^{\rho}}{\partial x^{\alpha}}\frac{\partial \tilde{x}^{\nu}}{\partial x^{\kappa}}}_{\Gamma^{\mu}_{\alpha\kappa}}A^{\kappa} = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}}\frac{\partial \tilde{x}^{\rho}}{\partial x^{\alpha}}\tilde{A}^{\nu}{}_{,\rho} \tag{16}$$

in another coordinate $(x'^{\mu} \to \tilde{x}^{\mu})$ we get again:

$$A'^{\mu}{}_{,\alpha} - \underbrace{\frac{\partial^2 x'^{\mu}}{\partial \tilde{x}^{\nu}\partial \tilde{x}^{\rho}}\frac{\partial \tilde{x}^{\rho}}{\partial x^{\alpha}}\frac{\partial \tilde{x}^{\nu}}{\partial x'^{\kappa}}}_{\Gamma'^{\mu}_{\alpha\kappa}}A'^{\kappa} = \frac{\partial x'^{\mu}}{\partial \tilde{x}^{\nu}}\frac{\partial \tilde{x}^{\rho}}{\partial x'^{\alpha}}\tilde{A}^{\nu}_{,\rho} \tag{17}$$

Suggesting that for a transformation ($x^\mu \to x'^\mu$) the following convention might work:

$$A^\mu{}_{,\alpha} + \Gamma^\mu{}_{\alpha\kappa} A^\kappa = \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x'^\rho}{\partial x^\alpha} \left(A'^\nu{}_{,\rho} + \Gamma'^\nu{}_{\sigma\rho} A'^\sigma \right) \tag{18}$$

The *necessary and sufficient condition* for $A^\mu{}_{,\alpha} + \Gamma^\mu{}_{\alpha\kappa} A^\kappa$ to be a tensor is:

$$\Gamma'^\lambda{}_{\rho\nu} = \frac{\partial^2 x^\mu}{\partial x'^\nu \partial x'^\rho} \frac{\partial x'^\lambda}{\partial x^\mu} + \frac{\partial x^\kappa}{\partial x'^\rho} \frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial x'^\lambda}{\partial x^\mu} \Gamma^\mu{}_{\kappa\sigma} \, . \tag{19}$$

The object $\Gamma^\lambda{}_{\rho\nu}$ is the called the **connection** of the space and **it is not tensor**.

Covariant Derivative

According to the previous assumptions, the following quantity transforms as a tensor of order 2

$$\boxed{A^{\mu}{}_{;\alpha} = A^{\mu}{}_{,\alpha} + \Gamma^{\mu}{}_{\alpha\lambda} A^{\lambda} \quad \text{or} \quad \nabla_{\alpha} A^{\mu} = \partial_{\alpha} A^{\mu} + \Gamma^{\mu}{}_{\alpha\lambda} A^{\lambda}} \tag{20}$$

and is called **absolute** or **covariant derivative** of the contravariant vector A^{μ}.

In similar way we get (how?) :

$$\phi_{;\lambda} = \phi_{,\lambda} \tag{21}$$

$$A_{\lambda;\mu} = A_{\lambda,\mu} - \Gamma^{\rho}_{\mu\lambda} A_{\rho} \tag{22}$$

$$T^{\lambda\mu}{}_{;\nu} = T^{\lambda\mu}{}_{,\nu} + \Gamma^{\lambda}_{\alpha\nu} T^{\alpha\mu} + \Gamma^{\mu}_{\alpha\nu} T^{\lambda\alpha} \tag{23}$$

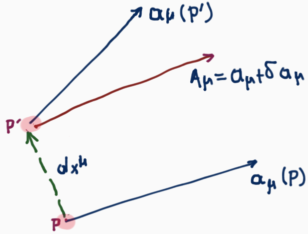
$$T^{\lambda}{}_{\mu;\nu} = T^{\lambda}{}_{\mu,\nu} + \Gamma^{\lambda}_{\alpha\nu} T^{\alpha}{}_{\mu} - \Gamma^{\alpha}_{\mu\nu} T^{\lambda}{}_{\alpha} \tag{24}$$

$$T_{\lambda\mu;\nu} = T_{\lambda\mu,\nu} - \Gamma^{\alpha}_{\lambda\nu} T_{\mu\alpha} - \Gamma^{\alpha}_{\mu\nu} T_{\lambda\alpha} \tag{25}$$

$$\begin{aligned} T^{\lambda\mu\cdots}{}_{\nu\rho\cdots;\sigma} &= T^{\lambda\mu\cdots}{}_{\nu\rho\cdots,\sigma} \\ &+ \Gamma^{\lambda}_{\alpha\sigma} T^{\alpha\mu\cdots}{}_{\nu\rho\cdots} + \Gamma^{\mu}_{\alpha\sigma} T^{\lambda\alpha\cdots}{}_{\nu\rho\cdots} + \cdots \\ &- \Gamma^{\alpha}_{\nu\sigma} T^{\lambda\mu\cdots}{}_{\alpha\rho\cdots} - \Gamma^{\alpha}_{\rho\sigma} T^{\lambda\mu\cdots}{}_{\nu\alpha\cdots} - \cdots \end{aligned} \tag{26}$$

Parallel Transport of a vector i

Let a_μ be some covariant vector field. Consider two neighbouring points P and P', and the displacement $\mathbf{PP'}$ having components dx^μ.



At the point P' we may define two vectors associated to $a_\mu(P)$:
a) The vector $a_\mu(P')$ defined as:

$$a_\mu(P') = a_\mu(P) + a_{\mu,\nu}(P)dx^\nu = a_\mu(P) + \Delta a_\mu(P) \tag{27}$$

The difference $a_\mu(P') - a_\mu(P) = a_{\mu,\nu}(P)dx^\nu \equiv \Delta a_\mu(P)$ is not a tensor.

b) The vector $A_\mu(P')$ after its "parallel transport" from P to P'

$$A_\mu(P') = a_\mu(P) + \delta a_\mu(P). \tag{28}$$

$\delta a_\mu(P)$ is not yet defined. Potentially it can be written as : $\delta a_\mu = C^\lambda_{\mu\nu} a_\lambda dx^\nu$.

Parallel Transport of a vector ii

The difference of the two vectors

$$\underbrace{a_\mu(P') - A_\mu(P')}_{\text{vector}} = \underbrace{a_\mu + \Delta a_\mu}_{\text{at point P}} - \underbrace{(a_\mu + \delta a_\mu)}_{\text{at point P}} = \underbrace{\Delta a_\mu - \delta a_\mu}_{\text{vector}}$$

$$= \underbrace{a_{\mu,\nu} dx^\nu - \delta a_\mu}_{\text{vector}} = \left(a_{\mu,\nu} - C^\lambda_{\mu\nu} a_\lambda\right) dx^\nu$$

Which leads to the obvious suggestion that $C^\lambda_{\mu\nu} \equiv \Gamma^\lambda{}_{\mu\nu}$.

The connection $\Gamma^\lambda_{\mu\nu}$ allows us to define a covariant vector $a_{\lambda;\mu}$ which is a tensor. In other words with the help of $\Gamma^\lambda_{\mu\nu}$ we can define a vector at a point P', which has to be considered as "equivalent" to the vector a_μ defined at P.

Parallel Transport of a vector iii

Parallel Transport

The connection ${\Gamma^{\lambda}}_{\mu\nu}$ allows to define the transport of a vector a_λ from a point P to a neighbouring point P' (**Parallel Transport**).

$$\delta a_\mu = {\Gamma^{\lambda}}_{\mu\nu} a_\lambda dx^\nu \quad \text{for covariant vectors} \tag{29}$$

$$\delta a^\mu = -{\Gamma^{\mu}}_{\lambda\nu} a^\lambda dx^\nu \quad \text{for contravariant vectors} \tag{30}$$

•The parallel transport of a scalar field is zero! $\delta\phi = 0$ (why?)

Review of the 1st Lecture

Tensor Transformations

$$\tilde{b}_\mu = \sum_\nu \frac{\partial x^\nu}{\partial \tilde{x}^\mu} b_\nu \quad \text{and} \quad \tilde{a}^\mu = \sum_\nu \frac{\partial \tilde{x}^\mu}{\partial x^\nu} a^\nu \tag{31}$$

$$\tilde{T}^{\alpha\beta} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} T^{\mu\nu}, \tilde{T}^\alpha_\beta = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} T^\mu{}_\nu \ \& \ \tilde{T}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} T_{\mu\nu}$$

Covariant Derivative

$$\phi_{;\lambda} = \phi_{,\lambda} \tag{32}$$

$$A_{\lambda;\mu} = A_{\lambda,\mu} - \Gamma^\rho_{\mu\lambda} A_\rho \tag{33}$$

$$T^{\lambda\mu}{}_{;\nu} = T^{\lambda\mu}{}_{,\nu} + \Gamma^\lambda_{\alpha\nu} T^{\alpha\mu} + \Gamma^\mu_{\alpha\nu} T^{\lambda\alpha} \tag{34}$$

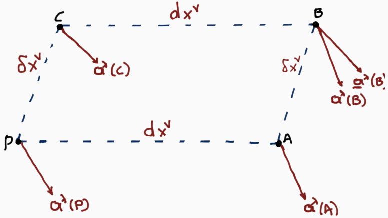
Parallel Transport

$$\delta a_\mu = \Gamma^\lambda{}_{\mu\nu} a_\lambda dx^\nu \quad \text{for covariant vectors} \tag{35}$$

$$\delta a^\mu = -\Gamma^\mu{}_{\lambda\nu} a^\lambda dx^\nu \quad \text{for contravariant vectors} \tag{36}$$

Measuring the Curvature of the Space i

The"expedition" of a vector parallel transported along a closed path



The parallel transport of the vector a^λ from the point P to A leads to a change of the vector by a quantity $\Gamma^{\lambda}{}_{\mu\nu}(P)a^\mu dx^\nu$ and the new vector is:

$$a^\lambda(A) = a^\lambda(P) - \Gamma^\lambda_{\mu\nu}(P)a^\mu(P)dx^\nu \tag{37}$$

A further parallel transport to the point B will lead the vector

$$\begin{aligned} a^\lambda(B) &= a^\lambda(A) - \Gamma^{\lambda}{}_{\rho\sigma}(A)a^\rho(A)\delta x^\sigma \\ &= a^\lambda(P) - \Gamma^{\lambda}{}_{\mu\nu}(P)a^\mu(P)dx^\nu - \Gamma^{\lambda}{}_{\rho\sigma}(A)\left[a^\rho(P) - \Gamma^{\rho}{}_{\beta\nu}(P)a^\beta dx^\nu\right]\delta x^\sigma \end{aligned}$$

Measuring the Curvature of the Space ii

Since both dx^ν and δx^ν assumed to be small we can use the following expression

$$\Gamma^\lambda{}_{\rho\sigma}(A) \approx \Gamma^\lambda{}_{\rho\sigma}(P) + \Gamma^\lambda{}_{\rho\sigma,\mu}(P)dx^\mu \,. \tag{38}$$

Thus we have estimated the total change of the vector a^λ from the point P to B via A (**all terms are defined at the point** P).

$$\begin{aligned} a^\lambda(B) &= a^\lambda - \Gamma^\lambda_{\mu\nu} a^\mu dx^\nu - \Gamma^\lambda_{\rho\sigma} a^\rho \delta x^\sigma + \Gamma^\lambda{}_{\rho\sigma}\Gamma^\rho_{\beta\nu} a^\beta dx^\nu \delta x^\beta + \Gamma^\lambda_{\rho\sigma,\tau} a^\rho dx^\tau \delta x^\sigma \\ &- \Gamma^\lambda{}_{\rho\sigma,\tau}\Gamma^\rho{}_{\beta\nu} a^\beta dx^\tau dx^\nu \delta x^\sigma \end{aligned}$$

If we follow the path $P \to C \to B$ we get:

$$\underline{a}^\lambda(B) = a^\lambda - \Gamma^\lambda{}_{\mu\nu} a^\mu \delta x^\nu - \Gamma^\lambda{}_{\rho\sigma} a^\rho dx^\sigma + \Gamma^\lambda{}_{\rho\sigma}\Gamma^\rho{}_{\beta\nu} a^\beta \delta x^\nu dx^\beta + \Gamma^\lambda{}_{\rho\sigma,\tau} a^\rho \delta x^\tau dx^\sigma$$

and the accumulated effect on the vector will be

$$\delta a^\lambda \equiv a^\lambda(B) - \underline{a}^\lambda(B) = a^\beta \left(dx^\nu \delta x^\sigma - dx^\sigma \delta x^\nu\right)\left(\Gamma^\lambda{}_{\rho\sigma}\Gamma^\rho_{\beta\nu} + \Gamma^\lambda{}_{\beta\sigma,\nu}\right) \tag{39}$$

Measuring the Curvature of the Space iii

By exchanging the indices $\nu \to \sigma$ and $\sigma \to \nu$ we construct a similar relation

$$\delta a^{\lambda} = a^{\beta} \left(dx^{\sigma} \delta x^{\nu} - dx^{\nu} \delta x^{\sigma} \right) \left(\Gamma^{\lambda}{}_{\rho\nu} \Gamma^{\rho}{}_{\beta\sigma} + \Gamma^{\lambda}{}_{\beta\nu,\sigma} \right) \tag{40}$$

and the total change will be given by the following relation:

$$\delta a^{\lambda} = -\frac{1}{2} a^{\beta} R^{\lambda}{}_{\beta\nu\sigma} \left(dx^{\sigma} \delta x^{\nu} - dx^{\nu} \delta x^{\sigma} \right) \tag{41}$$

where

$$\boxed{R^{\lambda}{}_{\beta\nu\sigma} = -\Gamma^{\lambda}{}_{\beta\nu,\sigma} + \Gamma^{\lambda}{}_{\beta\sigma,\nu} - \Gamma^{\mu}{}_{\beta\nu} \Gamma^{\lambda}{}_{\mu\sigma} + \Gamma^{\mu}{}_{\beta\sigma} \Gamma^{\lambda}{}_{\mu\nu}} \tag{42}$$

is the **curvature tensor**.

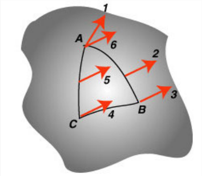


Figure 1: Measuring the curvature for the space.

Geodesics

- For a vector u^λ at point P we apply the parallel transport along a curve on an n-dimensional space which will be given by n equations of the form: $x^\mu = f^\mu(\lambda); \quad \mu = 1, 2, ..., n$
- If $u^\mu = \frac{dx^\mu}{d\lambda}$ is the *tangent vector* at P, the parallel transport of this vector will determine at another point of the curve a vector which **will not be in general** tangent to the curve.
- If the transported vector is tangent to any point of the curve then this curve is a **geodesic curve** of this space and is given by the equation :

$$\frac{du^\rho}{d\lambda} + \Gamma^\rho{}_{\mu\nu} u^\mu u^\nu = 0\,. \tag{43}$$

- Geodesic curves are the shortest curves connecting two points on a curved space.

Metric Tensor i

A space is called a **metric space** if a prescription is given attributing a scalar **distance** to each pair of neighbouring points

The distance ds of two points $P(x^\mu)$ and $P'(x^\mu + dx^\mu)$ is given by

$$ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \tag{44}$$

In another coordinate system, $\tilde{x}^\mu$, we will get

$$dx^\nu = \frac{\partial x^\nu}{\partial \tilde{x}^\alpha} d\tilde{x}^\alpha \tag{45}$$

which leads to:

$$ds^2 = \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = g_{\alpha\beta} dx^\alpha dx^\beta \,. \tag{46}$$

This gives the following transformation relation (why?):

$$\tilde{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta} \tag{47}$$

suggesting that the quantity $g_{\mu\nu}$ is a **symmetric** tensor, the so called **metric tensor**.

The relation (46) characterises a **Riemannian space**: This is a metric space in which the distance between neighbouring points is given by (46).

- If at some point P there are given 2 infinitesimal displacements $d^{(1)}x^{\alpha}$ and $d^{(2)}x^{\alpha}$, the metric tensor allows to construct the scalar

$$g_{\alpha\beta}\, d^{(1)}x^{\alpha}\, d^{(2)}x^{\beta}$$

which shall call **scalar product** of the two vectors.

- **Properties:**

$$g_{\mu\alpha}A^{\alpha} = A_{\mu}, \quad g_{\mu\alpha}T^{\lambda\alpha} = T^{\lambda}{}_{\mu}, \quad g_{\mu\nu}T^{\mu}{}_{\alpha} = T_{\nu\alpha}, \quad g_{\mu\nu}g_{\alpha\sigma}T^{\mu\alpha} = T_{\nu\sigma}$$

Metric Tensor iii

- Metric element for Minkowski spacetime

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad (48)$$

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad (49)$$

- For a sphere with radius R :

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad (50)$$

- The metric element of a torus with radii a and b

$$ds^2 = a^2 d\phi^2 + (b + a\sin\phi)^2\, d\theta^2 \quad (51)$$

Metric Tensor iv

- The **contravariant form** of the metric tensor:

$$g_{\mu\alpha}g^{\alpha\beta} = \delta^{\beta}{}_{\mu} \quad \text{where} \quad g^{\alpha\beta} = \frac{1}{\det|g_{\mu\nu}|} G^{\alpha\beta} \leftarrow \text{minor determinant} \tag{52}$$

With $g^{\mu\nu}$ we can now raise lower indices of tensors

$$A^{\mu} = g^{\mu\nu}A_{\nu}, \quad T^{\mu\nu} = g^{\mu\rho}T^{\nu}{}_{\rho} = g^{\mu\rho}g^{\nu\sigma}T_{\rho\sigma} \tag{53}$$

- The **angle**, ψ, between two infinitesimal vectors $d^{(1)}x^{\alpha}$ and $d^{(2)}x^{\alpha}$ is [1]:

$$\cos(\psi) = \frac{g_{\alpha\beta}\, d^{(1)}x^{\alpha}\, d^{(2)}x^{\beta}}{\sqrt{g_{\rho\sigma}\, d^{(1)}x^{\rho}\, d^{(1)}x^{\sigma}}\sqrt{g_{\mu\nu}\, d^{(2)}x^{\mu}\, d^{(2)}x^{\nu}}}\,. \tag{54}$$

[1] Remember the Euclidean relation: $\vec{A}\cdot\vec{B} = ||A||\cdot||B||\cos(\psi)$

The Determinant of $g_{\mu\nu}$

The quantity $g \equiv \det |g_{\mu\nu}|$ is the determinant of the metric tensor. The determinant transforms as :

$$\begin{aligned} \tilde{g} &= \det \tilde{g}_{\mu\nu} = \det \left(g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} \right) = \det g_{\alpha\beta} \cdot \det \left(\frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \right) \cdot \det \left(\frac{\partial x^\beta}{\partial \tilde{x}^\nu} \right) \\ &= \left(\det \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \right)^2 g = \mathcal{J}^2 g \end{aligned} \tag{55}$$

where $\mathcal{J}$ is the Jacobian of the transformation.

This relation can be written also as:

$$\sqrt{|\tilde{g}|} = \mathcal{J} \sqrt{|g|} \tag{56}$$

i.e. the quantity $\sqrt{|g|}$ is a **scalar density** of weight 1.

- The quantity

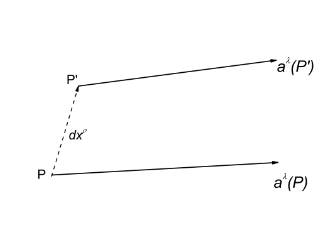
$$\sqrt{|g|} \delta V \equiv \sqrt{|g|} dx^1 dx^2 \ldots dx^n \tag{57}$$

is the **invariant volume element** of the Riemannian space.

- If the determinant vanishes at a point P the invariant volume is zero and this point will be called a **singular** point.

Christoffel Symbols

In Riemannian space there is a special connection derived directly from the metric tensor. This is based on a suggestion originating from Euclidean geometry that is :
" If a vector a^{λ} is given at some point P, its length must remain unchanged under parallel transport to neighboring points P'".



$$|\vec{a}|^2_P = |\vec{a}|^2_{P'} \quad \text{or} \quad g_{\mu\nu}(P)a^{\mu}(P)a^{\nu}(P) = g_{\mu\nu}(P')a^{\mu}(P')a^{\nu}(P') \qquad (58)$$

Since the distance between P and P' is $|dx^{\rho}|$ we can get

$$g_{\mu\nu}(P') \approx g_{\mu\nu}(P) + g_{\mu\nu,\rho}(P)dx^{\rho} \qquad (59)$$

$$a^{\mu}(P') \approx a^{\mu}(P) - \Gamma^{\mu}_{\sigma\rho}(P)a^{\sigma}(P)dx^{\rho} \qquad (60)$$

By substituting these two relation into equation (58) we get (how?)

$$\left(g_{\mu\nu,\rho} - g_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\rho} - g_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\rho}\right) a^{\mu} a^{\nu} dx^{\rho} = 0 \tag{61}$$

This relations must by valid for any vector a^{ν} and any displacement dx^{ν} which leads to the conclusion the relation in the parenthesis is zero. Closer observation shows that this is the covariant derivative of the metric tensor !

$$g_{\mu\nu;\rho} = g_{\mu\nu,\rho} - g_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\rho} - g_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\rho} = 0\,. \tag{62}$$

i.e. $g_{\mu\nu}$ is **covariantly constant**.

This leads to a unique determination of the connections of the space (Riemannian space) which will have the form (why?)

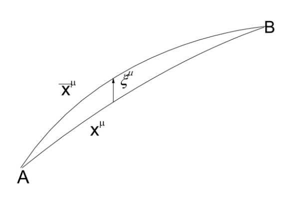
$$\boxed{\Gamma^{\alpha}{}_{\mu\rho} = \frac{1}{2} g^{\alpha\nu}\left(g_{\mu\nu,\rho} + g_{\nu\rho,\mu} - g_{\rho\mu,\nu}\right)} \tag{63}$$

and will be called **Christoffel Symbols** .

It is obvious that $\Gamma^{\alpha}{}_{\mu\rho} = \Gamma^{\alpha}{}_{\rho\mu}$.

The geodesics of a Riemannian space have the following important property. If a geodesic is connecting two points A and B is distinguished from the neighboring lines connecting these points as the line of minimum or maximum length. The length of a curve, $x^\mu(s)$, connecting A and B is:

$$S = \int_A^B ds = \int_A^B \left[g_{\mu\nu}(x^\alpha) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right]^{1/2} ds$$



A neighboring curve $\tilde{x}^\mu(s)$ connecting the same points will be described by the equation:

$$\tilde{x}^\mu(s) = x^\mu(s) + \epsilon\, \xi^\mu(s) \tag{64}$$

where $\xi^\mu(A) = \xi^\mu(B) = 0$.

Geodesics in a Riemann Space ii

The length of the new curve will be:

$$\tilde{S} = \int_A^B \left[g_{\mu\nu}(\tilde{x}^\alpha) \frac{d\tilde{x}^\mu}{ds} \frac{d\tilde{x}^\nu}{ds} \right]^{1/2} ds \tag{65}$$

For simplicity we set:
$f(x^\alpha, u^\alpha) = [g_{\mu\nu}(x^\alpha) u^\mu u^\nu]^{1/2}$ and $\tilde{f}(\tilde{x}^\alpha, \tilde{u}^\alpha) = [g_{\mu\nu}(\tilde{x}^\alpha) \tilde{u}^\mu \tilde{u}^\nu]^{1/2}$

$$u^\mu = \dot{x}^\mu = dx^\mu/ds \quad \text{and} \quad u^\mu = \dot{\tilde{x}}^\mu = d\tilde{x}^\mu/ds = \dot{x} + \epsilon \dot{\xi}^\mu \tag{66}$$

We can create the difference $\delta S = \tilde{S} - S$ and by making use of the relations:

$$\begin{aligned}
f(\tilde{x}^\alpha, \tilde{u}^\alpha) &= f(x^\alpha + \epsilon \xi^\alpha, u^\alpha + \epsilon \dot{\xi}^\alpha) \\
&= f(x^\alpha, u^\alpha) + \epsilon \left(\xi^\alpha \frac{\partial f}{\partial x^\alpha} + \dot{\xi}^\alpha \frac{\partial f}{\partial u^\alpha} \right) + O(\epsilon^2) \\
\frac{d}{ds}\left(\frac{\partial f}{\partial u^\alpha} \xi^\alpha \right) &= \frac{\partial f}{\partial u^\alpha} \dot{\xi}^\alpha + \frac{d}{ds}\left(\frac{\partial f}{\partial u^\alpha} \right) \xi^\alpha
\end{aligned}$$

we get

$$\tilde{f} - f = \epsilon \left[\frac{\partial f}{\partial x^\alpha} - \frac{d}{ds}\left(\frac{\partial f}{\partial u^\alpha}\right)\right]\xi^\alpha + \epsilon \frac{d}{ds}\left(\frac{\partial f}{\partial u^\alpha}\xi^\alpha\right) \tag{67}$$

Thus

$$\begin{aligned}\delta S &= \int_A^B \delta f\, ds = \int_A^B \left(\tilde{f} - f\right) ds \\ &= \epsilon \int_A^B \left[\frac{\partial f}{\partial x^\alpha} - \frac{d}{ds}\left(\frac{\partial f}{\partial u^\alpha}\right)\right]\xi^\alpha ds + \epsilon \int_A^B \frac{d}{ds}\left(\frac{\partial f}{\partial u^\alpha}\xi^\alpha\right) ds\end{aligned}$$

The last term does not contribute and the condition for the length of S to be an extremum will be expressed by the relation:

$$\delta S = \epsilon \int_A^B \left[\frac{\partial f}{\partial x^\alpha} - \frac{d}{ds}\left(\frac{\partial f}{\partial u^\alpha}\right)\right]\xi^\alpha ds = 0 \tag{68}$$

Since ξ^α is arbitrary, we must have for each point of the curve:

$$\frac{d}{ds}\left(\frac{\partial f}{\partial u^\mu}\right) - \frac{\partial f}{\partial x^\mu} = 0 \tag{69}$$

Notice that the Langrangian of a freely moving particle with mass $m = 2$, is: $\mathcal{L} = g_{\mu\nu} u^\mu u^\nu \equiv f^2$ this leads to the following relations

$$\frac{\partial f}{\partial u^\alpha} = \frac{1}{2}\frac{\partial \mathcal{L}}{\partial u^\alpha}\mathcal{L}^{-1/2} \quad \text{and} \quad \frac{\partial f}{\partial x^\alpha} = \frac{1}{2}\frac{\partial \mathcal{L}}{\partial x^\alpha}\mathcal{L}^{-1/2} \tag{70}$$

and by substitution in (69) we come to the condition for extremum of the distance (Euler-Lagrange)

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial u^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0\,. \tag{71}$$

Since $\mathcal{L} = g_{\mu\nu} u^\mu u^\nu$ we get:

$$\frac{\partial \mathcal{L}}{\partial u^\alpha} = g_{\mu\nu}\frac{\partial u^\mu}{\partial u^\alpha}u^\nu + g_{\mu\nu}u^\mu\frac{\partial u^\nu}{\partial u^\alpha} = g_{\mu\nu}\delta^\mu{}_\alpha u^\nu + g_{\mu\nu}u^\mu\delta^\nu{}_\alpha = 2g_{\mu\alpha}u^\mu \tag{72}$$

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = g_{\mu\nu,\alpha}u^\mu u^\nu \tag{73}$$

Geodesics in a Riemann Space v

thus

$$\frac{d}{ds}\left(2g_{\mu\alpha}u^{\mu}\right) = 2\frac{dg_{\mu\alpha}}{ds}u^{\mu} + 2g_{\mu\alpha}\frac{du^{\mu}}{ds} = 2g_{\mu\alpha,\nu}u^{\nu}u^{\mu} + 2g_{\mu\alpha}\frac{du^{\mu}}{ds}$$

$$= g_{\mu\alpha,\nu}u^{\nu}u^{\mu} + g_{\nu\alpha,\mu}u^{\mu}u^{\nu} + 2g_{\mu\alpha}\frac{du^{\mu}}{ds} \quad (74)$$

and by substitution (73) and (74) in (71) we get

$$g_{\mu\alpha}\frac{du^{\mu}}{ds} + \frac{1}{2}\left[g_{\mu\alpha,\nu} + g_{\alpha\mu,\nu} - g_{\mu\nu,\alpha}\right]u^{\mu}u^{\nu} = 0$$

if we multiply with $g^{\rho\alpha}$ the geodesic equations

$$\frac{du^{\rho}}{ds} + \Gamma^{\rho}_{\mu\nu}u^{\mu}u^{\nu} = 0\,, \quad \text{or} \quad u^{\rho}{}_{;\nu}u^{\nu} = 0 \quad (75)$$

because $du^{\rho}/ds = u^{\rho}{}_{,\nu}u^{\nu}$.

Euler-Lagrange Eqns vs Geodesic Eqns

By setting the Lagrangian for a freely moving particle to be $\mathcal{L} = g_{\mu\nu} u^\mu u^\nu$ we get the Euler-Lagrange equations

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial u^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0$$

which are equivalent to the geodesic equations

$$\frac{du^\rho}{ds} + \Gamma^\rho_{\mu\nu} u^\mu u^\nu = 0 \quad \text{or} \quad \frac{d^2 x^\rho}{ds^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0\,.$$

Notice that if the metric tensor does not depend from a specific coordinate e.g. x^κ then

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial u^\kappa}\right) = 0$$

which means that the quantity $\partial \mathcal{L}/\partial u^\kappa$ is constant along the geodesic.

Then eq (72) implies that $\frac{\partial \mathcal{L}}{\partial u^\kappa} = g_{\mu\kappa} u^\mu$ that is the κ component of the generalized momentum $p_\kappa = g_{\mu\kappa} u^\mu$ remains constant along the geodesic.

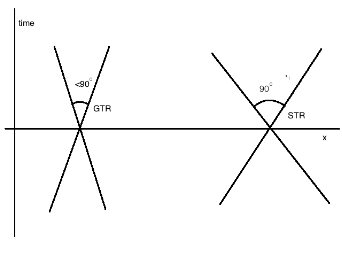
Types of Geodesics

If we know the tangent vector u^ρ at a given point of a known space we can determine the geodesic curve.

Which will be characterized as:

- **timelike** if $|\vec{u}|^2 > 0$
- **null** if $|\vec{u}|^2 = 0$
- **spacelike** if $|\vec{u}|^2 < 0$

where $|\vec{u}|^2 = g_{\mu\nu} u^\mu u^\nu$



If $g_{\mu\nu} \neq \eta_{\mu\nu}$ then the light cone is affected by the curvature of the spacetime. For example, in a space with metric $ds^2 = -f(t,x)dt^2 + g(t,x)dx^2$ the light cone will be drawn from the relation $dt/dx = \pm\sqrt{g/f}$ which leads to STR results for f, $g \to 1$.

Null Geodesics

For null geodesics $ds = 0$ and the proper length s cannot be used to parametrize the geodesic curves.

Instead we will use another parameter λ and the equations will be written as:

$$\frac{d^2 x^\kappa}{d\lambda^2} + \Gamma^\kappa_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \tag{76}$$

and obiously:

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0\,. \tag{77}$$

Geodesic Eqns & Affine Parameter

In deriving the geodesic equations we have chosen to parametrize the curve via the proper length s. This choice simplifies the form of the equation but it is not a unique choice. If we chose a new parameter, $\sigma = \sigma(s)$ then the geodesic equations will be written:

$$\frac{d^2 x^\mu}{d\sigma^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} = -\frac{d^2\sigma/ds^2}{(d\sigma/ds)^2} \frac{dx^\mu}{d\sigma} \tag{78}$$

where we have used

$$\frac{dx^\mu}{ds} = \frac{dx^\mu}{d\sigma}\frac{d\sigma}{ds} \quad \text{and} \quad \frac{d^2 x^\mu}{ds^2} = \frac{d^2 x^\mu}{d\sigma^2}\left(\frac{d\sigma}{ds}\right)^2 + \frac{dx^\mu}{d\sigma}\frac{d^2\sigma}{ds^2} \tag{79}$$

The new geodesic equation (78), reduces to the original equation (76) when the right hand side is zero. This is possible if

$$\frac{d^2\sigma}{ds^2} = 0 \tag{80}$$

which leads to a linear relation between s and σ i.e. $\sigma = \alpha s + \beta$ where α and β are arbitrary constants. σ is called **affine parameter**.

Riemann - Ricci & Einstein Tensors i

When in a space we define a metric then is called **metric space** or **Riemann space**. For such a space the curvature tensor

$$\boxed{R^{\lambda}{}_{\beta\nu\mu} = -\Gamma^{\lambda}_{\beta\nu,\mu} + \Gamma^{\lambda}_{\beta\mu,\nu} - \Gamma^{\sigma}_{\beta\nu}\Gamma^{\lambda}_{\sigma\mu} + \Gamma^{\sigma}_{\beta\mu}\Gamma^{\lambda}_{\sigma\nu}} \tag{81}$$

is called **Riemann Tensor** and can be also written as:

$$\begin{aligned} R_{\kappa\beta\nu\mu} = g_{\kappa\lambda} R^{\lambda}{}_{\beta\nu\mu} &= \frac{1}{2}\left(g_{\kappa\mu,\beta\nu} + g_{\beta\nu,\kappa\mu} - g_{\kappa\nu,\beta\mu} - g_{\beta\mu,\kappa\nu}\right) \\ &+ g_{\alpha\rho}\left(\Gamma^{\alpha}_{\kappa\mu}\Gamma^{\rho}_{\beta\nu} - \Gamma^{\alpha}_{\kappa\nu}\Gamma^{\rho}_{\beta\mu}\right) \end{aligned}$$

- **Properties of the Riemann Tensor**:

$$R_{\kappa\beta\nu\mu} = -R_{\kappa\beta\mu\nu}\ ,\ R_{\kappa\beta\nu\mu} = -R_{\beta\kappa\nu\mu}\ ,\ R_{\kappa\beta\nu\mu} = R_{\nu\mu\kappa\beta}\ ,\ R_{\kappa[\beta\mu\nu]} = 0$$

Thus in an n-dim space the number of independent components is (how?):

$$n^2(n^2-1)/12 \tag{82}$$

Riemann - Ricci & Einstein Tensors ii

Thus in a 4-dimensional space the Riemann tensor has only 20 independent components.

- The contraction of the Riemann tensor leads to **Ricci Tensor**

$$\begin{aligned} R_{\alpha\beta} &= R^{\lambda}{}_{\alpha\lambda\beta} = g^{\lambda\mu} R_{\lambda\alpha\mu\beta} \\ &= \Gamma^{\mu}_{\alpha\beta,\mu} - \Gamma^{\mu}_{\alpha\mu,\beta} + \Gamma^{\mu}_{\alpha\beta}\Gamma^{\nu}_{\nu\mu} - \Gamma^{\mu}_{\alpha\nu}\Gamma^{\nu}_{\beta\mu} \end{aligned} \tag{83}$$

which is symmetric i.e. $R_{\alpha\beta} = R_{\beta\alpha}$.

- Further contraction leads to the **Ricci or Curvature Scalar**

$$R = R^{\alpha}{}_{\alpha} = g^{\alpha\beta} R_{\alpha\beta} = g^{\alpha\beta} g^{\mu\nu} R_{\mu\alpha\nu\beta} \,. \tag{84}$$

- The following combination of Riemann and Ricci tensors is called **Einstein Tensor**

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R} \tag{85}$$

with the very important property:

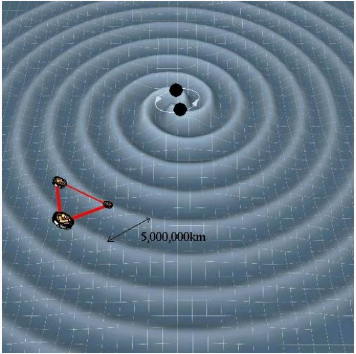
$$\boxed{G^{\mu}{}_{\nu;\mu} = \left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R\right)_{;\mu} = 0\,.} \tag{86}$$

This results from the **Bianchi Identity** (how?)

$$\boxed{R^{\lambda}{}_{\mu[\nu\rho;\sigma]} = 0} \tag{87}$$

Flat & Empty Spacetimes

- When $R_{\alpha\beta\mu\nu} = 0$ the spacetime is flat
- When $R_{\mu\nu} = 0$ the spacetime is empty



Prove that :

$$a^{\lambda}{}_{;\mu;\nu} - a^{\lambda}{}_{;\nu;\mu} = -R^{\lambda}{}_{\kappa\mu\nu} a^{\kappa}$$

Weyl Tensor i

Important relations can be obtained when we try to express the Riemann or Ricci tensor in terms of **trace-free quantities**.

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \quad \Rightarrow \quad S = S^{\mu}{}_{\mu} = g^{\mu\nu} S_{\mu\nu} = 0 \,. \tag{88}$$

or the **Weyl** tensor $C_{\lambda\mu\nu\rho}$:

$$\begin{aligned} R_{\lambda\mu\nu\rho} = C_{\lambda\mu\nu\rho} \quad &+ \quad \frac{1}{2}\left(g_{\lambda\rho} S_{\mu\nu} + g_{\mu\nu} S_{\lambda\rho} - g_{\lambda\nu} S_{\mu\rho} - g_{\mu\rho} S_{\lambda\nu}\right) \\ &+ \quad \frac{1}{12} R \left(g_{\lambda\rho} g_{\mu\nu} - g_{\lambda\nu} g_{\mu\rho}\right) \end{aligned} \tag{89}$$

$$\begin{aligned} R_{\lambda\mu\nu\rho} = C_{\lambda\mu\nu\rho} \quad &- \quad \frac{1}{2}\left(g_{\lambda\rho} R_{\mu\nu} + g_{\mu\nu} R_{\lambda\rho} - g_{\lambda\nu} R_{\mu\rho} - g_{\mu\rho} R_{\lambda\nu}\right) \\ &- \quad \frac{1}{12} R \left(g_{\lambda\rho} g_{\mu\nu} - g_{\lambda\nu} g_{\mu\rho}\right) \,. \end{aligned} \tag{90}$$

and we can prove (how?) that :

$$g^{\lambda\rho} C_{\lambda\mu\rho\nu} = 0 \tag{91}$$

Weyl Tensor ii

- The Weyl tensor is the trace-free part of the Riemann tensor. In the absence of sources, the trace part of the Riemann tensor will vanish due to the Einstein equations, but the Weyl tensor can still be non-zero.
This is the case for gravitational waves propagating in vacuum.

- The Weyl tensor expresses the tidal force (as the Riemann curvature tensor does) that a body feels when moving along a geodesic.
The Weyl tensor differs from the Riemann curvature tensor in that it does not convey information on how the volume of the body changes, but rather only how the shape of the body is distorted by the tidal force.

- The Weyl tensor is called also **conformal curvature tensor** because it has the following property :

If we consider besides the Riemannian space M with metric $g_{\mu\nu}$ a second Riemannian space $\tilde{M}$ with metric

$$\tilde{g}_{\mu\nu} = e^{2A} g_{\mu\nu}$$

where A is a function of the coordinates. The space $\tilde{M}$ is said to be **conformal** to M.

One can prove that

$$R^{\alpha}_{\beta\gamma\delta} \neq \tilde{R}^{\alpha}_{\beta\gamma\delta} \quad \text{while} \quad C^{\alpha}_{\beta\gamma\delta} = \tilde{C}^{\alpha}_{\beta\gamma\delta} \tag{92}$$

i.e. a "conformal transformation" does not change the Weyl tensor.

Weyl Tensor iv

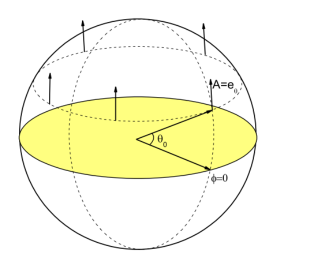
- It can be verified at once from equation (90) that the Weyl tensor has the same symmetries as the Riemann tensor. Thus it should have 20 independent components, but because it is traceless [condition (91)] there are 10 more conditions for the components therefore the Weyl tensor has **10** independent components. In general, the number of independent components is given by

$$\frac{1}{12}n(n+1)(n+2)(n-3) \tag{93}$$

- In 2D and 3D spacetimes the Weyl curvature tensor vanishes identically. Only for dimensions ≥ 4, the Weyl curvature is generally nonzero.
- If in a spacetime with ≥ 4 dimensions the Weyl tensor vanishes then the metric is locally conformally flat, i.e. there exists a local coordinate system in which the metric tensor is proportional to a constant tensor.
- On the symmetries of the Weyl tensor is based the Petrov classification of the space times (any volunteer?).

Tensors : An example for parallel transport i

A vector $\vec{A} = A^\theta \vec{e}_\theta + A^\phi \vec{e}_\phi$ is parallel transported along a closed line on the surface of a sphere with metric $ds^2 = d\theta^2 + sin^2\theta d\phi^2$ and Christoffel symbols $\Gamma^1_{22} = \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$ and $\Gamma^2_{12} = \Gamma^\phi_{\theta\phi} = \cot\theta$.



The eqns $\delta A^\alpha = -\Gamma^\alpha_{\mu\nu} A^\mu dx^\nu$ for parallel transport will be written as:

$$\frac{\partial A^1}{\partial x^2} = -\Gamma^1_{22} A^2 \Rightarrow \frac{\partial A^\theta}{\partial \phi} = \sin\theta\cos\theta A^\phi$$

$$\frac{\partial A^2}{\partial x^2} = -\Gamma^2_{12} A^1 \Rightarrow \frac{\partial A^\phi}{\partial \phi} = -\cot\theta A^\theta$$

The solutions will be:

$$\frac{\partial^2 A^\theta}{\partial\phi^2} = -\cos^2\theta A^\theta \quad \Rightarrow \quad A^\theta = \alpha\cos(\phi\cos\theta) + \beta\sin(\phi\cos\theta)$$

$$\Rightarrow \quad A^\phi = -\left[\alpha\sin(\phi\cos\theta) - \beta\cos(\phi\cos\theta)\right]\sin^{-1}\theta$$

and for an initial unit vector $(A^\theta, A^\phi) = (1, 0)$ at $(\theta, \phi) = (\theta_0, 0)$ the integration constants will be $\alpha = 1$ and $\beta = 0$.

Thus $A^\theta = \cos(\phi\cos\theta)$ and $A^\phi = -\sin(\phi\cos\theta)$

- The solution is:

$$\vec{A} = A^\theta\vec{e}_\theta + A^\phi\vec{e}_\phi = \cos(2\pi\cos\theta)\vec{e}_\theta - \frac{\sin(2\pi\cos\theta)}{\sin\theta}\vec{e}_\phi$$

Tensors : An example for parallel transport iii

- i.e. different components but the measure is still the same

$$
\begin{aligned}
|\vec{A}|^2 &= g_{\mu\nu} A^\mu A^\nu = \left(A^\theta\right)^2 + \sin^2\theta \left(A^\phi\right)^2 \\
&= \cos^2(2\pi\cos\theta) + \sin^2\theta \frac{\sin^2(2\pi\cos\theta)}{\sin^2\theta} = 1
\end{aligned}
$$

Question : What is the condition for the path followed by the vector to be a geodesic?

Extension

1-forms (*) i

- A 1-form can be defined as a linear, real valued function of vectors. [2]

- A 1-form $\tilde{\omega}$ at a point P associates with a vector $\mathbf{v}$ at P a **real number**, which maybe called $\tilde{\omega}(\mathbf{v})$. We may say that $\tilde{\omega}$ is a function of vectors while the linearity of this function means:

  - $\tilde{\omega}(a\mathbf{v} + b\mathbf{w}) = a\tilde{\omega}(\mathbf{v}) + b\tilde{\omega}(\mathbf{w}) \quad a, b \in \mathbb{R}$
  - $(a\tilde{\omega})(\mathbf{v}) = a[\tilde{\omega}(\mathbf{v})]$
  - $(\tilde{\omega} + \tilde{\sigma})(\mathbf{v}) = \tilde{\omega}(\mathbf{v}) + \tilde{\sigma}(\mathbf{v})$

- Thus 1-forms at a point P satisfy the axioms of a vector space, which is called the **dual space** to T_P, and is denoted by T_P^*.

- The linearity also allows to consider the vector as function of 1-forms: Thus vectors and 1-forms are thus said to be **dual** to each other.

1-forms (*) ii

- Their value on one another can be represented as follows:

$$\tilde{\omega}(\mathbf{v}) \equiv \mathbf{v}(\tilde{\omega}) \equiv <\tilde{\omega}, \mathbf{v}> \ . \tag{94}$$

- We may introduce a set of basis 1-forms $\tilde{\omega}^{\alpha}$ **dual** to the basis vectors $\mathbf{e}_{\alpha}$.
- An arbitrary 1-form $\tilde{\mathbf{B}}$ can be expanded in its covariant components according to

$$\tilde{\mathbf{B}} = B_{\alpha}\tilde{\omega}^{\alpha} \, . \tag{95}$$

- The scalar product of two 1-forms $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ is

$$\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}} = (A_{\alpha}\tilde{\omega}^{\alpha}) \cdot \left(B_{\beta}\tilde{\omega}^{\beta}\right) = g^{\alpha\beta}A_{\alpha}B_{\beta} \quad \text{where} \quad g^{\alpha\beta} = \tilde{\omega}^{\alpha} \cdot \tilde{\omega}^{\beta} \, . \tag{96}$$

- A basis of 1-forms dual to the basis $\mathbf{e}_{\alpha}$ always satisfies

$$\tilde{\omega}^{\alpha} \cdot \mathbf{e}_{\beta} = \delta^{\alpha}{}_{\beta} \, . \tag{97}$$

1-forms (*) iii

- The scalar product of a vector with a 1-form does not involve the metric, but only a summation over an index

$$\mathbf{A} \cdot \tilde{\mathbf{B}} = (A^{\alpha} \mathbf{e}_{\alpha}) \cdot \left(B_{\beta} \tilde{\omega}^{\beta} \right) = A^{\alpha} \delta_{\alpha}{}^{\beta} B_{\beta} = A^{\alpha} B_{\alpha} \,. \tag{98}$$

A vector $\mathbf{A}$ carries the same information as the corresponding 1- form $\tilde{\mathbf{A}}$, and we often make no distinction between them, recall that :

$$A_{\alpha} = g_{\alpha\beta} A^{\beta} \quad \text{and} \quad A^{\alpha} = g^{\alpha\beta} A_{\beta} \,. \tag{99}$$

A coordinate basis of 1-forms may be written $\tilde{\omega}^{\alpha} = \tilde{\mathbf{d}} x^{\alpha}$, remember that $\mathbf{e}_{\alpha} = \partial / \partial x^{\alpha} \equiv \partial_{\alpha}$.

Geometrically the basis form $\tilde{\mathbf{d}} x^{\alpha}$ may be thought of as surfaces of constant x^{α}.

- An orthonormal basis $\tilde{\omega}^{\hat{\alpha}}$ satisfies the relation

$$\tilde{\omega}^{\hat{\alpha}} \cdot \tilde{\omega}^{\hat{\beta}} = \eta^{\hat{\alpha}\hat{\beta}} \,. \tag{100}$$

1-forms (*) iv

- The gradient, $\tilde{\mathbf{d}}f$, of an arbitrary scalar function f is particularly useful 1-form.
- In a coordinate basis, it may be expanded according to $\tilde{\mathbf{d}}f = \partial_\alpha f \tilde{\mathbf{d}x}^\alpha$ (its components are ordinary partial derivatives).
- The scalar product between an arbitrary vector $\mathbf{v}$ and the 1-form $\tilde{\mathbf{d}}f$ gives the directional derivative of f along $\mathbf{v}$

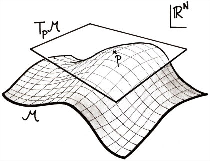
$$\mathbf{v} \cdot \tilde{\mathbf{d}}f = (v^\alpha \mathbf{e}_\alpha) \cdot \left(\partial_\beta f \tilde{\mathbf{d}x}^\beta\right) = v^\alpha \partial_\alpha f \, . \tag{101}$$

[2] See B. F. Schutz "Geometrical methods of mathematical physics" Cambridge, 1980.

Based on the previous, any 4-vector **A** can be expanded in contravariant components [3]

$$\mathbf{A} = A^{\alpha}\mathbf{e}_{\alpha} \,. \tag{102}$$

Here the four basis vectors $\mathbf{e}_{\alpha}$ span the *vector space* $T_P\mathcal{M}$ tangent to the spacetime manifold $\mathcal{M}$



1-forms or co-vectors (*) ii

and

$$g_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta \,. \tag{103}$$

In a coordinate basis, the basis vectors are *tangent vectors* to coordinate lines and can be written as $\mathbf{e}_\alpha = \partial/\partial x^\alpha \equiv \partial_\alpha$. The coordinate vectors commute.

We may also set an orthonormal basis vectors at a point (orthonormal tetrad) for which

$$\mathbf{e}_{\hat{\alpha}} \cdot \mathbf{e}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} \,. \tag{104}$$

- In general, orthonormal basis vectors do not form a coordinate basis and do not commute.
- In a flat spacetime it is always possible to transform to coordinates which are everywhere orthonormal i.e. $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$.
- **For a genaral spacetime this is not possible**, but one may select an event in the spacetime to be the origin of a local inertial coordinate frame, where

1-forms or co-vectors (*) iii

$g_{\alpha\beta} = \eta_{\alpha\beta}$ at this point and in addition the 1st derivatives of the metric tensor at that point vanish, i.e. $\partial_\gamma g_{\alpha\beta} = 0$.

- An observer in such a coordinate frame is called a local inertial and can use a coordinate basis that forms that forms a local orthonormal tetrad to make measurements in SR.
- Such n observer will find that all the (non-gravitational) laws of physics in this frame are the same as in special relativity.
- The scalar product of two 4-vectors **A** and **B** is

$$\mathbf{A} \cdot \mathbf{B} = (A^\alpha \mathbf{e}_\alpha) \cdot \left(B^\beta \mathbf{e}_\beta\right) = g_{\alpha\beta} A^\alpha B^\beta \,. \tag{105}$$

[3] Baumgarte-Shapiro "Numerical Relativity", Cambridge 2010

Lie Derivatives, Isometries & Killing Vectors i

Up to now we consider coordinate transformations $x^\mu \to \tilde{x}^\mu$ with the following meaning: to the point P with initial coordinate values x^μ we assign new coordinates $\tilde{x}^\mu$ determined from x^μ via the n functions

$$\tilde{x}^\mu = f^\mu(x^\alpha)\,,\; \mu = 1 \ldots n \tag{106}$$

We shall give to the transformation $x^\mu \to \tilde{x}^\mu$ the following meaning:
To the point P having the coordinate values x^μ we correspond another point Q of the same space having the coordinate values $\tilde{x}^\mu$ **in the same coordinate system**.

This operations is a **mapping of the space into itself**.

In the infinitesimal mapping

$$\tilde{x}^\mu = x^\mu + \epsilon\xi^\mu \tag{107}$$

where $\xi^\mu = \xi^\mu(x^\alpha)$ is a given vector field and ϵ an infinitesimal parameter.

The meaning is: in the point P with coordinate values x^μ we correspond the point Q with coordinate values $x^\mu + \epsilon\xi^\mu$ (in the same coordinate system).

Thus we can define the difference between two tensors defined at the point Q this will be the **Lie derivative**.

The first tensor is the one defined by the field at Q while the second will be the one defined if we "take over" the value of the field at P and map it via (107) into the point Q.

Lie Derivatives, Isometries & Killing Vectors iii

The Lie derivative of a Scalar Field

Lets assume a scalar field at a point P i.e. $\phi_P = \phi(x^\mu)$, since it is a scalar we postulate that the mapping $P \to Q$ will leave it unchanged.

Thus we have at Q two scalars

$$\phi_Q = \phi(\tilde{x}^\mu) + \epsilon \phi_{,\mu} \xi^\mu \quad \text{and} \quad \phi_{P\to Q} = \phi_P \tag{108}$$

The Lie derivative of the scalar ϕ with respect to ξ^μ is defined as follows:

$$\mathcal{L}_\xi \phi = \lim_{\epsilon \to 0} \frac{\phi_Q - \phi_{P\to Q}}{\epsilon} \tag{109}$$

therefore due to (108) we get:

$$\mathcal{L}_\xi \phi = \phi_{,\mu} \xi^\mu \tag{110}$$

The Lie derivative of a Contravariant Vector Field

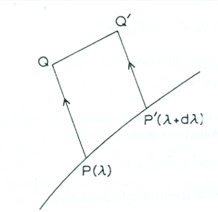
Let's assume that the contravariant vector $k^\mu = k^\mu(x^\alpha)$ is the tangent vector to a curve passing from the point at P.
If this curve is described by a parameter λ we have:

$$k^\mu = \frac{dx^\mu}{d\lambda} \tag{111}$$

where dx^μ are the components of the vector PP'. If the points Q and Q' correspond to P and P' via the mapping (107) then the components of the vector QQ' will be

$$\delta x^\mu = dx^\mu + \epsilon\xi^\mu_{\mathrm{P}'} - \epsilon\xi^\mu_{\mathrm{P}} = dx^\mu + \epsilon\xi^\mu{}_{,\nu} dx^\nu \tag{112}$$

Lie Derivatives, Isometries & Killing Vectors v



The transport of the vector k^μ from P to Q by the mapping (107) will be:

$$k^\mu_{P\to Q} = \frac{\delta x^\mu}{d\lambda} = k^\mu_P + \epsilon \xi^\mu{}_{,\nu} k^\nu \tag{113}$$

and the definition of the Lie derivative of the vector k^μ is:

$$\mathcal{L}_\xi k^\mu = \lim_{\epsilon\to 0} \frac{k^\mu_Q - k^\mu_{P\to Q}}{\epsilon} \tag{114}$$

and since

$$k^\mu_Q = k^\mu(\tilde{x}^\alpha) = k^\mu_P + \epsilon k^\mu{}_{,\nu}\xi^\nu$$

we finally get

$$\mathcal{L}_\xi k^\mu = k^\mu{}_{,\nu}\xi^\nu - \xi^\mu{}_{,\nu}k^\nu \,. \tag{115}$$

The Lie derivative of any other tensor $T^{\cdots}_{\cdots}$ will be defined as

$$\mathcal{L}_\xi T^{\cdots}_{\cdots} = \lim_{\epsilon\to 0} \frac{(T^{\cdots}_{\cdots})_Q - (T^{\cdots}_{\cdots})_{P\to Q}}{\epsilon} \tag{116}$$

The detailed formulae will be derived from the formulae that we have already derived for the scalar and the contravariant vector.

The Lie derivative of a Covariant Vector Field

For a covariant vector field p_μ with the help of a contravariant vector k^μ we get

$$\mathcal{L}_\xi (p_\mu k^\mu) = \mathcal{L}_\xi p_\mu k^\mu + p_\mu \mathcal{L}_\xi k^\mu \tag{117}$$

For the left hand side due to (110) we get

$$\mathcal{L}_\xi (p_\mu k^\mu) = (p_\mu k^\mu)_{,\nu} \xi^\nu = (p_{\mu,\nu} k^\mu + p_\mu k^\mu{}_{,\nu}) \xi^\nu \tag{118}$$

Then by introducing in the last term of (117) the expression (115) we get:

$$k^\mu \left[\mathcal{L}_\xi p_\mu - p_{\mu,\nu}\xi^\nu - p_\nu \xi^\nu{}_{,\mu}\right] = 0 \tag{119}$$

and since k^μ is arbitrary we get:

$$\mathcal{L}_\xi p_\mu = p_{\mu,\nu}\xi^\nu + p_\nu \xi^\nu{}_{,\mu} \tag{120}$$

In a similar way we can find the formulae for the Lie derivative of any tensor

$$\mathcal{L}_\xi T_{\mu\nu} = T_{\mu\nu,\rho}\xi^\rho + T_{\rho\nu}\xi^\rho{}_{,\mu} + T_{\mu\rho}\xi^\rho{}_{,\nu} \tag{121}$$

The Lie derivative of the metric tensor

If we apply the formula (121) to the metric tensor we get:

$$\mathcal{L}_\xi g_{\mu\nu} = g_{\mu\nu,\rho}\xi^\rho + g_{\rho\nu}\xi^\rho{}_{,\mu} + g_{\mu\rho}\xi^\rho{}_{,\nu} \tag{122}$$

and if we use the relation $g_{\lambda\mu;\nu} = g_{\lambda\mu,\nu} - g_{\alpha\mu}\Gamma^\alpha_{\lambda\nu} - g_{\lambda\alpha}\Gamma^\alpha_{\mu\nu} = 0$ we get

$$\mathcal{L}_\xi g_{\mu\nu} = g_{\rho\nu}\xi^\rho{}_{;\mu} + g_{\mu\rho}\xi^\rho{}_{;\nu} \tag{123}$$

but since $g_{\rho\nu}\xi^\rho{}_{;\mu} = (g_{\rho\nu}\xi^\rho)_{;\mu} = \xi_{\nu;\mu}$ we get

$$\mathcal{L}_\xi g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} \tag{124}$$

QUESTION: If the neighboring points P and P' go over, under the mapping (107), to the points Q and Q' what is the condition ensuring that

$$ds^2_{PP'} = ds^2_{QQ'} \tag{125}$$

for any pair of points P and P'.

If such a mapping exist, it will be called **isometric mapping**.

We have

$$ds^2_{PP'} = (g_{\mu\nu})_P \, dx^\mu dx^\nu \tag{126}$$

But then according to the second of (108):

$$ds^2_{PP'} = (g_{\mu\nu} dx^\mu dx^\nu)_{P\to Q} = (g_{\mu\nu})_{P\to Q} \delta x^\mu \delta x^\nu \tag{127}$$

where dx^μ are the components of $\mathbf{PP}'$ and δx^μ the components of $\mathbf{QQ}'$.

On the other side

$$ds^2_{QQ'} = (g_{\mu\nu})_Q \, \delta x^\mu \delta x^\nu \tag{128}$$

Isometries & Killing Vectors ii

Therefore the condition for the validity of (125) is

$$(g_{\mu\nu})_{P\to Q} = (g_{\mu\nu})_Q \,.$$

and according to the definition (116) of the Lie derivative:

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \tag{129}$$

Therefore we found that the condition for the existence of isometric mappings is the existence of solutions ξ^μ of the equation

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \tag{130}$$

This is the **Killing equation** and the vectors ξ^μ which satisfy it are called **Killing vectors**.

The existence of a Killing vector, i.e. of an isometric mapping of the space onto itself, it is an expression of a certain intrinsic symmetry property of the space.

Isometries & Killing Vectors iii

According to Noether's theorem, to every continuous symmetry of a physical system corresponds a conservation law.

- Symmetry under spatial translations $\rightarrow$ conservation of momentum
- Symmetry under time translations $\rightarrow$ conservation of energy
- Symmetry under rotations $\rightarrow$ conservation of angular momentum

PROVE: If ξ^μ is a Killing vector, then, for a particle moving along a geodesic, the scalar product of this killing vector and the momentum $P^\mu = m\, dx^\mu / d\tau$ of the particle is a constant.