Physics on Curved Spaces[a]

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[a]Based on:
"General Relativity" M.P.Hobson, G. Efstahiou and A.N. Lasenby, Cambridge 2006 (Chapter 6)
"Gravity" E. Poisson, C.M. Will, Cambridge 2014

The electromagnetic (EM) force of a moving charge

In an inertial frame S, the 3-force on a particle of charge q moving in an EM field is:

$$\vec{f} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

This equation suggests that the force is proportional to the velocity and this means that the extension of this equation in a 4-dim spacetime should be of the form:

$$\mathbf{f} = q\mathbf{F} \cdot \mathbf{u} \quad \Rightarrow \quad f_\mu = qF_{\mu\nu}u^\nu \tag{1}$$

where $\mathbf{F}$ (or $F_{\mu\nu}$) is the **electromagnetic field tensor**, the scalar q is some property of the particle that determines the strength of the EM force upon it (i.e. the **charge**).

In order that the rest mass of a particle is not altered by the action of the EM force we require the latter to be a pure force or rest-mass preserving [1]. This requires that $\mathbf{u} \cdot \mathbf{f} = 0$ or

$$f_\mu u^\mu = qF_{\mu\nu}u^\mu u^\nu = 0 \tag{2}$$

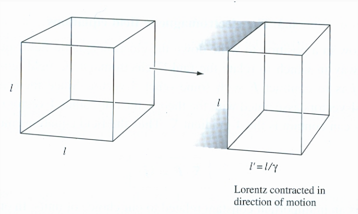
This implies that EM field tensor must be antisymmetric (why?)

$$F_{\mu\nu} = -F_{\nu\mu} \tag{3}$$

[1] A force which does not change the rest mass of the object on which it acts is called a *pure force* (see Appendix at the end)

The 4-current density i

- In order to construct the field equations of the theory which will determine $\mathbf{F}(x)$ at any point in spacetime in terms of charges and currents we must find a **covariant** way of expressing the **source term**.
- Let us consider some general time-dependent charge distribution. At each event P of the spacetime we give the charge density ρ and 3-velocity $\vec{u}$ as measured in some inertial frame e.g. the frame in which $\vec{u} = \vec{0}$ at P.
- In this frame the **proper charge density** is given by $\rho_0 = qn_0$, where q is the charge of each particle and n_0 is the number of particles in a unit volume.



The 4-current density ii

- In some other frame S' moving with speed u relative to S the volume will be **Lorentz contracted** along the direction of motion hence the number density of particles will be $n' = \gamma_u n_0$ and

$$\rho' = \gamma_u \rho_0 \tag{4}$$

i.e the charge density is not a scalar but transforms like the component of a vector.

Thus the obvious choice of a source term for the EM field equations should be a 4-vector

$$\mathbf{j} = \rho_0(x)\mathbf{u}(x) \quad \text{and} \quad \mathbf{j}\cdot\mathbf{j} = \rho_0^2 c^2 \tag{5}$$

In an inertial frame S the components of the 4-current density $\mathbf{j}$

$$j^\mu = \rho_0 \gamma_u (c, \vec{u}\,) = (c\rho, \vec{j}\,) \tag{6}$$

Thus we see that $c^2\rho^2 - j^2$ is a Lorentz invariant ($j^2 = \vec{j}\cdot\vec{j}$).

The 4-current density iii

NOTE:

Suppose that there is a reference frame in which there is a charge density but no currents.

In another moving reference frame there will be both a charge density

$$\tilde{j}^0 = \frac{\partial \tilde{x}^0}{\partial x^0} j^0$$

and a current density

$$\tilde{j}^k = \frac{\partial \tilde{x}^k}{\partial x^0} j^0$$

In other words the charge density of the first frame becomes a combination of the charge density and current density.

Any law that will be valid in all reference frames must include both interactions of charge densities and current densities.

If the charge density j^0 generates some kind of force field, so will the current density j^k.

The EM field equations

The general form of the field equations will be

$$\nabla \cdot \mathbf{F} = k\,\mathbf{j}\,.$$

In Cartesian inertial coordinates x^μ in some inertial frame S

$$F^{\mu\nu}{}_{,\mu} = k\,j^\nu\,. \qquad (7)$$

This equation can be used to find a law for conservation of charges

$$F^{\mu\nu}{}_{,\mu\nu} = k\,j^\nu{}_{,\nu} \quad \Rightarrow \quad j^\nu{}_{,\nu} = 0 \quad (\text{why?})^2 \qquad (8)$$

In a 3-vector notation we may write this in a well known form:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

which is similar to the non-relativistic equation of charge continuity, but we should use the relativistic expressions for ρ and $\vec{j}$.

[2] $$F^{\mu\nu}{}_{,\mu\nu} = -F^{\nu\mu}{}_{,\mu\nu} = -F^{\mu\nu}{}_{,\nu\mu} = -F^{\mu\nu}{}_{,\mu\nu} = 0$$

Do we have a viable theory? i

- The field equations are given by eqn (7). But $F_{\mu\nu}$ has 6 independent components and from eqn (7) we get only 4 field equations.
- Our theory is underdetermined as it stands and this suggests that one can use a 4-vector "potential" A_μ to construct $F_{\mu\nu}$.
- In Cartesian coordinates we may write:

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \tag{9}$$

i.e. $F_{\mu\nu}$ is antisymmetric by construction and contains ONLY 4 independent fields A_μ.

- Thus we can write the field equations (7) in terms of the 4-vector potential A_μ

$$\eta^{\mu\sigma} \left(A_{\lambda,\sigma\mu} - A_{\sigma,\lambda\mu}\right) = k\, j_\lambda \tag{10}$$

Do we have a viable theory? ii

Alternatively, we can express electromagnetism entirely in terms of the EM field tensor and in this case, we require the 2 field equations:

$$F^{\mu\nu}{}_{,\mu} = k\, j^{\mu} \qquad (11)$$

$$F_{[\mu\nu,\sigma]} \equiv F_{\mu\nu,\sigma} + F_{\sigma\mu,\nu} + F_{\nu\sigma,\mu} = 0 \qquad (12)$$

The last equation can be easily derived from equation (9) (how?).

Finally, the constant k can be found by demanding consistency with the standard Maxwell equations.

In SI units we have $k = \mu_0$, where $\epsilon_0\mu_0 = 1/c^2$.

Electromagnetism in the Lorenz gauge i

We can always add an arbitrary 4-vector Q_μ to the 4-potential A_μ. Thus in Cartesian inertial coordinates x^μ we have

$$A_\mu^{(new)} = A_\mu + Q_\mu \tag{13}$$

This is **not** a coordinate transformation. The new EM field tensor will be:

$$F_{\mu\nu}^{(new)} = A_{\nu,\mu}^{(new)} - A_{\mu,\nu}^{(new)} = A_{\nu,\mu} - A_{\mu,\nu} + Q_{\nu,\mu} - Q_{\mu,\nu} \tag{14}$$

and the original EM field tensor will be recovered if $Q_{\nu,\mu} = Q_{\mu,\nu}$.

This is possible if Q_μ is the gradient of a scaler field i.e. $Q_\mu = \psi_{,\mu}$.
This is called **gauge freedom** of the theory i.e. we are free to add the gradient of **any** scalar field ψ to the 4-vector potential A_μ i.e.

$$A_\mu^{(new)} = A_\mu + \psi_{,\mu} \tag{15}$$

and still recover the same EM field tensor and the same EM field equations.

Electromagnetism in the Lorenz gauge ii

This is an example of a **gauge transformation**, which is not a coordinate transformation.

In the field equations

$$\eta^{\mu\sigma}\left(A_{\lambda,\sigma\mu} - A_{\sigma,\lambda\mu}\right) = \mu_0 j_\lambda \tag{16}$$

the 2nd term in the left hand side can be written as $A^{\mu}{}_{,\mu\lambda}$ thus this term can become zero by choosing a scalar field ψ such that

$$A^{\mu}{}_{,\mu} = 0 \tag{17}$$

This condition is called **Lorenz gauge**[3] and simplifies the EM field equations

$$\eta^{\mu\sigma} A_{\lambda,\sigma\mu} = A_{\lambda}{}^{,\mu}{}_{,\mu} = \Box A_\lambda = \mu_0 j_\lambda \tag{18}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial^\nu\partial_\nu$

- The last 2 equations are the EM field equations in the Lorenz gauge.

Electromagnetism in the Lorenz gauge iii

- In the absence of charges and currents A_μ admits wave solutions traveling with the speed of light.
- This is also true for the components of $F_{\mu\nu}$ since in this case we also have $\Box F_{\mu\nu} = 0$.

[3] Prove that this condition is preserved by any further gauge transformation $A_\mu \to A_\mu + \psi_{,\mu}$ if and only if $\eta^{\nu\mu}\psi_{,\nu\mu} \equiv \psi_{,\mu}{}^{,\mu} = 0$.

Electric and Magnetic Fields in inertial frames i

It is useful to define the components of $F^{\mu\nu}$ and A^{μ} in terms of the familiar electric and magnetic 3-vector fields $\vec{E}$ and $\vec{B}$ in some inertial frame S.

Thus the components of A^{μ} can be written in terms of the electrostatic potential ϕ and the 3-dimesional vector potential $\vec{A}$:

$$A^{\mu} = \left(\phi/c, \vec{A}\right) \tag{19}$$

Then the Lorenz gauge condition becomes (why?):

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0 \tag{20}$$

and in this gauge, the field equations become [4]

$$\Box \vec{A} = \mu_0 \vec{j} \quad \text{and} \quad \Box \phi = \frac{\rho}{\epsilon_0} \tag{21}$$

In terms of ϕ and $\vec{A}$, the electric and magnetic fields in S are given by:

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \tag{22}$$

Electric and Magnetic Fields in inertial frames ii

These equations lead to **Maxwell equations** in the familiar form (prove it):

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{23}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{24}$$

Finally, by using eqns (9) and (22) we can find (how?) that the covariant components of $F^{\mu\nu}$ in S are:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & -B^3 & B^2 \\ -E^2/c & B^3 & 0 & -B^1 \\ -E^3/c & -B^2 & B^1 & 0 \end{pmatrix} \tag{25}$$

The corresponding electric and magnetic fields $\vec{E}'$ and $\vec{B}'$ in some other Cartesian inertial frame S' are most easily obtained by calculating the components of the EM field tensor $F^{\mu\nu}$ or the 4-potential A^{μ} in this frame.

[4] ϵ_0 : permittivity of free space, μ_0 magnetic permeability of free space, both are universal constants. SI UNITS: henries per meter, or newtons per ampere squared. NOTE: $\epsilon_0 \cdot \mu_0 = 1/c^2$.

Electromagnetism in arbitrary coordinates i

The EM field equations in Cartesian inertial coordinates are given by:

$$F^{\mu\nu}{}_{,\mu} = \mu_0 j^\mu \tag{26}$$

$$F_{[\mu\nu,\sigma]} \equiv F_{\mu\nu,\sigma} + F_{\sigma\mu,\nu} + F_{\nu\sigma,\mu} = 0 \tag{27}$$

In such a coordinate system, the partial derivative is identical to the covariant derivative and we can rewrite the equations as:

$$F^{\mu\nu}{}_{;\mu} = \mu_0 j^\mu \tag{28}$$

$$F_{[\mu\nu;\sigma]} \equiv F_{\mu\nu;\sigma} + F_{\sigma\mu;\nu} + F_{\nu\sigma;\mu} = 0 \tag{29}$$

These new equations are fully covariant and if they are valid in one system of coordinates then they will be valid in all coordinate systems.

In the same way we can write the EM field equations in terms of the 4-potential A_μ

$$\eta^{\mu\sigma}\left(A_{\lambda,\sigma\mu} - A_{\sigma,\lambda\mu}\right) = k\, j_\lambda \quad \Rightarrow \quad g^{\mu\sigma}\left(A_{\lambda;\sigma\mu} - A_{\sigma;\lambda\mu}\right) = \mu_0\, j_\lambda \tag{30}$$

which is again **a fully covariant equation**.

Electromagnetism in arbitrary coordinates ii

Gauge transformations are also permitted in arbitrary coordinates:

$$A_\mu^{(new)} = A_\mu + \psi_{,\mu} \equiv A_\mu + \psi_{;\mu} \tag{31}$$

The **Lorenz gauge** condition in abritrary coordinates will be written as:

$$A^\mu{}_{;\mu} = 0$$

and in this case the **EM field equations** will be written as:

$$\Box_g A_\mu = \mu_0 j_\mu$$

where the d'Alembertian operator is: $\Box_g = g^{\mu\nu}\nabla_\mu\nabla_\nu = \nabla^\nu\nabla_\nu$ and in vacuuo we may again have:

$$\Box_g A_\mu = 0 \quad \text{and} \quad \Box_g F_{\mu\nu} = 0 \tag{32}$$

Finally, the **charge conservation** in arbitrary coordinates will be expressed as:

$$j^\mu{}_{;\mu} = 0\,.$$

Obviously, the components of $F^{\mu\nu}$ and A^μ in two different arbitrary coordinate systems x^μ and x'^μ are related by:

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu \quad \text{and} \quad F'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x'^\mu}{\partial x^\rho} F^{\sigma\rho} \tag{33}$$

Equations of motion for charged particles

The equation of motion of a charged particle in an EM field in a coordinate invariant form is:

$$\frac{d\mathbf{p}}{d\tau} = m_0 \frac{d\mathbf{u}}{d\tau} = q\,\mathbf{F}\cdot\mathbf{u}$$

where m_0 is the rest mass of the particle, $\mathbf{p}$ is its 4-momentum, $\mathbf{u}$ is its 4-velocity and τ is the proper time measured along its worldline.

In Cartesian inertial coordinates this will be written

$$m_0 \frac{du^\mu}{d\tau} = q\,F^\mu{}_\nu\,u^\nu$$

in an arbitrary coordinate system the derivative in left hand side must be replaced with the **absolute** (or intrinsic) derivative:

$$m_0 \frac{Du^\mu}{D\tau} = m_0\left(\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\sigma} u^\nu u^\sigma\right) = q\,F^\mu{}_\nu\,u^\nu$$

where $u^\mu = dx^\mu/d\tau$, since the 4-velocity is tangent to the particle's worldline $x^\mu(\tau)$. In arbitrary coordinates will be:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{q}{m_0}\,F^\mu{}_\nu\,\frac{dx^\nu}{d\tau} \tag{34}$$

Conclusion

The general procedure for converting an equation valid in Cartesian inertial coordinates into one that is valid in an arbitrary coordinate system is as follows:

- replace partial with covariant derivatives
- replace ordinary derivatives along curves with absolute (intrinsic) derivatives
- replace $\eta_{\mu\nu}$ with $g_{\mu\nu}$.

Energy-Momentum Tensor of the EM field i

The energy momentum tensor $T^{\mu\nu}$ in Cartesian coortinates will be

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu\beta}\eta_{\alpha\beta} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{35}$$

The explicit form of the various components is:

$$\begin{aligned}
T^{00} &= \frac{1}{8\pi}\left(|\vec{E}|^2 + |\vec{B}|^2\right) \\
T^{0i} &= \frac{1}{4\pi}\left(\delta^{ij}\varepsilon_{jkm}E^k B^m\right) = \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)^i \\
T^{ij} &= \frac{1}{8\pi}\left[\left(|\vec{E}|^2 + |\vec{B}|^2\right)\delta^{ij} - 2\left(E^iE^j + B^iB^j\right)\right]
\end{aligned} \tag{36}$$

- The component T^{00} is the electromagnetic energy density
- The component $T^{0i} = T^{i0}$ gives the momentum flux density of the EM field, referred as the Poynting vector.

$$\vec{P} = \frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right)$$

Energy-Momentum Tensor of the EM field ii

- The spacelike diagonal component with $i = j$

$$T^{ii} = \frac{1}{8\pi}\left[\left(|\vec{E}|^2 + |\vec{B}|^2\right) - 2\left(E^i E^i + B^i B^i\right)\right]$$

gives the pressure normal to the face of a small cube of EM field

- The off-diagonal spacelike component with $i \neq j$

$$T^{ij} = -\frac{1}{4\pi}\left(E^i E^j + B^i B^j\right)$$

gives the shear stresses that act parallel to the faces of an infinitesimal region of the EM field

EXAMPLE: The EM stress energy tensor for a constant electric field in the x direction will be (prove it):

$$T^{00} = \frac{E^2}{8\pi}, \quad T^{xx} = -\frac{E^2}{8\pi}, \quad T^{yy} = T^{zz} = \frac{E^2}{8\pi}$$

what will be the components of this tensor in a frame S' moving with a velocity v at the x-direction?

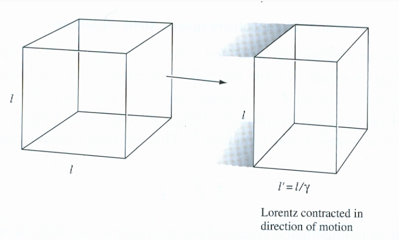
Energy-Momentum Tensor i

To construct the gravitational field equations, we must find a covariant way of expressing the **source term**. We must find a tensor that describes the matter distribution at each event in spacetime.

• Let's start with **dust**, this is a time-dependent distribution of electrically neutral non-interacting particles, each of rest mass m_0.

• At each event P of the spacetime we can characterize the distribution completely by giving the matter density ρ and 3-velocity $\vec{u}$ as measured in some inertial frame.

• For example in an instantaneous rest frame S at a given point P, $\vec{u} = \vec{0}$ while the proper density is given by $\rho_0 = m_0 n_0$, where m_0 is the rest mass of each particle and n_0 is the number of particles in the unit volume.



• In a moving frame S', the volume containing a fixed number of particles is Lorentz contracted along the direction of the motion thus $n' = \gamma_u n$ and the mass of each particle in S' is $m' = \gamma_u m_0$ thus the matter density in S' is

$$\rho' = \gamma_u^2 \rho_0$$

• This means that the matter density is **not** a scalar but does transform as the 00 component of rank-2 tensor. This suggests that the source term in the gravitational field equations should be a rank-2 tensor

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu \qquad (37)$$

In a local Cartesian coordinate frame the components of the 4-velocity for the fluid is $u^\mu = \gamma_u(c, \vec{u})$.

In this frame the components of the energy-momentum tensor are:

Energy - Momentum Tensor

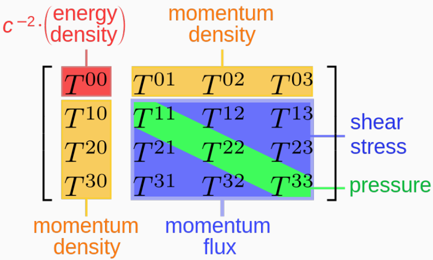
$$T^{00} = \rho u^0 u^0 = \gamma_u^2 \rho c^2$$ energy density of the particles

$$T^{0k} = \rho u^0 u^k = \gamma_u^2 \rho c u^k$$ energy flux $\times c^{-1}$ in the k-direction

$$T^{k0} = \rho u^0 u^k = \gamma_u^2 \rho c u^k$$ momentum density $\times c$ in the k-direction

$$T^{ij} = \rho u^i u^j = \gamma_u^2 \rho u^i u^j$$ the rate of flow of the i-component of momentum per unit area in the j-direction

Due to these identifications the tensor **T** is known as the energy - momentum or stress - energy tensor.[5]



[5]wikipedia

The Energy-Momentum tensor of a perfect fluid

A perfect fluid is defined as one for which there are no forces between the particles and no heat conduction or viscosity. Thus in an inertial reference frame the components of **T** for a perfect fluid are given by:

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

while we can show that for inertial observers (or in STR)

$$T^{\mu\nu} = (\rho + p/c^2)u^\mu u^\nu + \eta^{\mu\nu} p \tag{38}$$

while in an arbitrary coordinate system it will be:

$$T^{\mu\nu} = (\rho + p/c^2)u^\mu u^\nu + g^{\mu\nu} p \tag{39}$$

Note : the $T^{\mu\nu}$ is **symmetric** and is made up from 2 scalar fields ρ and p and the vector field $\vec{u}$ that uniquely characterize the perfect fluid.

The Energy-Momentum tensor of a "real fluid"

For the general real fluid we have to take into account that:

- besides the bulk motion of the fluid, each particle has some random (thermal) velocity
- there may be various forces between the particles that contribute potential energies to the total.

In this case the components of the energy momentum tensor will be:

- T^{00} : is the total energy density, including any potential energy contributions from forces between the particles and kinetic energy from the random thermal motions.
- T^{0i} : although there is NO bulk motion, energy might transmitted by heat conduction i.e. it is a heat conduction term in the inertial frame of reference
- T^{i0} : if heat is being conducted then the energy will carry momentum
- T^{ij} : the random thermal motions of the particles will give rise to momentum flow, so that T^{ii} is the isotropic pressure in the i-direction and T^{ij} $(i \neq j)$ are the viscous stresses in the fluid.

Conservation of Energy & Momentum for a perfect fluid

By analogy with equation ${j^{\mu}}_{,\mu} = 0$ for the conservation of charge, the conservation of energy and momentum is represented by

$$ {T^{\mu\nu}}_{,\nu} = 0 \qquad (40) $$

We will show that this relation leads to well known equations of motion and continuity for a fluid in the Newtonian limit.

Substituting eqn (38) into equation (40) we get (how?):

$$ \left(\rho + p/c^2\right)_{,\nu} u^{\mu} u^{\nu} + \left(\rho + p/c^2\right) \left[({u^{\mu}}_{,\mu}) u^{\nu} + u^{\mu}({u^{\nu}}_{,\mu})\right] - \eta^{\mu\nu} p_{,\mu} = 0 \qquad (41) $$

from the normalization condition for the 4-velocity $u^{\nu} u_{\nu} = c^2$ by differentiation we get the relativistic **equation of continuity** for a perfect fluid in local inertial coordinates at a point P:

$$ \left(\rho u^{\mu}\right)_{,\mu} + (p/c^2){u^{\mu}}_{,\mu} = 0 \qquad (42) $$

and equation (41) becomes the relativistic **equation of motion** for a perfect fluid in local inertial coordinates at some point P:

$$ \left(\rho + p/c^2\right) ({u^{\mu}}_{,\mu}) u^{\nu} = \left(\eta^{\mu\nu} - u^{\mu} u^{\nu}/c^2\right) p_{,\mu} \qquad (43) $$

A slowly moving fluid is the one for which we may neglect u/c and so take $\gamma_u \approx 1$ and $u^\mu \approx (c, \vec{u})$ and the proper density becomes the "normal" density. In this limit we get

$$T^{0\nu}{}_{,\nu} = 0 \quad \rightarrow \quad \text{eqn (42)} \;\Rightarrow\; \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \tag{44}$$

which is the classical equation of continuity for a fluid, and

$$T^{j\nu}{}_{,\nu} = 0 \quad \rightarrow \quad \text{eqn (43)} \;\Rightarrow\; \frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right) \vec{u} = -\frac{1}{\rho} \vec{\nabla} p \tag{45}$$

which Euler's equations of motion for a perfect fluid.

We can easily obtain the condition for energy and momentum conservation in arbitrary coordinates by replacing the partial with the covariant derivative, and we get:

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{46}$$

This is a fundamental equation which imposes tight restrictions on the possible forms that the gravitational field equations may take.

Linear Field Equations for Gravitation

- In the linear approximation for the gravitational field, we neglect the effects of the gravitational field on itself. [6]
- The approximation applies to all phenomena that lie in the region of overlap between Newton's and Einstein's theories.
- Although, it is true that the most spectacular results of gravitational theory depend in a crucial way on the nonlinearity of the field equations, **almost all** of the results that have been the subject of experimental investigation can be described by the linear approximation.
- The deflection of the light, the retardation of the light, the grav. time dilation and gravitational radiation emerge in the linear approximation.
- It will be instructive if we follow the same path as for constructing the Electromagnetic field equations which are linear.

[6] From "Gravitation & Spacetime" H. Ohanian & R. Ruffini, Cambridge (2013)

Linear Field Equations for Electromagnetism

The most general form of the EM field equations will have the form:

$$A^{\nu,\mu}{}_{,\mu} + bA^{\mu,\nu}{}_{,\mu} + aA^{\nu} = \kappa j^{\nu} \tag{47}$$

where a, b and $\kappa(= 4\pi)$ are constants.

We can actually assume $a = 0$ since it leads to unphysical solutions.

By taking the partial derivative on both sides we get

$$(A^{\nu,\mu}{}_{,\mu} + bA^{\mu,\nu}{}_{,\mu})_{,\nu} = 4\pi j^{\nu}{}_{,\nu} \tag{48}$$

and by rearranging the indices we get

$$(1 + b)A^{\nu}{}_{,\nu}{}^{,\mu}{}_{,\mu} = 4\pi j^{\nu}{}_{,\nu} \tag{49}$$

and because $j^{\nu}{}_{,\nu} = 0$ we get that $b = 1$ and the form of the field equations is

$$A^{\nu,\mu}{}_{,\mu} - A^{\mu,\nu}{}_{,\mu} = 4\pi j^{\nu} \tag{50}$$

or

$$F^{\mu\nu}{}_{,\mu} = 4\pi j^{\nu} \quad \text{for} \quad F^{\mu\nu} = A^{\mu,\nu} - A^{\nu,\mu} \tag{51}$$

and obviously

$$F_{[\mu\nu,\sigma]} \equiv F_{\mu\nu,\sigma} + F_{\sigma\mu,\nu} + F_{\nu\sigma,\mu} = 0 \tag{52}$$

Linear Field Equations for Gravitation i

- We will assume that our spacetime is the flat spacetime of STR and we look for a suitable field equation for the gravitational field.
- The first thing that we should decide is the source of the gravitation, and from Newtonian theory we know that the energy density is the best candidate. However, it is impossible to construct a Lorentz invariant theory of gravitation. As we have seen earlier, the energy density is just a component of the the energy density tensor $T^{\mu\nu}$ and if in one system of coordinates there exists only one component the T^{00} this will not be true in any system. This means that the whole $T^{\mu\nu}$ will be the source of the gravitation field. [7]
- Since the source of gravitation is a 2nd-rank symmetric tensor the obvious choice is a 2nd rank symmetric field tensor lets say $h^{\mu\nu}$. Then the most general field equation, linear in $h^{\mu\nu}$ is of 2nd differential order and contains the $T^{\mu\nu}$ as source term:

Linear Field Equations for Gravitation ii

$$h^{\mu\nu}{}_{,\lambda}{}^{,\lambda} + \alpha h^{,\mu\nu} - \beta\left(h^{\mu\lambda}{}_{,\lambda}{}^{,\nu} + h^{\nu\lambda}{}_{,\lambda}{}^{,\mu}\right) + \gamma\eta^{\mu\nu}h^{,\lambda}{}_{,\lambda} + \delta\eta^{\mu\nu}h^{\lambda\sigma}{}_{,\lambda\sigma} = -\kappa T^{\mu\nu} \tag{53}$$

where α, β, γ , δ and κ are arbitrary constants.

- If we operate on both sides with $\partial/\partial x^\mu$ the right hand side should vanish due to energy-momentum conservation ($T^{\mu\nu}{}_{,\mu} = 0$). [8]

$$\begin{aligned} \partial_\lambda\partial^\lambda\partial_\mu h^{\mu\nu} \quad &+ \quad \alpha\,\partial_\mu\partial^\mu\partial_\nu h - \beta\left(\partial_\lambda\partial^\nu\partial_\mu h^{\mu\lambda} + \partial_\lambda\partial^\mu\partial_\mu h^{\nu\lambda}\right) \\ &+ \quad \gamma\,\partial^\nu\partial_\lambda\partial^\lambda h + \delta\,\partial^\nu\partial_\lambda\partial_\sigma h^{\lambda\sigma} = 0 \end{aligned} \tag{54}$$

- Then by rearranging the terms on the left hand side we get the following set of equations for the constants: $-\beta + \delta = 0$, $1 - \beta = 0$ and $\alpha + \gamma = 0$. (How?)

- Thus $\beta = \delta = 1$ & $\gamma = -\alpha$ and the field equations will be written as

$$h^{\mu\nu}{}_{,\lambda}{}^{,\lambda} + \alpha h^{,\mu\nu} - \left(h^{\mu\lambda}{}_{,\lambda}{}^{,\nu} + h^{\nu\lambda}{}_{,\lambda}{}^{,\mu}\right) - \alpha\eta^{\mu\nu}h^{,\lambda}{}_{,\lambda} + \eta^{\mu\nu}h^{\lambda\sigma}{}_{,\lambda\sigma} = -\kappa T^{\mu\nu} \tag{55}$$

even α can be be eliminated if we introduce a new tensor of the form

$$h^{\mu\nu} = \tilde{h}^{\mu\nu} - C\eta^{\mu\nu}\tilde{h}$$

where C is a constant.

Then the equations will get exactly the same form as eqn (55) but instead of α we will have $\tilde{a} = a(1 - C) + 2C$.

By choosing $C = (1 - \alpha)/(2 - \alpha)$ we get $\tilde{a} = 1$(prove it?).
$C = (1 - \alpha)/(2 - 4\alpha)$

- Thus the relativistic field equation for gravitation in the linear approximation becomes:

$$h^{\mu\nu}{}_{,\lambda}{}^{,\lambda} + h^{,\mu\nu} - \left(h^{\mu\lambda}{}_{,\lambda}{}^{,\nu} + h^{\nu\lambda}{}_{,\lambda}{}^{,\mu}\right) - \eta^{\mu\nu} h^{,\lambda}{}_{,\lambda} + \eta^{\mu\nu} h^{\lambda\sigma}{}_{,\lambda\sigma} = -\kappa T^{\mu\nu} \quad (56)$$

while κ is the coupling constant of the theory and will be estimated from the Newtonian limit of the above equation.

- We will call $h^{\mu\nu}$ **gravitational tensor potential** in analogy to EM potential A^{μ}.
- Besides the tensor field $h^{\mu\nu}$ one may consider extra **scalar**, **vector** or even **tensor** fields and to construct endlessly complicated theories.
- Jordan (1951) and Brans-Dicke (1961) suggested the least complicated scalar-tensor theory containing only an extra scalar field. This new scalar field has the effect of making the gravitational constant dependent on position. The experimental evidence (up to now) speaks against the scalar-tensor theory

Linear Field Equations for Gravitation V

GAUGE CONDITION

- In EM we got a simplified field equation by taking advantage of the gauge invariance. The same can be done for gravitation and it can be shown that the field equations are invariant under the gauge transformation

$$h^{\mu\nu}_{(new)} = h^{\mu\nu} + Q^{\mu,\nu} + Q^{\nu,\mu} \tag{57}$$

where $\mathbf{Q}$ is an arbitrary vector field.

- In analogy to EM (Lorenz gauge) using the gauge condition known as **Hilbert gauge**[9]

$$\phi^{\mu\nu}{}_{,\mu} = \left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h\right)_{,\mu} = 0 \tag{58}$$

If the Hilbert gauge condition holds; the field equations simplifies to:

$$\left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h\right)_{,\lambda}^{\;,\lambda} = -\kappa T^{\mu\nu} \tag{59}$$

and by substituting

$$\phi^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h \tag{60}$$

the field equation becomes

$$\Box\phi^{\mu\nu} = \phi^{\mu\nu}{}_{,\lambda}{}^{,\lambda} = -\kappa T^{\mu\nu} \tag{61}$$

and the gauge condition

$$\phi^{\mu\nu}{}_{,\mu} = 0 \tag{62}$$

Table 3.1 Analogy between the electromagnetic and gravitational field theories

Description	Electromagnetism	Gravitation (Linear Approximation)
Source of field	j^ν	$T^{\mu\nu}$
Conservation law	$\partial_\nu j^\nu = 0$	$\partial_\nu T^{\mu\nu} = 0$
Field	A^ν	$h^{\mu\nu}$
Field equation	$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = 4\pi j^\nu$	$\partial_\lambda \partial^\lambda h^{\mu\nu} + \partial^\mu \partial^\nu h - (\partial_\lambda \partial^\nu h^{\mu\lambda} + \partial_\lambda \partial^\mu h^{\nu\lambda}) - \eta^{\mu\nu} \partial_\lambda \partial^\lambda h + \eta^{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} = -\kappa T^{\mu\nu}$
Gauge transformation	$A^\mu \to A^\mu + \partial^\mu \Lambda$	$h^{\mu\nu} \to h^{\mu\nu} + \partial^\mu \Lambda^\nu + \partial^\nu \Lambda^\mu$
Gauge condition	$\partial_\mu A^\mu = 0$	$\partial_\mu (h^{\mu\nu} - 1/2 \eta^{\mu\nu} h) = 0$
Field equation in this gauge	$\partial_\mu \partial^\mu A^\nu = 4\pi j^\nu$	$\partial_\lambda \partial^\lambda (h^{\mu\nu} - 1/2 \eta^{\mu\nu} h) = -\kappa T^{\mu\nu}$
Particle equation of motion	$m\, du_\mu / d\tau - q F_{\mu\nu} u^\nu = 0$	$du_\mu / d\tau + \kappa h_{\mu\alpha,\beta} u^\alpha u^\beta - 1/2 \kappa h_{\alpha\beta,\mu} u^\alpha u^\beta = 0$
Proper time interval	$d\tau^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$	$d\tau^2 = (\eta_{\alpha\beta} + \kappa h_{\alpha\beta}) dx^\alpha dx^\beta$

Figure 1: Adapted from H.C. Ohanian - R. Ruffini “Gravitation and Spacetime”, Cambridge (2013)

[7] Can you find a good reason why the trace of $T^{\mu\nu}$ alone cannot be the source of the gravitational field?

[8] Actually, we can demand the this conservation law - as in the case of electromagnetism - to emerge directly from the field equations.

[9] Show that if Q^ν is solution of the differential equation $Q^\nu{}_{,\mu}{}^{,\mu} = -h^{\mu\nu}{}_{,\mu} + \frac{1}{2} h^{,\nu}$ then $h^{\mu\nu}_{(new)}$ will satisfy the Hilbert condition.

Interaction of Gravitation and Matter

In writing field eqn (61) we assumed that $T^{\mu\nu}$ was the energy - momentum tensor of matter. I.e in order to obtain the linear field equations we left out the effect of the gravitational field upon itself.

Due to this our linear field equations have two drawbacks:

- According to eqn (61) matter acts on the gravitational fields and changes the fields, but there is no reciprocal action of the gravitational fields on matter, and the energy-momentum of matter is conserved $T^{\mu\nu}{}_{,\nu} = 0$! This is an incosistency.
- Gravitational energy does not act as source of gravitation, in contradiction to the principle of equivalence.

Thus eqn (61) is only an approximation for weak gravitational fields.

The way to correct it is to include the energy-momentum of the gravitational field in $T^{\mu\nu}$ i.e.

$$T^{\mu\nu} = T^{\mu\nu}_{(m)} + \mathcal{T}^{\mu\nu} \tag{63}$$

where $T^{\mu\nu}_{(m)}$ and $\mathcal{T}^{\mu\nu}$ are the energy momentum tensors of **matter** and **gravitation**.

- The field equations become:

$$\Box\phi^{\mu\nu} = \phi^{\mu\nu}{}_{,\lambda}{}^{,\lambda} = -\kappa\left(T_{(m)}{}^{\mu\nu} + \mathcal{T}^{\mu\nu}\right) \tag{64}$$

- If we act on both side with $\partial/\partial x^{\nu}$ we get (because of the Hilbert gauge)

$$\left(T_{(m)}{}^{\mu\nu} + \mathcal{T}^{\mu\nu}\right)_{,\nu} = 0 \tag{65}$$

which expresses the conservation of the total energy-momentum, and raises the inconsistencies mentioned earlier.

- But, $\mathcal{T}^{\mu\nu}$ is still unknown and we can find it only approximately.

- If we assume a solution of the above eqn free of sources i.e. $\phi^{\mu\nu}{}_{,\lambda}{}^{,\lambda} = 0$ from field theory we get a first order approximation to the energy-momentum tensor of the gravitational field

$$\mathcal{T}_{(1)}{}^{\mu\nu} = \frac{1}{4}\left[2\phi^{\alpha\beta,\mu}\phi_{\alpha\beta}{}^{,\nu} - \phi^{,\mu}\phi^{,\nu} - \eta^{\mu\nu}\left(\phi^{\alpha\beta,\sigma}\phi_{\alpha\beta,\sigma} - \frac{1}{2}\phi_{,\sigma}\phi^{,\sigma}\right)\right] \tag{66}$$

Thus the new approximate form of the grav. field equations will be:

$$\Box\phi^{\mu\nu} = \phi^{\mu\nu}{}_{,\lambda}{}^{,\lambda} = -\kappa\left(T^{\mu\nu}_{(m)} + \mathcal{T}_{(1)}{}^{\mu\nu}\right) \tag{67}$$

The corresponding equation for the conservation of the total energy-momentum is

$$\left(T_{(m)}{}^{\mu\nu} + \mathcal{T}_{(1)}{}^{\mu\nu} \right)_{,\nu} \approx 0 \tag{68}$$

this can be used to derive the equation of motion.

Thus by expanding the above equation we get (how?):

$$\mathcal{T}_{(1)}{}^{\mu\nu}{}_{,\nu} = \frac{1}{4}\left(2\phi^{\alpha\beta,\mu}\phi_{\alpha\beta}{}^{,\nu}{}_{,\nu} - \phi^{,\mu}\phi^{,\nu}{}_{,\nu}\right) \approx -\frac{\kappa}{4}\left(2\phi^{\alpha\beta,\mu}T_{(m)\alpha\beta} - \phi^{,\mu}T_{(m)}\right) + \ldots \tag{69}$$

we can obtain (how?):

$$T_{(m)}{}^{\mu\nu}{}_{,\nu} - \frac{\kappa}{2}T_{(m)\alpha\beta}\left(\phi^{\alpha\beta,\mu} - \frac{1}{2}\eta^{\alpha\beta}\phi^{,\mu}\right) = 0 \tag{70}$$

or

$$T_{(m)\,\mu}{}^{\nu}{}_{,\nu} - \frac{\kappa}{2}h_{\alpha\beta,\mu}T_{(m)}{}^{\alpha\beta} = 0 \tag{71}$$

This equation tells how much energy and momentum the gravitational field transfers to the matter on which it acts.

In other words it determines the rate at which the momentum of a particle changes and therefore it determines the **equation of motion** of a particle acted upon a gravitational field.

To derive this equation of motion we integrate equation (71) over the volume of the "particle" (a small system of finite size).

The 1st term gives:

$$\frac{d}{dt}\int T_{(m)\mu}{}^{0} d^3x \equiv \frac{d}{dt} P_\mu \qquad (72)$$

For the 2nd term of (71) we should remember that:

$$T_{(m)}{}^{\alpha\beta} = \rho_0 u^\alpha u^\beta + \ldots \qquad (73)$$

then[10]

$$-\frac{\kappa}{2} h_{\alpha\beta,\mu} \int T_{(m)}{}^{\alpha\beta} d^3x \approx -\frac{\kappa}{2} h_{\alpha\beta,\mu} \sqrt{1-u^2}\, m u^\alpha u^\beta \qquad (74)$$

Thus we finally get (how?)

$$\boxed{\frac{d}{d\tau} P_\mu = -\frac{\kappa}{2} m h_{\alpha\beta,\mu} u^\alpha u^\beta}$$

which has the right form for an equation of motion

[10]Show that $\int \rho_0 u^\alpha u^\beta d^3x = m u^\alpha u^\beta \sqrt{1-u^2}$

If we assume that :

$$\frac{\partial L}{\partial x^\mu} = -\frac{\kappa}{2} m h_{\alpha\beta,\mu} u^\alpha u^\beta \quad \text{and} \quad \frac{\partial L}{\partial u^\mu} = P_\mu \tag{75}$$

we get the Euler -Lagrange form of equations of motion.

Actually, in the absence of forces the Lagrangian will have the form

$$L_0 = \frac{1}{2} m \, \eta_{\alpha\beta} u^\alpha u^\beta$$

while in the presence of a field becomes:

$$L = \frac{1}{2} m \left(\eta_{\alpha\beta} + \kappa \tilde{h}_{\alpha\beta} \right) u^\alpha u^\beta$$

notice that $\tilde{h}_{\alpha\beta}$ is a quantity related to the spacetime while $h_{\alpha\beta}$ is the "tensor potential" of our theory.

But because of the Euler-Lagrange equations these two quantities are identical and this suggests that a relation between the "geometrical" quantity $\tilde{h}_{\alpha\beta}$ and the " physical" quantity" $h_{\alpha\beta}$!

NOTE 1 : We have created a relation between physical and geometrical quantities

Alternatively, if in the geodesic equations one assumes that $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ then it comes to the equation

$$\frac{d}{d\tau}\left(u_{\mu} + \kappa h_{\mu\alpha} u^{\alpha}\right) - \frac{\kappa}{2} h_{\alpha\beta,\mu} u^{\alpha} u^{\beta} = 0\,. \tag{76}$$

NOTE 2 : That the equation of motion of a particle in a gravitational field is independent of the mass of the particle, in agreement with Galileo principle.

NOTE 3 : The equations of motion of a particle is no more than a statement about the exchange of momentum between particle and field. Hence the equations of motion cannot be independent of the conservation law for the energy-momentum tensor of the fields.

APPENDIX

Pure forces i

$$U^{\mu} = \gamma\left(c, u^{j}\right) \quad \text{4-velocity} \tag{77}$$

$$F^{\mu} = \frac{dP^{\mu}}{d\tau} = \frac{d}{d\tau}\left(m_0 U^{\mu}\right) = \gamma \frac{d}{dt}\left(mc, p^{j}\right) \quad \text{force 4-vector} \tag{78}$$

$$f^{j} = \frac{dp^{j}}{dt} = \frac{d}{dt}\left(mu^{j}\right) \quad \text{relativistic force 3-vector} \tag{79}$$

where F^{μ} is the force 4-vector, P^{μ} the momentum 4-vector, U^{μ} the velocity 4-vector, p^{j} the relativistic momentum and m the mass of the particle on which f^{j} acts.

Then the equation (78) can be written as

$$F^{\mu} = \gamma \frac{d}{dt}\left(mc, p^{j}\right) = \gamma\left(c\frac{dm}{dt}, f^{j}\right) \tag{80}$$

Pure forces ii

If the force F^μ acts on a particle with 4-velocity $U^\mu = \gamma\left(c, u^j\right)$ their inner product in Minkowski space is

$$\begin{aligned} \eta_{\mu\nu} U^\mu F^\nu &= \gamma^2 \left(c^2 \frac{dm}{dt} - \eta_{ii} f^i u^i \right) \\ &= c^2 \frac{dm_0}{d\tau} = \gamma c^2 \frac{dm_0}{dt} \end{aligned} \tag{81}$$

where the second equation results from specializing the first to the rest frame of the particle, and the third is a consequence of

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - u^2/c^2}} = \gamma \tag{82}$$

We shall call a force **pure** if it does not change a particle's rest mass

The necessary and sufficient condition for a force to be **pure** follows at once from (81):

$$\eta_{\mu\nu} U^\mu F^\nu = 0 \quad \Longleftrightarrow \quad m_0 = \text{constant} \tag{83}$$

The antisymmetry of the electromagnetic tensor

$$f_\mu = \frac{q}{c} F_{\mu\nu} U^\nu \tag{84}$$

where the "coeffeicients" $F_{\mu\nu}$ in this linear relation must be tensorial to make it Lorenz-invariant if we take the charge q of the particle in question to be a scalar invariant.

If we demand that the force f_μ to be pure, we need

$$f_\mu U^\mu = \frac{q}{c} F_{\mu\nu} U^\mu U^\nu = 0 \tag{85}$$

for all U^μ, and hence

$$F_{\mu\nu} = -F_{\nu\mu} \tag{86}$$

i.e. the field tensor must be anti-symmetric.