Einstein's Theory of Gravity

Kostas Kokkotas

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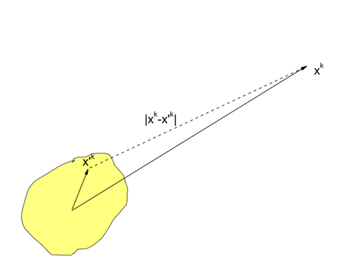
Newtonian Gravity

Poisson equation

$$\nabla^2 U(\vec{x}) = 4\pi G\rho(\vec{x}) \rightarrow U(\vec{x}) = -G\int \frac{\rho(\vec{x})}{|\vec{x}-\vec{x'}|} d^3\vec{x'}$$

For a spherically symmetric mass distribution of radius R

$$\begin{aligned} U(r) &= -\frac{1}{r}\int_0^R r'^2\rho(r')dr' \quad \text{for} \quad r > R \\ U(r) &= -\frac{1}{r}\int_0^r r'^2\rho(r')dr' - \int_r^R r'\rho(r')dr' \quad \text{for} \quad r < R \end{aligned}$$



For a non-spherical distribution the term $1/|\vec{x} - \vec{x'}|$ can be expanded as

$$\frac{1}{|\vec{x} - \vec{x'}|} = \frac{1}{r} + \sum_k \frac{x^k x'^k}{r^3} + \frac{1}{2} \sum_k \sum_l \left(3x'^k x'^l - \vec{r}'^2 \delta_k^l\right) \frac{x^k x^l}{r^5} + \ldots$$

$$U(\vec{x}) = -\frac{GM}{r} - \frac{G}{r^3} \sum_k x^k D^k - \frac{G}{2} \sum_{kl} Q^{kl} \frac{x^k x^l}{r^5} + \ldots$$

Gravitational Multipoles

$$\begin{aligned}
M &= \int \rho(\vec{x'}) d^3x' \quad \text{Mass} \\
D^k &= \int x'^k \rho(\vec{x'}) d^3x' \quad \text{Mass Dipole moment}^1 \\
Q^{kl} &= \int \left(3x'^k x'^l - \vec{r'}^2 \delta_k^l\right) \rho(\vec{x'}) d^3x' \quad \text{Mass Quadrupole tensor}^2
\end{aligned}$$

[1] If the center of mas is chosen to coincide with the origin of the coordinates then $D^k = 0$ (no mass dipole).

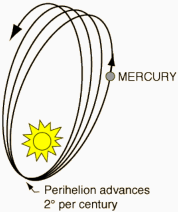
[2] If $Q^{kl} \neq 0$ the potential will contain a term proportional to $\sim 1/r^3$ and the gravitational force will deviate from the inverse square law by a term $\sim 1/r^4$.

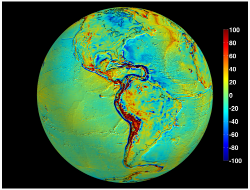
The Earth's polar and equatorial diameters differ by 3/1000, while for the Sun we measure 10^{-5}. This deviation produces a quadrupole term in the gravitational potential, which causes perturbations in the elliptical Kepler orbits of satellites. Usually, we define the dimensionless parameter

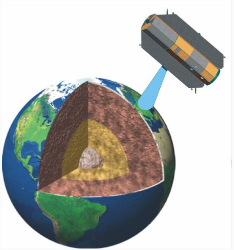
$$J_2 = -\frac{Q^{33}}{2M_\odot R_\odot^2} \tag{1}$$

as a convenient measure of the oblateness of a nearly spherical body. For the Sun the oblateness due to rotation (if uniform) would be $J_2 \approx 10^{-7}$. The most recent measurements (based on helioseismology) suggest $J_2 \approx 2.2 \times 10^{-7}$.

The main perturbation is the precession of Kepler's ellipse and this can be used for precise determinations of the multipole moments and the mass distribution in the Earth.







Equivalence Principle i

- In GR gravitational phenomena arise not from forces and fields, but from the curvature of the 4-dim spacetime.
- The starting point for this consideration is the Equivalence Principle which states that the Gravitational and the Inertial masses are equal.
- The equality $m_G = m_I$ is one of the most accurately tested principles in physics!
- Now it is known experimentally that: $\gamma = \frac{m_G - m_I}{m_G} < 10^{-13}$

Equivalence Principle ii

Experimental verification

- Galileo (1610) $\gamma < 10^{-3}$
- Newton (1680) $\gamma < 10^{-3}$
- Bessel (19th century) $\gamma < 2 \times 10^{-5}$
- Eötvös (1890) & (1908) $\gamma < 3 \times 10^{-9}$
- Dicke et al. (1964) $\gamma < 3 \times 10^{-11}$
- Braginsky et al. (1971) $\gamma < 9 \times 10^{-13}$,
- Kuroda and Mio (1989) $\gamma < 8 \times 10^{-10}$,
- Adelberger et al. (1990) $\gamma < 1 \times 10^{-11}$
- Su et al. (1994) $\gamma < 1 \times 10^{-12}$ (torsional)
- Williams et al. (‘96), Anderson & Williams (‘01) $\gamma < 1 \times 10^{-13}$
- Schlamminger et al. (2008) $\gamma < 2 \times 10^{-13}$

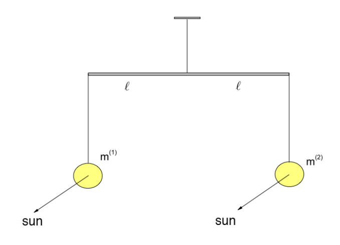
Equivalence Principle iii

MICROSCOPE: On 4 December 2017, the first results were published. The equivalence principle was measured to hold true within a precision of 10^{-15}, improving prior measurements by an order of magnitude.

- Weak Equivalence Principle :
  - The motion of a neutral test body released at a given point in space-time is independent of its composition
- Strong Equivalence Principle :
  - The results of all local experiments in a frame in free fall are independent of the motion
  - The results are the same for all such frames at all places and all times
  - The results of local experiments in free fall are consistent with STR

Equivalence Principle : Dicke's Experiment

The experiment is based on measuring the effect of the gravitational field on two masses of different material in a torsional pendulum.



$$\frac{GMm_G^{(E)}}{R^2} = \frac{m_I^{(E)} v^2}{R} \quad \Rightarrow \quad v^2 = \frac{GM}{R}\left(\frac{m_G}{m_I}\right)^{(E)}$$

The forces acting on both masses are:

$$F^{(j)} + \frac{GMm_G^{(j)}}{R^2} = \frac{m_I^{(j)} v^2}{R} \Rightarrow \quad F^{(j)} = \frac{GMm_I^{(j)}}{R^2}\left[\left(\frac{m_G}{m_I}\right)^{(E)} - \left(\frac{m_G}{m_I}\right)^{(j)}\right]$$

and the total torque applied is: $L = \left(F^{(1)} - F^{(2)}\right)\ell = \frac{GM}{R^2}\left[m_G^{(1)} - m_G^{(2)}\right]\ell$ here we assumed that : $m_I^{(1)} = m_I^{(2)} = m$.

Towards a New Theory for Gravity i

Because of the success of Newton's theory of gravity, we should demand from our new theory :

- In an appropriate first approximation (weak field) the new theory should reduce to the newtonian one.
- Beyond this approximation the new theory should predict small deviations from newtonian theory which must be verified by experiments/observations

Towards a New Theory for Gravity ii

Einstein's equivalence principle:

Gravitational *and* **Inertial** *forces/accelerations are equivalent and they cannot be distinguished by any physical experiment*.

This statement has 3 implications:

1. Gravitational forces/accelerations are described in the same way as the inertial ones.
2. When gravitational accelerations are present the space cannot be flat.
3. Consequence: if gravity is present there cannot exist inertial frames.

Towards a New Theory for Gravity iii

1. **Gravitational forces/accelerations are described in the same way as the inertial ones**.

- This means that the motion of a freely moving particle, observed from an inertial frame, will be described by

$$\frac{d^2x^\mu}{dt^2} = 0$$

while from a non-inertial frame its movement will be described by the geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} = 0\,.$$

- The 2nd term appeared due to the use of a non-inertial frame, i.e. the inertial accelerations will be described by the Christoffel symbols.

- But according to Einstein the gravitational accelerations as well will be described by the Christoffel symbols.

Towards a New Theory for Gravity iv

- This leads to the following conclusion:

The metric tensor should play the role of the gravitational potential since $\Gamma^{\mu}_{\rho\sigma}$ is a function of the metric tensor and its derivatives.

Towards a New Theory for Gravity v

2. **When gravitational accelerations are present the space cannot be flat** since the Christoffel symbols are non-zero and the Riemann tensor is not zero as well.

In other words, the presence of the gravitational field forces the space to be curved, i.e. there is a direct link between the presence of a gravitational field and the geometry of the space.

3. **Consequence: if gravity is present there cannot exist inertial frames.**

If it was possible then one would have been able to discriminate among inertial and gravitational accelerations which against the "*generalized equivalence principle*" of Einstein.

The absence of "special" coordinate frames (like the inertial) and their substitution from "general" (non-inertial) coordinate systems lead in naming Einstein's theory for gravity "General Theory of Relativity".

Einstein's Equations i

- Since the source of the gravitational field is a tensor ($T_{\mu\nu}$) the field should be also described by a 2nd order tensor e.g. $\mathcal{E}_{\mu\nu}$.
- Since the role of the gravitational potential is played by the metric tensor then $\mathcal{E}_{\mu\nu}$ should be a function of the metric tensor $g_{\mu\nu}$ and its 1st and 2nd order derivatives.
- Moreover, the law of energy-momenum conservation implies that ${T^{\mu\nu}}_{;\mu} = 0$ which suggests that ${\mathcal{E}^{\mu\nu}}_{;\mu} = 0$.
- Then, since $\mathcal{E}^{\mu\nu}$ should be a linear function of the 2nd derivative of $g_{\mu\nu}$ we come to the following form of the field equations (how & why?):

$$\mathcal{E}^{\mu\nu} = R^{\mu\nu} + a g^{\mu\nu} R + b g^{\mu\nu} = \kappa T^{\mu\nu} \tag{2}$$

where $\kappa = \frac{8\pi G}{c^4}$. Then since ${\mathcal{E}^{\mu\nu}}_{;\mu} = 0$ there should be

$$\left(R^{\mu\nu} + a g^{\mu\nu} R + b g^{\mu\nu}\right)_{;\mu} = 0 \tag{3}$$

which is possible only for $a = -1/2$.

Einstein's Equations ii

Thus the final form of Einstein's equations is:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = \kappa T^{\mu\nu}\,, \tag{4}$$

where $\Lambda = \frac{8\pi G}{c^2}\rho_{\mathrm{vac}}$ is the so called cosmological constant.

Example: Another writing of Einstein Equations

We will show that Einstein's equations can be written as

$$R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \tag{5}$$

This is true because if we multiply

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}$$

with $g^{\rho\nu}$ they can be written as:

$$2R^{\rho}{}_{\nu} - \delta^{\rho}{}_{\nu} R = -2\kappa T^{\rho}{}_{\nu}$$

and by contracting ρ and ν we get $2R - 4R = -2\kappa T$ i.e. $R = \kappa T$. Thus:

$$R_{\mu\nu} = -\kappa T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = -\kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) . \tag{6}$$

Newtonian Limit of Einstein's Equations

In the absence of strong gravitational fields and for small velocities both Einstein's & geodesic equations reduce to the Newtonian ones.

- **Geodesic equations:**

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = -\frac{d^2t/ds^2}{(dt/ds)^2}\frac{dx^\mu}{dt} \quad \Rightarrow \quad \frac{d^2x^j}{dt^2} \approx g_{00,j} \tag{7}$$

  If $g_{00} \approx \eta_{00} + h_{00} = 1 + h_{00} = 1 + \frac{2U}{c^2}$ then $\frac{d^2x^k}{dt^2} \approx -\frac{\partial U}{\partial x^k}$

- **Einstein's equations**

$$R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \quad \Rightarrow \nabla^2 U = \frac{1}{2}\kappa\rho \tag{8}$$

  where $\kappa = 8\pi G$ or $\kappa = 8\pi G/c^4$.

Here we have used the following approximations:

$$\Gamma^j_{00} \approx \frac{1}{2}g_{00,j} \quad \text{and} \quad R_{00} \approx \Gamma^j_{00,j} \approx \nabla^2 U$$

Newtonian Limit: The geodesic equations i

- The geodesic equations are typically written with respect to the proper time τ or the proper length s.

- In Newtonian theory the absolute time and the proper time are identical thus the equations need to be written with respect to the coordinate time t, which is not an affine parameter!

- Thus we will use the form of the geodesic equations (for non affine parameters) presented in Chapter 1 i.e. eqn (78)

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma} = -\frac{d^2\sigma/ds^2}{(d\sigma/ds)^2}\frac{dx^\mu}{d\sigma}\,. \tag{9}$$

and if we select $\sigma = x^0 = ct$ the above relation will be written:

$$\frac{d^2x^\mu}{c^2dt^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{cdt}\frac{dx^\beta}{cdt} = -\frac{cd^2t/ds^2}{(cdt/ds)^2}\frac{dx^\mu}{cdt}\,. \tag{10}$$

Newtonian Limit: The geodesic equations ii

The 1st of the above equations (the one for x^0) simplified because $dx^0/cdt = d(ct)/cdt = 1$ and $d^2x^0/c^2dt^2 = 0$ and thus

$$\Gamma^0_{\alpha\beta}\frac{dx^\alpha}{cdt}\frac{dx^\beta}{cdt} = -\frac{cd^2t/ds^2}{(cdt/ds)^2}\,. \tag{11}$$

This can be substituted in (10) and the remaining 3 equations will have the form for the coordinates x^k for $(k = 1, 2, 3)$ become:

$$\begin{aligned} \frac{d^2x^k}{c^2dt^2} + \left(\Gamma^k_{\alpha\beta} - \Gamma^0_{\alpha\beta}\frac{dx^k}{cdt}\right)\frac{dx^\alpha}{cdt}\frac{dx^\beta}{cdt} &= 0 \\ \frac{d^2x^k}{c^2dt^2} + \left(\Gamma^k_{\alpha\beta} - \Gamma^0_{\alpha\beta}\frac{u^k}{c}\right)\frac{u^\alpha}{c}\frac{u^\beta}{c} &= 0 \end{aligned} \tag{12}$$

Then we will use the approximation that all components of the 3-velocities are much smaller than the speed of light i.e. $dx^k/cdt = u^k/c << 1$.

Newtonian Limit: The geodesic equations iii

Thus $\Gamma^k_{\alpha\beta} >> \Gamma^0_{\alpha\beta}\frac{u^k}{c}$ and the previous equation will be approximately (...) written as:

$$\frac{1}{c^2}\frac{d^2x^k}{dt^2} \approx -\Gamma^k{}_{00} \tag{13}$$

In other words the Christoffel symbol Γ^k_{00} corresponds to the Newtonian force per unit of mass.

Let's assume that the space is "slightly curved" i.e.

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad \eta_{\mu\nu} >> h_{\mu\nu}. \tag{14}$$

Under this assumption the Christoffel $\Gamma^k{}_{00}$, gets the form

$$\Gamma^k{}_{00} = \frac{1}{2}g^{k\lambda}\left(2\frac{\partial g_{\lambda 0}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\lambda}\right) \approx -\frac{1}{2}\eta^{k\lambda}\frac{\partial g_{00}}{\partial x^\lambda} = \frac{1}{2}\delta^{kj}\frac{\partial g_{00}}{\partial x^j} = \frac{1}{2}\frac{\partial g_{00}}{\partial x^k} \equiv \frac{1}{2}g_{00,k} \tag{15}$$

Thus in the Newtonian limit the geodesic equations reduced to:

$$\frac{1}{c^2}\frac{d^2x^k}{dt^2} \approx \frac{1}{2}\frac{\partial g_{00}}{\partial x^k}\,. \tag{16}$$

Newtonian Limit: The geodesic equations iv

Which reminds the Newtonian relation

$$\frac{d^2x^k}{dt^2} \approx \frac{\partial U}{\partial x^k} \tag{17}$$

suggesting that g_{00} has the form

$$g_{00} \approx \eta_{00} + h_{00} = 1 + h_{00} = 1 + \frac{2U}{c^2} \tag{18}$$

here we set $h_{00} = \frac{2U}{c^2}$.

Newtonian Limit: Einstein equations i

We will show that Einstein's equations in the Newtonian limit reduce to the well known Poisson equation of Newtonian gravity.

For small concentrations of masses the dominant component of the energy momentum tensor is the T_{00}. Thus the dominant component of the Einstein's equations is

$$\begin{aligned} R_{00} &= -\kappa\left(T_{00} - \frac{1}{2}g_{00}T\right) \approx -\kappa\left(T_{00} - \frac{1}{2}\eta_{00}T\right) \approx -\frac{1}{2}\kappa T_{00} \\ &= -\frac{1}{2}\kappa\rho c^2 \,. \end{aligned} \tag{19}$$

where we assumed that $T = g^{\mu\nu}T_{\mu\nu} \approx \eta^{\mu\nu}T_{\mu\nu} \approx \eta^{00}T_{00} = T_{00}$.

The 00 component of the Ricci tensor is given by the relation

$$R_{00} = \Gamma^{\mu}_{00,\mu} - \Gamma^{\mu}_{0\mu,0} + \Gamma^{\mu}_{00}\Gamma^{\nu}_{\nu\mu} - \Gamma^{\mu}_{0\nu}\Gamma^{\nu}_{0\mu} \approx \Gamma^{\mu}_{00,\mu} \approx \Gamma^{j}_{00,j} \,. \tag{20}$$

Newtonian Limit: Einstein equations ii

But as we have shown $\Gamma^j_{00} \approx g_{00,j}/2$ and thus:

$$R_{00} \approx \Gamma^j_{00,j} \approx \frac{1}{2}\frac{\partial^2 g_{00}}{\partial x^j \partial x^j} = \frac{1}{2}\nabla^2 g_{00} \approx \frac{1}{c^2}\nabla^2 U\,. \tag{21}$$

leading to:

$$\nabla^2 U = \frac{1}{2}\kappa c^4 \rho \tag{22}$$

From which by comparing with Poisson's equation (1) we get the value of the coupling constant κ that is:

$$\kappa = \frac{8\pi G}{c^4}\,. \tag{23}$$

A few words about the cosmological-constant term i

NOTE: In eqn (3) there exists the term $bg^{\mu\nu}$.
There is freedom in adding constant multiples of $g^{\mu\nu}$ since we know from equation (62) (1st Chapter) that ${g^{\mu\nu}}_{;\mu} = 0$.

- This is the reason that we kept this term in (4) and we wrote it as Λ and named it as cosmological constant.

- We already know that Einstein's equations (5) can be written as

$$R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) + \Lambda g_{\mu\nu} \,. \tag{24}$$

which in the Newtonian limit, eqn (22), will be written in the form

$$\nabla^2 U = 4\pi G\rho - \Lambda c^2 \tag{25}$$

and for a spherically symmetric mass M the gravitational field strength will be written as

$$\vec{g} = \vec{\nabla} U = -\frac{GM}{r^2}\hat{\vec{r}} + \frac{c^2 \Lambda r}{3}\hat{\vec{r}} \tag{26}$$

A few words about the cosmological-constant term ii

- Thus, if Λ is positive corresponds to a **gravitational repulsion** whose strengh **increases linearly** with r.

The energy-momentum tensor of a perfect fluid as defined earlier is:

$$T^{\mu\nu} = \left(\rho + p/c^2\right) u^\mu u^\nu - p g^{\mu\nu} \tag{27}$$

If we consider some **strange** form of **something** with equation of state $p = -\rho c^2$ (negative pressure !!!) its energy momentum tensor will be written as

$$T^{\mu\nu} = \rho c^2 g^{\mu\nu} \tag{28}$$

Notice that:

- Its energy-momentum tensor depends only on the **metric tensor** (not on the velocities). This means that it is property of the vacuum itself and can be called **the energy density of the vacuum**.
- Its $T^{\mu\nu}$ is similar to the cosmological-constant term in eqn (4).

A few words about the cosmological-constant term iii

Thus we can view the cosmological constant as a universal constant that fixes the energy density of the vacuum

$$\rho_{\mathrm{vac}} c^2 = \frac{\Lambda c^4}{8\pi G} \tag{29}$$

The Scalar Theory of Gravity

It is the simplest generalization of Newton's gravity.

Matter relativistically is described by the energy-momentum tensor $T_{\mu\nu}$, the only scalar with the dimensions of mass density is T^{μ}_{μ}.

Thus a "consistent" scalar relativistic theory of gravity will be given by the field equation

$$\Box^2 U = -\frac{4\pi G}{c^2} T^{\mu}_{\mu} \tag{30}$$

This theory cannot be accepted because:

- Predicts a **retardation** of the perihelion of Mercury in contradiction to observations
- It does not allow one to **couple gravity to electromagnetism** since ${}^{(EM)}T^{\mu}_{\mu} = 0$.
- Does not predict **gravitational redshift**.
- Does not predict **deflection of light by matter** .

NOTE: A gravitational theory based on a **vector field** cannot be accepted since such a theory predicts that two massive particles would repel and not attract one another.

The Scalar-Tensor Theory of Gravity (Jordan-Brans-Dicke) i

In principle it is possible to construct theories which combine **tensor**, **vector** *and* **scalar** *fields.*

Brans-Dicke assumed equivalence principle and also that gravity is described as spacetime curvature.
In addition, they introduced a **scalar field** ϕ that determines the strength of gravitational "constant" G.
The key ingredients of the theory are:

- **Matter** is represented by the energy-momentum tensor ${}^{(M)}T_{\mu\nu}$, while a coupling constant λ fixes the scalar field ϕ
- The scalar field ϕ fixes the value of G.
- The gravitational field equations relate the curvature to the energy momentum tensors of the **scalar field** and **matter**.

The Scalar-Tensor Theory of Gravity (Jordan-Brans-Dicke) ii

The field equations are:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi}{c^4 \phi} \left[{}^{(M)}T_{\mu\nu} + {}^{(\phi)}T_{\mu\nu} \right] \tag{31}$$

$$\Box\phi = -4\pi\lambda \, {}^{(M)}T^{\mu}_{\mu} \tag{32}$$

while the energy-momentum tensor of the scalar field is given by

$$ {}^{(\phi)}T_{\alpha\beta} = \frac{\omega}{\phi^2} \left(\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2} g_{\alpha\beta} \phi_{,\gamma}\phi^{,\gamma} \right) + \frac{1}{\phi} \left(\phi_{;\alpha\beta} - g_{\alpha\beta} \Box\phi \right) \tag{33}$$

- Customary the coupling constant λ is written as:

$$\lambda = \frac{2}{3 + 2\omega} \tag{34}$$

- In the limit $\omega \to \infty$, $\lambda \to 0$ so the scalar field is not affected by the matter distribution and can be set equal to a constant $\phi = 1/G$ (!).

The Scalar-Tensor Theory of Gravity (Jordan-Brans-Dicke) iii

In this case ${}^{(\phi)}T\mu\nu = 0$ and Brans - Dicke theory reduces to Einstein's in the limit $\omega \to \infty$.

- One of the key features of Brans - Dicke theory is that the effective gravitational "constant" G varies with time and is determined by the scalar field ϕ.
- Brans - Dicke theory predicts **light deflection** and the **precession of perihelia** of planets orbiting the Sun but the formulae depend on ω.
- A variation of the G affects the orbits of planets e.g. altering the dates of eclipses.
- The experiments as for today predict that $\omega \geq \mathbf{40,000}$ (Cassini-Huygens experiment, 2003). In 1973 $\omega \geq 5$, in 1981 $\omega \geq 30$!

Solutions of Einstein's Equations i

• A **static spacetime** is one for which a timelike coordinate x^0 has the following properties:

(I) all metric components $g_{\mu\nu}$ are independent of x^0

(II) the line element ds^2 is invariant under the transformation $x^0 \to -x^0$.

Note that the 1st property does not imply the 2nd (e.g. the time reversal on a rotating star changes the sense of rotation, but the metric components are constant in time).

• A spacetime that satisfies (I) but not (II) is called **stationary**.

• The line element ds^2 of a static metric depends only on **rotational invariants** of the spacelike coordinates x^i and their differentials, i.e. the metric is **isotropic**.

Solutions of Einstein's Equations ii

The only rotational invariants of the spacelike coordinates x^i and their differentials are

$$\vec{x}\cdot\vec{x}\equiv r^2, \quad \vec{x}\cdot d\vec{x}\equiv r dr, \quad d\vec{x}\cdot d\vec{x}\equiv dr^2+r^2 d\theta^2+r^2\sin^2\theta d\phi^2$$

Thus the more general form of a spatially isotropic metric is:

$$\begin{aligned} ds^2 &= A(t,r)dt^2 - B(t,r)dt\,(\vec{x}\cdot d\vec{x}) - C(t,r)(\vec{x}\cdot d\vec{x})^2 - D(t,r)d\vec{x}^2 \\ &= A(t,r)dt^2 - B(t,r)r\,dt\,dr - C(t,r)r^2dr^2 \\ &\qquad -D(t,r)\left(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2\right) \qquad (35) \\ &= A(t,r)dt^2 - \tilde{B}(t,r)dt\,dr - \tilde{C}(t,r)dr^2 - \tilde{D}(t,r)\left(d\theta^2+\sin^2\theta d\phi^2\right) \\ &= A'(t,\tilde{r})dt^2 - B'(t,\tilde{r})dt\,d\tilde{r} - C'(t,\tilde{r})d\tilde{r}^2 - \tilde{r}^2\left(d\theta^2+\sin^2\theta d\phi^2\right) \qquad (36) \end{aligned}$$

where we have set $\tilde{r}^2 = D(t,r)$ and redefined the A, $\tilde{B}$ and $\tilde{C}$.

The next step will be to introduce a new timelike coordinate $\tilde{t}$ as

$$d\tilde{t} = \Phi(t,\tilde{r})\left[A'(t,\tilde{r})dt - \frac{1}{2}B'(t,\tilde{r})d\tilde{r}\right]$$

Solutions of Einstein's Equations iii

where $\Phi(t, \tilde{r})$ is an integrating factor that makes the right-hand side an exact differential.

By squaring we obtain

$$d\tilde{t}^2 = \Phi^2 \left(A'^2 dt^2 - A' B' dt d\tilde{r} + \frac{1}{4} B'^2 d\tilde{r}^2 \right)$$

from which we find

$$A' dt^2 - B' dt d\tilde{r} = \frac{1}{A'\Phi^2} d\tilde{t}^2 - \frac{B'^2}{4A'} d\tilde{r}^2$$

Thus by defining the new functions $\hat{A} = 1/(A'\Phi)^2$ and $\hat{B} = C + B'^2/(4A')$ the metric (36) becomes **diagonal**

$$ds^2 = \hat{A}(\tilde{t}, \tilde{r}) d\tilde{t}^2 - \hat{B}(\tilde{t}, \tilde{r}) d\tilde{r}^2 - \tilde{r}^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \tag{37}$$

or by dropping the 'hats' and 'tildes'

$$ds^2 = A(t, r) dt^2 - B(t, r) dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \tag{38}$$

Solutions of Einstein's Equations iv

- Thus the **general isotropic metric** is specified by two functions of t and r, namely $A(t,r)$ and $B(t,r)$.
- Also, surfaces for t and r constant are 2-spheres (isotropy of the metric).
- Since $B(t,r)$ is not unity we cannot assume that r is the radial distance.

The metric tensor in matrix form is:

$$g_{\mu\nu} = \begin{pmatrix} A(t,r) & 0 & 0 & 0 \\ 0 & -B(t,r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{39}$$

Schwarzschild Solution i

A typical solution of Einstein's equations describing spherically symmetric spacetimes has the form:

$$ds^2 = e^{\nu(t,r)}dt^2 - e^{\lambda(t,r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{40}$$

We next calculate the components of the Ricci tensor

$$R_{\mu\nu} = 0\,. \tag{41}$$

Then the Christoffel symbols for the metric (40) ($\dot{} = \partial/\partial t$) & ($' = \partial/\partial r$) are

$$\Gamma^0_{00} = \frac{\dot\nu}{2}\,, \qquad \Gamma^1_{00} = \frac{\nu'}{2}e^{\nu-\lambda}\,, \qquad \Gamma^0_{01} = \frac{\nu'}{2}\,, \tag{42}$$

$$\Gamma^1_{01} = \frac{\dot\lambda}{2}\,, \qquad \Gamma^0_{11} = \frac{\dot\lambda}{2}e^{\lambda-\nu}\,, \qquad \Gamma^1_{11} = \frac{\lambda'}{2}\,, \tag{43}$$

$$\Gamma^1_{12} = \frac{1}{r} \qquad \Gamma^3_{13} = \frac{1}{r}\,, \qquad \Gamma^1_{22} = -re^{-\lambda}\,, \tag{44}$$

$$\Gamma^3_{23} = \cot\theta\,, \qquad \Gamma^1_{33} = -re^{-\lambda}\sin^2\theta\,, \qquad \Gamma^2_{33} = -\sin\theta\ \cos\theta\,. \tag{45}$$

Schwarzschild Solution ii

Then the components of the Ricci tensor will be:

$$
\begin{aligned}
R_{00} &= e^{\nu-\lambda}\left[\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'\lambda'}{4}+\frac{\nu'}{r}\right]-\left[\frac{\ddot{\lambda}}{2}+\frac{\dot{\lambda}^2}{4}-\frac{\dot{\lambda}\dot{\nu}}{4}\right] && (46)\\
R_{11} &= -\frac{\nu''}{2}-\frac{\nu'^2}{4}+\frac{\nu'\lambda'}{4}+\frac{\lambda'}{r}+e^{\lambda-\nu}\left[\frac{\ddot{\lambda}}{2}+\frac{\dot{\lambda}^2}{4}-\frac{\dot{\lambda}\dot{\nu}}{4}\right] && (47)\\
R_{10} &= \frac{\dot{\lambda}}{r} && (48)\\
R_{22} &= = -e^{-\lambda}\left[1+\frac{r}{2}(\nu'-\lambda')\right]+1 && (49)\\
R_{33} &= \sin^2\theta R_{22}\,. && (50)
\end{aligned}
$$

In the absence of matter (outside the source) $R_{\mu\nu}=0$ we can prove that $\lambda(r)=-\nu(r)$, i.e. the solution is independent of time (how and why?).

We will use the above relations to find the functional form of $\nu(t,r)$ and $\lambda(t,r)$ in an empty space i.e. when $R_{\mu\nu}=0$.

Then from the sum: $R_{11} + e^{\lambda-\nu} R_{00} = 0$ we get:

$$\nu' + \lambda' = 0\,. \tag{51}$$

While from eqn. (48)

$$R_{10} = \frac{\dot{\lambda}}{r} = 0 \tag{52}$$

we conclude that $\lambda = \lambda(r)$.

From $R_{22} = 0$, eqn. (49), we get:

$$\nu' - \lambda' = \frac{2}{r}(e^{\lambda} - 1) \tag{53}$$

Thus by combining eqns (51) and (53) we get:

$$\nu' = \frac{1}{r}(e^{\lambda} - 1) \tag{54}$$

$$\lambda' = -\frac{1}{r}(e^{\lambda} - 1) \tag{55}$$

Schwarzschild Solution iv

Since the right side of the above differential equation (54) for ν is function only of r the function $\nu = \nu(t, r)$ can be written as (why?):

$$\nu(t,r) = \alpha(t) + \tilde{\nu}(r) \tag{56}$$

which suggests that we may redefine the time coordinate t to $\tilde{t}$ as follows;

$$d\tilde{t} = e^{\alpha/2} dt \tag{57}$$

i.e the time dependence of $\nu(t, r)$ is "absorbed" by the change of variable from t to $\tilde{t}$.

Thus from (51) we lead to the relation:

$$\lambda(r) = -\tilde{\nu}(r) \tag{58}$$

i.e. both unknown components of the metric tensor are independent of time and for simplicity we will write $\nu(r)$ instead of $\tilde{\nu}(r)$.

Schwarzschild Solution v

BIRKOFF's THEOREM

- *If the geometry of a spacetime is* **spherically symmetric** *and* **solution of the Einstein's equations**, *then it is described by the Schwarzschild solution*.

OR

- The spacetime geometry outside a general spherically symmetric matter distribution is the Schwarzschild geometry.

The solution of equation (55) provides the function $\lambda(r)$.
We can get it by substituting $f = e^{-\lambda}$ then eqn (55) will be written as:

$$rf' + f = 1 \tag{59}$$

with obvious solution:

$$f = 1 - \frac{k}{r} = e^{-\lambda} \equiv e^{\nu} \tag{60}$$

where k and will be determined by the boundary conditions.

Schwarzschild Solution vi

In the present case for $r \to \infty$ our solutions should lead to Newton's solution that is

$$g_{00} = 1 + \frac{2}{c^2} U(r) \tag{61}$$

where $U(r)$ is the Newtonian potential and for the case of spherical symmetry is

$$U(r) = -\frac{GM}{r} \tag{62}$$

which leads to

$$k = \frac{2GM}{c^2} \tag{63}$$

and Schwarzschild solutions gets the form:

$$ds^2 = \left(1 - \frac{2}{c^2}\frac{GM}{r}\right)c^2 dt^2 - \left(1 - \frac{2}{c^2}\frac{GM}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- Sun: $M_\odot \approx 2 \times 10^{33} gr$ and $R_\odot = 696.000 km$

$$\frac{2GM}{rc^2} \approx 4 \times 10^{-6}$$

- Neutron star : $M \approx 1.4 M_\odot$ and $R_\odot \approx 10 - 15 km$

$$\frac{2GM}{rc^2} \approx 0.3 - 0.5$$

- Newtonian limit:

$$g_{00} \approx \eta_{00} + h_{00} = 1 + \frac{2U}{c^2} \quad \Rightarrow \quad U = \frac{GM}{r} \tag{64}$$