The Variational Principle in Field Theory & the Canonical Energy Momentum Tensor

A brief introduction based on Ohanian-Ruffini

Kostas Kokkotas

January 30, 2020

Lagrange Equations for a system of particles i

Lagrangian formalism for a system of fields is a generalization of the formalism for a system of particles.

- A system of particles can be described by a finite set of **generalized coordinates** $q_i(t)$, $(i = 1, 2, 3, ..., N)$.
- The Lagrangian formalism rests on the assumption that the dynamical equations of motion can be derived from Hamilton's variational principle.
- That is, we define the integral I

$$I = \int_{t_1}^{t_2} L\left(q_i(t), \dot{q}_i(t)\right) dt \qquad (1)$$

where the Lagrangian is a function of the coordinates q_i and the velocities $\dot{q}_i = dq_i/dt$.

- The equations of motion can be derived by requiring that the action remains stationery for infinitesimal variations of the functions $q_i(t)$.

Lagrange Equations for a system of particles ii

- These variations are arbitrary except for the constraint that they vanish at the times t_1 and t_2.
- If the variation of q_i at the time t is $\delta q_i(t)$, then the velocities will change accordingly as $\delta \dot{q}_i(t) = (d/dt)\delta q_i(t)$ and the action will change as

$$\delta I = \int_{t_1}^{t_2} \sum_{i=1}^{N} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt \tag{2}$$

the second term (in blue color) can be written as $\frac{\partial L}{\partial \dot{q}_i}(\frac{d}{dt}\delta q_i)$ and be integrated by parts. Since the variation δq_i vanishes at $t = t_1$ and at $t = t_2$ we get

$$\delta I = \int_{t_1}^{t_2} \sum_{i=1}^{N} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \, dt \tag{3}$$

- The action will be stationary for arbitrary choices of the functions $\delta q_i(t)$ if and only if all these functions have zero coefficients, that is, if

Lagrange Equations for a system of particles iii

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0} \tag{4}$$

These are the **Euler-Lagrange** equations for the system.

- The number of eqns is the same as the degrees of freedom of the system (N).
- In eqn(4) we may use also the **canonical momentum** (conjugate to δq_i)

$$\pi_i = \frac{\partial L}{\partial \dot{q}_i} \quad \to \quad \frac{d}{dt}\pi_i - \frac{\partial L}{\partial q_i} = 0. \tag{5}$$

- Note that here we used the time t as independent coordinate but later we will use the proper time τ

Based on the above we can prove that the Hamiltonian is a constant of motion.

The **Hamiltonian** is:

Lagrange Equations for a system of particles iv

$$H = \sum_{i=1}^{N} \left[\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right] - L \equiv \sum_{i=1}^{N} \dot{q}_i \pi_i - L \tag{6}$$

Then

$$\begin{aligned} \frac{dH}{dt} &= \sum_{i=1}^{N} \left[\cancel{\ddot{q}_i \frac{\partial L}{\partial \dot{q}_i}} + \dot{q}_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right] - \sum_{i=1}^{N} \left[\frac{\partial L}{\partial q_i} \dot{q}_i + \cancel{\frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i} \right] \\ &= \sum_{i=1}^{N} \dot{q}_i \left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \right] = 0 \end{aligned} \tag{7}$$

- The conservation law of the Hamiltonian is directly related to invariance under **time translation** ($dH/dt = 0$) which is direct consequence of the absence of an explicit time dependence in the Lagrangian (it depends only implicit via $\delta q_i(t)$).

Lagrange Equations for a system of particles v

- Momentum conservation is a direct consequence of invariance under **space translations** ($d\pi_i/dt = 0$) if L is independent of q_i (see eqn (5)).

Thus the conservation of energy and momentum is intimately connected with symmetry under time and space translations (Noether's Theorem).

The Lagrange Equations for fields i

Let's assume the case of one-component field, a **scalar field**.

- A one-component field is described by a function $\psi(\mathbf{x}, t)$.
- At a given time, the function $\psi(\mathbf{x}, t)$ gives the amplitude of the field at the point $\mathbf{x}$. This analogous to the set of the generalized coordinates q_i used earlier.
- To describe the field system we must specify $\psi(\mathbf{x}, t)$ for all $\mathbf{x}$; to describe the particle system we must specify q_i for all i. The correspondence can be used if we try to prescribe the field in a discretized set of points.
- If we divide the space in cubical shells of volume ΔV and consider only the amplitudes of the field at the centers of the cubes which have coordinates $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ then we can replace the field by a set of generalized coordinates

$$q_i = \psi(\mathbf{x}) \qquad i = 1, 2, 3, \ldots \tag{8}$$

The Lagrange Equations for fields ii

- In this way we replace approximately the field equations by a set of Euler-Lagrange equations given by (4). The exact field can me recovered in the limit $\Delta V \to 0$.

The action integral for the field can be written by analogy to the point particle case

$$L = \int \mathcal{L}\left(\psi, \partial\psi/\partial t, \partial\psi/\partial x^k\right) d^3x \tag{9}$$

where $\mathcal{L}\left(\psi, \partial\psi/\partial t, \partial\psi/\partial x^k\right)$ is the **Lagrangian density**.

The action integral is then

$$I = \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \int \mathcal{L}\left(\psi, \partial\psi/\partial t, \partial\psi/\partial x^k\right) d^3x\, dt \tag{10}$$

The variation $\delta\psi(\mathbf{x}, t)$ in the field produces corresponding variations

$$\delta\left(\frac{\partial\psi}{\partial t}\right) = \frac{\partial}{\partial t}\delta\psi \quad \text{and} \quad \delta\left(\frac{\partial\psi}{\partial x^k}\right) = \frac{\partial}{\partial x^k}\delta\psi \tag{11}$$

Then the action can be written:

$$
\begin{aligned}
\delta I &= \int_{t_1}^{t_2} \int \left[\frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial (\partial \psi / \partial t)} \delta \left(\frac{\partial \psi}{\partial t} \right) + \frac{\partial \mathcal{L}}{\partial (\partial \psi / \partial x^k)} \delta \left(\frac{\partial \psi}{\partial x^k} \right) \right] d^3x\, dt \\
&= \int_{t_1}^{t_2} \int \left[\frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial (\partial \psi / \partial t)} \frac{\partial}{\partial t} \delta \psi + \frac{\partial \mathcal{L}}{\partial (\partial \psi / \partial x^k)} \frac{\partial}{\partial x^k} \delta \psi \right] d^3x\, dt \quad (12)
\end{aligned}
$$

By integrating the 2nd term (in blue) by parts and imposing the constraint that $\delta\psi = 0$ at $t = t_1$ and at $t = t_2$ and the 3rd term (in red) by parts using the assumption that $\delta\psi = 0$ at the limits of the spatial domain we get:

$$
\delta I = \int_{t_1}^{t_2} \int \left[\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial (\partial \psi / \partial t)} - \frac{\partial}{\partial x^k} \frac{\partial \mathcal{L}}{\partial (\partial \psi / \partial x^k)} \right] \delta \psi \, d^3x \, dt \qquad (13)
$$

Since $\delta\psi$ is an arbitrary function, the action will be stationary if and only if the term in brackets is zero.

The Lagrange Equations for fields iv

The previous statement leads to the equations of motion for the fields, which contain both space and time derivatives ($\mu = 0, ..., 3$) and resembles to eq (4)

$$\boxed{\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \psi_{,\mu}}\right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0} \tag{14}$$

The **Hamiltonian** for a system of fields resembles Eq (6), but the summation is replaced by integration over x.

$$H = \int \psi_{,0} \frac{\partial \mathcal{L}}{\partial \psi_{,0}} d^3x - L = \int \left(\psi_{,0} \frac{\partial \mathcal{L}}{\partial \psi_{,0}} - \mathcal{L}\right) d^3x \tag{15}$$

From the field equations one can show that $dH/dt = 0$, which says that **the Hamiltonian of a closed system is the total energy of the system**.

Correspondence between particle and field systems

	Particle System	Field System
State of the system described by	$q_i(t)$	$\psi(\mathbf{x}, t)$
Independent variables	$i\ ,\ t$	$\mathbf{x}\ ,\ t$
Lagrangian	$L = L(q_i, \dot{q}_i)$	$L = \int \mathcal{L}\left(\psi, \partial\psi/\partial t, \partial\psi/\partial x^k\right) d^3x$
Equations of motion	$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$	$\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \psi_{,\mu}} - \frac{\partial \mathcal{L}}{\partial \psi} = 0$
Hamiltonian Energy	$H = \sum_{i=1}^{N}\left[\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right] - L$	$H = \int \left(\psi_{,0}\frac{\partial \mathcal{L}}{\partial \psi_{,0}} - \mathcal{L}\right) d^3x$

The Energy-Momentum Tensor i

Since H is the energy, the integrant appearing in (15) can be regarded as the energy density

$$t_0^0 = \psi_{,0} \frac{\partial \mathcal{L}}{\partial \psi_{,0}} - \mathcal{L} \tag{16}$$

which leads to an *educated guess* for the complete energy-momentum tensor

$$t_\mu{}^\nu = \psi_{,\mu} \frac{\partial \mathcal{L}}{\partial \psi_{,\nu}} - \mathcal{L} \tag{17}$$

This tensor is called **the canonical energy-momentum tensor**.

The next step is to prove that obeys to the conservation law

$$t_\mu{}^\nu{}_{,\nu} = 0 \tag{18}$$

The proof is as follows:

$$t_\mu{}^\nu{}_{,\nu} = \cancel{\psi_{,\mu\nu} \frac{\partial \mathcal{L}}{\partial \psi_{,\nu}}} + \psi_{,\mu} \frac{\partial}{\partial x^\nu} \left(\frac{\partial \mathcal{L}}{\partial \psi_{,\nu}} \right) - \psi_{,\mu} \frac{\partial \mathcal{L}}{\partial \psi} - \cancel{\psi_{,\alpha\mu} \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}} \tag{19}$$

The Energy-Momentum Tensor ii

The 1st and 4th term cancel out while the remaining terms are zero due to (14).

The differential conservation law (18)

$$\frac{\partial}{\partial t} {t_0}^0 + \frac{\partial}{\partial x^k} {t_0}^k = 0 \tag{20}$$

implies the conservation of energy (Hamiltonian)

$$\frac{dH}{dt} = \frac{d}{dt} \int {t_0}^0 d^3 x = \int \frac{\partial}{\partial t} {t_0}^0 d^3 x = - \int \frac{\partial}{\partial x^k} {t_0}^k d^3 x \tag{21}$$

By using the Gauss' theorem, we can change the volume integral of the divergence term $(\partial/\partial x^k){t_0}^k$ into a surface integral.

- Under the assumption that t_0^k is exactly zero beyond some large distance or at least tends to zero faster than $1/r^2$, the surface integral vanishes implying that H is constant.

In a similar way it can be shown that the **total momentum**

$$P_k = \int {t_0}^k d^3x \tag{22}$$

is constant. (HOW?)

DISCUSSION ...

The Energy-Momentum Tensor for Scalar Fields i

Let us consider the case of the one-component field $\psi(\mathbf{x}, t)$ with the **Lagrangian density**

$$\mathcal{L} = \frac{1}{2}\left(\psi_{,\alpha}\psi^{,\alpha} - m^2\psi^2\right) = \frac{1}{2}\left(\eta^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} - m^2\psi^2\right) \tag{23}$$

where m is a constant.

The Lagrangian field equation is:

$$\frac{\partial}{\partial x^\alpha}\eta^{\mu\alpha}\psi_{,\alpha} + m^2\psi = 0 \quad \text{that is} \quad \psi_{,\mu}{}^{,\mu} + m^2\psi = \left(\Box + m^2\right)\psi = 0 \tag{24}$$

This is a very well known equation in quantum mechanics, is the **Klein-Gordon** equation, for a free scalar field.

The corresponding **canonical energy-momentum tensor** (according to eqn (17)) is

$$t_\mu{}^\nu = \psi_{,\mu}\eta^{\nu\alpha}\psi_{,\alpha} - \delta_\mu{}^\nu\mathcal{L} = \psi_{,\mu}\psi^{,\nu} - \frac{1}{2}\delta_\mu^\nu\left(\psi_{,\alpha}\psi^{,\alpha} - m^2\psi^2\right) \tag{25}$$

The Energy-Momentum Tensor for Scalar Fields ii

and the **energy density** is

$$t_0{}^0 = \frac{1}{2}\left(\psi_{,0}\right)^2 + \frac{1}{2}\left(\nabla\psi\right)^2 + \frac{1}{2}m^2\psi^2 \tag{26}$$

The Energy-Momentum Tensor for Electromagnetic Fields i

For multi-component fields (e.g. A_μ, $h_{\mu\nu}$) if we perform the variation of the action we end up with equations of the type (13) for every component.

Since the different components have independent variations, each of these terms must vanish separately.

Thus we get as many equations as they are the components of the field.

For a **4-component** (vector) field we get

$$\boxed{\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial A^\nu{}_{,\mu}} - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0 \quad \text{for} \quad \nu = 0,1,2,3} \tag{27}$$

Then equation (27) yields the field equations for a free EM field provided we take

$$\mathcal{L}_{(\mathrm{em})} = -\frac{1}{16\pi}\left(A_{\mu,\nu} - A_{\nu,\mu}\right)\left(A^{\mu,\nu} - A^{\nu,\mu}\right) = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \tag{28}$$

The Energy-Momentum Tensor for Tensor Fields i

For a **16-component** (tensor) field we get

$$\boxed{\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial h^{\alpha\beta}{}_{,\mu}} - \frac{\partial \mathcal{L}}{\partial h^{\alpha\beta}} = 0 \quad \text{for} \quad \alpha, \beta = 0, 1, 2, 3} \tag{29}$$

Equation (29) yields the linear field equations for the free gravitational field (in the linear approximation) provided we take [1]

$$\mathcal{L}_{(0)} = \frac{1}{4}\left(h_{\mu\nu,\lambda}h^{\mu\nu,\lambda} - 2h_{\mu\nu}{}^{,\mu}h^{\nu\lambda}{}_{,\lambda} + 2h_{\mu\nu}{}^{,\mu}h^{,\nu} - h_{,\nu}h^{,\nu}\right) \tag{30}$$

The canonical energy-momentum tensors can be calculated from the generalization of (15):

$$t_{(\mathrm{em})}{}_{\mu}{}^{\nu} = A^{\alpha}{}_{,\mu}\frac{\partial \mathcal{L}_{(\mathrm{em})}}{\partial A^{\alpha}{}_{,\nu}} - \delta_{\mu}{}^{\nu}\mathcal{L}_{(\mathrm{em})} \tag{31}$$

$$t_{(0)\mu}{}^{\nu} = h^{\alpha\beta}{}_{,\mu}\frac{\partial\mathcal{L}_{(1)}}{\partial h^{\alpha\beta}{}_{,\nu}} - \delta_{\mu}{}^{\nu}\mathcal{L}_{(1)} \tag{32}$$

Then by combining equations (28) and (31) we get:

$$t_{(\mathrm{em})\mu}{}^{\nu} = -\frac{1}{4\pi}\left[-A^{\alpha}{}_{,\mu}F^{\nu}{}_{\alpha} - \frac{1}{4}\delta_{\mu}{}^{\nu}F^{\alpha\beta}F_{\alpha\beta}\right] \tag{33}$$

where

$$F^{\alpha\beta} = A^{\alpha,\beta} - A^{\beta,\alpha} \tag{34}$$

In the same way by using equations (30) and (32) plus the gauge condition $h^{\mu\nu}{}_{,\nu} - (1/2)h^{,\nu} = 0$ we get:

$$t_{(1)\mu}{}^{\nu} = \frac{1}{4}\left[2h^{\alpha\beta}{}_{,\mu}h_{\alpha\beta}{}^{,\nu} - h_{,\mu}h^{,\nu} - \delta_{\mu}{}^{\nu}\left(h^{\alpha\beta}{}_{,\lambda}h_{\alpha\beta}{}^{,\lambda} - \frac{1}{2}h_{,\lambda}h^{,\lambda}\right)\right] \tag{35}$$

this equation is identical to the one used in Chapter 2 if we substitute $h^{\alpha\beta} = \phi^{\alpha\beta} - (1/2)\eta^{\alpha\beta}\phi$.

DISCUSSION

The equation (33) for the energy momentum-tensor of the EM field is similar to the equation (28) of the second set of slides i.e.

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu\beta}\eta_{\alpha\beta} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{36}$$

and it is not gauge invariant. But by adding an extra term

$$\frac{1}{4\pi}\frac{\partial}{\partial x^{\alpha}}\left(A_{\mu}F^{\nu\alpha}\right) \tag{37}$$

to ${t_{(\mathrm{em})}}_{\mu}{}^{\nu}$ of equation (33) everything can be repaired (HOW?)

[1]The subscript (0) indicates that we are dealing with the linear approximation

The Variational Principle for Einstein's Equations i

In the previous section we have seen that we could take the field equations for gravitation (linear approximation) from the variational principle with the Lagrangian $\mathcal{L}_{(0)}$ of eqn (30).

In the same way the exact nonlinear Einstein equations can also be obtained from a variational principle by taking the following Lagrangian

$$\mathcal{L}_G = \frac{1}{\kappa^2} R\sqrt{-g} \tag{38}$$

where R is the curvature scalar and $\kappa^2 = 16\pi G/c^4$.

This Lagrangian contains both 1st and 2nd derivatives of the fields $g_{\mu\nu}$. Analytically it can be written as:

$$R\sqrt{-g} = \left[g^{\mu\nu}\sqrt{-g}\left(-\Gamma^{\alpha}_{\mu\nu} + \delta_{\mu}{}^{\alpha}\Gamma^{\beta}_{\nu\beta}\right)\right]_{,\alpha} - g^{\mu\nu}\sqrt{-g}\left(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}\right) \tag{39}$$

The Variational Principle for Einstein's Equations ii

The first term contains 2nd derivatives of $g_{\mu\nu}$, but since it is a divergence its contribution to the action

$$I = \frac{1}{\kappa^2} \int R\sqrt{-g} d^3x \, dt \tag{40}$$

is of the form

$$\frac{1}{\kappa^2} \int \int \frac{\partial}{\partial x^\alpha} \left[g^{\mu\nu} \sqrt{-g} \left(-\Gamma^\alpha_{\mu\nu} + \delta^\alpha_\mu \Gamma^\beta_{\nu\beta} \right) \right] d^3x \, dt \tag{41}$$

which is the 4-dim volume integral of a divergence and can be converted to a surface integral over the boundary of the volume.

Since the variational principle assumes that the variation $\delta g_{\mu\nu}$ vanishes on the boundary it follows that the equation (41) gives no-contribution to the variation of the action and we will use the following form of the (effective) Lagrangian which includes only 1st derivatives of $g_{\mu\nu}$

The Variational Principle for Einstein's Equations iii

$$\mathcal{L}_G = \frac{1}{\kappa^2} g^{\mu\nu} \sqrt{-g} \left(\Gamma^{\beta}{}_{\mu\alpha} \Gamma^{\alpha}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\mu\nu} \Gamma^{\beta}{}_{\alpha\beta} \right) . \tag{42}$$

Einstein's equations can be derived using this Lagrangian (after a lengthy calculation).

It is easier to show it by using the linear approximation

The Euler-Lagrange equations for our case will be tensor equations.

This can be seen by writing the variation of the action in the form:

$$\delta \int \frac{1}{\kappa^2} R \sqrt{-g} d^3x\, dt = \int \left[\frac{1}{\sqrt{-g}} \left(\frac{\partial \mathcal{L}_G}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x^\alpha} \frac{\partial \mathcal{L}_G}{\partial g_{\mu\nu,\alpha}} \right) \delta g_{\mu\nu} \right] \sqrt{-g}\, d^3x\, dt \tag{43}$$

- Since R is a scalar the quantity in the brackets should be also a scalar but $\delta g_{\mu\nu}$ is an arbitrary tensor and thus the quantity in the parenthesis (in blue) is also a tensor

$$\frac{1}{\sqrt{-g}} \left(\frac{\partial \mathcal{L}_G}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x^\alpha} \frac{\partial \mathcal{L}_G}{\partial g_{\mu\nu,\alpha}} \right) . \tag{44}$$

The Variational Principle for Einstein's Equations iv

We want to check whether the Einstein's equations coincide with the Euler-Lagrance equations derived from equation (42).

- The tensor character of this equation guarantees that if agreement is obtained in a special coordinate system this will hold in general coordinates.
- We choose the **local geodesic coordinates**, in these coordinates, the field equations will contain only 2nd-order derivatives linearly.

To single out these terms we write:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \quad \text{and} \quad \Gamma^{\alpha}{}_{\mu\nu} = \frac{\kappa}{2}\eta^{\alpha\beta}\left(h_{\nu\beta,\mu} + h_{\beta\mu,\nu} - h_{\mu\nu,\beta}\right) \tag{45}$$

and the Lagrangian (42) is expresses as

$$\begin{aligned} \mathcal{L}_G &= -\frac{1}{4}\eta^{\mu\nu}\left[\left(h_{\alpha}{}^{\beta}{}_{,\mu} + h^{\beta}_{\mu,\alpha} - h_{\mu\alpha}{}^{,\beta}\right)\left(h_{\beta}{}^{\alpha}{}_{,\nu} + h^{\alpha}{}_{\nu,\beta} - h_{\nu\beta}{}^{,\alpha}\right)\right] \\ &+ \frac{1}{4}\eta^{\mu\nu}\left(h_{\nu}{}^{\alpha}{}_{,\mu} + h^{\alpha}_{\mu,\nu} - h^{;\alpha}_{\mu\nu}\right)h^{\beta}{}_{\beta,\alpha} + \dots \end{aligned} \tag{46}$$

the dots stand for non relevant terms. By expanding the above relation we get

$$\mathcal{L}_G = \frac{1}{4}\left(h_{\alpha\beta,\mu}h^{\alpha\beta,\mu} - 2h_{\alpha\beta,\mu}h^{\mu\alpha,\beta} + 2h_{\mu\alpha}{}^{,\mu}h^{,\alpha} - h_{,\alpha}h^{,\alpha}\right) + \dots \tag{47}$$

The above Lagrangian agrees with $\mathcal{L}_{(0)}$ in eqn (30) except from the term $(1/2)h_{\mu\nu}{}^{,\mu}h^{\nu\lambda}{}_{,\lambda}$ which is replaced by the term $(1/2)h_{\alpha\beta,\mu}h^{\mu\alpha,\beta}$. But it does not makes any difference for the differential equations derived at the end.

In this way we establish that in the local geodesic coordinates the Euler-Lagrange equations obtained from (42) agree with the usual equations of the linear approximation.

A coordinate transformation from geodesic to general coordinates then tells us that the Euler-Lagrange equations always coincide with the Einstein equations in vacuum.

$$\frac{1}{\sqrt{g}}\left(\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x^\alpha}\frac{\partial\mathcal{L}}{\partial g_{\mu\nu,\alpha}}\right) = -\frac{1}{\kappa^2}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) = 0 \tag{48}$$

Var. Principle for Einstein's Equations (Palatini method) i

Einstein's equations can be derived also via the Palatini method, according to which the metric tensor $g_{\mu\nu}$ and the Christoffel symbols $\Gamma^{\lambda}{}_{\mu\nu}$ are treated as independent variables.

In terms of these the Lagrangian (38) can rewritten

$$\begin{aligned} \mathcal{L} &= \frac{1}{\kappa^2}\sqrt{-g}\, g^{\beta\alpha} R_{\beta\alpha} \\ &= \frac{1}{\kappa^2}\sqrt{-g}\, g^{\beta\alpha} \left(\Gamma^{\mu}{}_{\beta\mu,\alpha} - \Gamma^{\mu}{}_{\beta\alpha,\mu} + \Gamma^{\sigma}{}_{\beta\mu}\Gamma^{\mu}{}_{\sigma\beta} - \Gamma^{\sigma}{}_{\beta\alpha}\Gamma^{\mu}{}_{\sigma\mu}\right) \end{aligned} \tag{49}$$

The variation of the action $I = \int \mathcal{L} d^3x dt$ is therefore

$$\begin{aligned} \delta I &= \frac{1}{\kappa^2}\int R_{\beta\alpha}\left[\frac{\partial\left(g^{\beta\alpha}\sqrt{-g}\right)}{\partial g^{\mu\nu}}\right]\delta g^{\mu\nu}\, d^3x\, dt \\ &+ \frac{1}{\kappa^2}\int\left[\sqrt{-g}g^{\beta\alpha}\frac{\partial R_{\beta\alpha}}{\partial\Gamma^{\lambda}_{\mu\nu}} - \frac{\partial}{\partial x^{\rho}}\left(\sqrt{-g}g_{\alpha\beta}\frac{\partial R_{\beta\alpha}}{\partial\Gamma^{\lambda}_{\mu\nu,\rho}}\right)\right]\delta\Gamma^{\lambda}_{\mu\nu}\, d^3x\, dt \end{aligned} \tag{50}$$

Var. Principle for Einstein's Equations (Palatini method) ii

The first of the previous integrals can be reduced to (use also $\partial g / \partial g^{\mu\nu} = -g g_{\mu\nu}$)

$$\frac{1}{\kappa^2} \int \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \sqrt{-g}\, \delta g^{\mu\nu} d^3x dt \tag{51}$$

and the condition that this vanish for an arbitrary variation $\delta g^{\mu\nu}$ leads to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

The other term of the variation (51) leads to a condition that the covariant derivative of the metric tensor is zero (HOW?), and practically restores the relation between the Christoffel symbols and the metric that we abandoned initially.

Var. Principle for Einstein's Equations (Palatini method) iii

NOTICE: According to (40) we can say that the Einstein equations represent the condition for an extremum in the (4-dim) volume integral of the curvature.

"Gravitation simply represents a continuum effort of the universe to straighten itself out" (Whittaker).