Gravitational Waves

Theory

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Suggested Reading

Books

- *Gravitation* by Charles W. Misner, Kip S. Thorne and John Archibald Wheeler (Sep 15, 1973) W.H. Freeman

Review articles

- *Gravitational wave astronomy: in anticipation of first sources to be detected* L P Grishchuk, V M Lipunov, K A Postnov, M E Prokhorov, B S Sathyaprakash, Physics- Uspekhi 44 (1) 1 -51 (2001)
Linearized Theory I

Weak gravitational fields can be represented by a slightly deformed Minkowski spacetime:

\[ g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu} + O(h_{\mu\nu})^2, \quad |h_{\mu\nu}| \ll 1 \]  \hspace{1cm} (1)

here \( h_{\mu\nu} \) is a small metric perturbation.

The indices will be raised and lowered by \( \eta_{\mu\nu} \) i.e.

\[ h^{\alpha\beta} = \eta^{\alpha\mu} \eta^{\beta\nu} h_{\mu\nu} \]  \hspace{1cm} (2)

\[ h = \eta^{\mu\nu} h_{\mu\nu} \]  \hspace{1cm} (3)

\[ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad \text{why?} \]  \hspace{1cm} (4)

and we will define the traceless (\( \phi_{\mu\nu} \)) tensor:

\[ \phi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \]  \hspace{1cm} (5)

for which we get \( \phi = \eta^{\mu\nu} \phi_{\mu\nu} = h - 2h = -h \) and

\[ h_{\mu\nu} = \phi_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \phi. \]  \hspace{1cm} (6)
The Christoffel symbols & the Ricci tensor will become:

$$\Gamma^{\lambda}_{\mu\nu} \approx \frac{1}{2} \eta^{\lambda\rho} (h_{\rho\nu,\mu} + h_{\mu\rho,\nu} - h_{\mu\nu,\rho})$$  \hspace{1cm} (7)

$$R^\sigma_{\mu\nu\rho} \approx \frac{1}{2} \left( h^{\sigma}_{\rho,\nu\mu} + h_{\mu\nu}^\sigma - h_{\mu\rho}^\sigma - h^{\sigma}_{\nu,\rho\mu} \right)$$  \hspace{1cm} (8)

$$R_{\mu\nu} \approx \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} \approx \frac{1}{2} \left( h^{\alpha}_{\nu,\mu\alpha} + h^{\alpha}_{\mu,\nu\alpha} - h_{\mu\nu}^{\alpha\alpha} - h^{\alpha}_{\alpha,\mu\nu} \right)$$  \hspace{1cm} (9)

$$R = \eta^{\mu\nu} R_{\mu\nu} \approx \frac{1}{2} \left( h^{\alpha\beta}_{\alpha\beta} - h^{\alpha}_{\alpha,\beta} \right)$$  \hspace{1cm} (10)

Finally, Einstein tensor gets the form:

$$G^{(1)}_{\mu\nu} = \frac{1}{2} \left( h^{\alpha}_{\nu,\mu\alpha} + h^{\alpha}_{\mu,\nu\alpha} - h_{\mu\nu}^{\alpha\alpha} - h^{\alpha}_{\alpha,\mu\nu} \right) - \eta_{\mu\nu} \left( h^{\alpha\beta}_{\alpha\beta} - h^{\alpha}_{\alpha,\beta} \right)$$  \hspace{1cm} (11)

Einstein’s equations reduce to (how?):

$$-\phi_{\mu\nu,\alpha}^{,\alpha} - \eta_{\mu\nu} \phi_{\alpha\beta}^{,\alpha\beta} + \phi_{\mu\alpha}^{,\alpha} + \phi_{\nu\alpha}^{,\alpha}_{,\mu} = \kappa T_{\mu\nu}$$  \hspace{1cm} (12)
Then by using the so called Hilbert (or Lorenz or Harmonic or De Donder) gauge similar to Lorenz gauge \((A^\alpha,_{\alpha} = A^\alpha,_{\alpha} = 0)\) in EM \(^1\)

\[
\phi^{\mu\alpha},_{\alpha} = \phi_{\mu\alpha},_{\alpha} = 0
\] (13)

we come to the following equation:

\[
\phi_{\mu\nu,_{\alpha}} \equiv \Box \phi_{\mu\nu} \equiv - \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi_{\mu\nu} = -\kappa T_{\mu\nu}
\] (14)

which is a simple wave equations describing ripples of spacetime propagating with the speed of light (why?).

These ripples are called gravitational waves.

\(^1\)The De Donder gauge is defined in a curved background by the condition

\[
\partial_{\mu}(g^{\mu\nu} \sqrt{-g}) = 0
\]
General Relativity is invariant under the group of all possible coordinate transformations \( x^\mu \rightarrow x'^\mu(x) \) where \( x'^\mu \) is an arbitrary function of \( x^\mu \). Under this transformation (which should be invertible and differentiable) the metric transforms as

\[
g_{\mu\nu} \rightarrow g'_{\mu\nu}(x') = \frac{\partial x^\kappa}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\nu} g_{\kappa\lambda}(x).
\]

(15)

This is usually referred to as the gauge symmetry of GR.

When we make the assumption (1) we practically fix the coordinate system and we assume that the approximation is valid in an extended region of the space. Still a gauge symmetry remains and by careful choice of coordinates the linearized Einstein equations can be simplified.

We can fix \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \) and make small changes in the coordinates that leave \( \eta_{\mu\nu} \) unchanged but induce small changes in \( h_{\mu\nu} \).
For example, let's consider a change of the form:

\[ x'\mu = x\mu + \xi^\mu(x) \]  

(16)

where \( \xi^\mu \) are 4 small arbitrary functions of the same order as \( h^{\mu \nu} \).

Then

\[
\frac{\partial x'\mu}{\partial x\nu} = \delta^\mu_\nu + \partial_\nu \xi^\mu \quad \text{and} \quad \frac{\partial x^\mu}{\partial x'\nu} = \delta^\mu_\nu - \partial_\nu \xi^\mu
\]

Thus, the metric transforms as:

\[
g'_{\mu \nu} = \frac{\partial x^\rho}{\partial x'\mu} \frac{\partial x^\sigma}{\partial x'\nu} g_{\rho \sigma} = \left( \delta^\rho_\mu - \partial_\rho \xi^\mu \right) \left( \delta^\sigma_\nu - \partial_\sigma \xi^\nu \right) \left( \eta_{\rho \sigma} + h_{\rho \sigma} \right)
\]

\[ \approx \eta_{\mu \nu} + h_{\mu \nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu = \eta_{\mu \nu} + h'_{\mu \nu} \]  

(17)

Then in the new coordinate system we get

\[ h'_{\mu \nu} = h_{\mu \nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \]  

(18)

This transformation is called **gauge transformation**.
This is analogous to the gauge transformation in Electromagnetism \((A^\mu, \mu = 0)\).

That is, if \(A_\mu\) is a solution of the EM field equations then another solution that describes precisely the same physical situation is given by

\[
A^{(\text{new})\mu} = A_\mu - \psi,\mu
\]  

where \(\psi\) is any scalar field.

Then the gauge condition \(A^{(\text{new})\mu, \mu} = A^{(\text{new})\, \mu, \mu} = 0\) means that

\[
\psi,\mu, \mu = \psi,\mu, \mu = \Box \psi = A^\mu, \mu = f(x). \tag{18}
\]

From (18) it is clear that if \(h_{\mu\nu}\) is a solution to the linearised field equations then the same physical situation is also described by

\[
\phi^{(\text{new})}_{\mu\nu} = \phi_{\mu\nu} - (\xi_{\mu, \nu} + \xi_{\nu, \mu}) = \phi_{\mu\nu} - \Xi_{\mu\nu} \tag{20}
\]

**NOTE**

- This is a gauge transformation and not a coordinate one
- We are still working on the same set of coordinates \(x^\mu\) and have defined a new tensor \(\phi^{(\text{new})}_{\mu\nu}\) whose components in this basis are given by (20).
We can easily see that from (18) or (20) we can get

\[ \phi^{(\text{new})\mu\rho},\rho = \phi^{\mu\rho},\rho - \Box \xi^{\mu} \] (21)

Therefore, if we assume that \( \phi^{\mu\rho},\rho = f^{\mu}(x) \) we can choose the function \( \xi^{\mu} \) so that to satisfy

\[ \Box \xi^{\mu} = f^{\mu}(x) \] (22)

Then we can get the Hilbert gauge

\[ \phi^{(\text{new})\mu\rho},\rho = 0 \] (23)

**NOTE:** This gauge condition is preserved by any further gauge transformation of the form (20) provided that the functions \( \xi^{\mu} \) satisfy \( \Box \xi^{\mu} = 0 \) or equivalently \( \Box \Xi^{\mu\nu} = 0 \).
**NOTE:** Equation (22) always admits a solution, because the d’Alembertian operator is invertible. If \( G(x) \) is the Green’s function of the d’Alembertian operator so that
\[
\Box_x G(x - y) = \delta^4(x - y)
\] (24)
then the corresponding solution is:
\[
\xi^\mu(x) = \int G(x - y)f^\mu(y)d^4x
\] (25)

- The choice of the Hilbert gauge \( \phi^\mu_{\ \nu,\nu} = 0 \), gives 4 conditions that reduce the 10 independent components of the symmetric tensor \( h_{\mu\nu} \) to 6!
- Eqn (20) tells us that, from the 6 independent components of \( \phi_{\mu\nu} \) which satisfy \( \Box \phi_{\mu\nu} = 0 \), we can subtract the functions \( \Xi_{\mu\nu} \), which depend on 4 independent arbitrary functions \( \xi_{\mu} \) satisfying the same equation \( \Box \Xi_{\mu\nu} = 0 \).
- This means that we can choose the functions \( \xi_{\mu} \) so that as to impose 4 conditions on \( \phi_{\mu\nu} \).
- We can choose $\xi^0$ such that the trace $\phi = 0$ (TRACELESS). Note that if $\phi = 0$ then $\phi_{\mu\nu} = h_{\mu\nu}$.
- The 3 functions $\xi_i$ can be chosen so that $\phi^0_i = 0$.
- Then the Hilbert condition for $\mu = 0$ will be written $\phi^{00} + \phi^0_{0i} = 0$. But since we fixed $\phi^0_i = 0$ we get $\phi^{00} = 0$, i.e. $\phi^{00}$ is a constant in time.
- A time-independent part term $\phi^{00}$ corresponds to the static part of the grav. interactions i.e. to the Newtonian potential of the source.
- The GW itself is the time-dependent part and therefore as far as the GW concerns $h^{00}_0 = 0$ means $h_{00} = 0$.

In conclusion, we set

$$h^{0\mu} = 0, \quad h^{i}_i = 0, \quad h^{ij},_i = 0$$

(26)
Equation (14) is the basis for computing the generation of GWs within the linearised theory.

To study the propagation of GWs as well as the interaction with test masses (and therefore the GW detector) we are interested for the equations outside the source, i.e. where $T_{\mu\nu} = 0$.

GWs are periodic changes of spacetime curvature and for weak gravitational fields far away from sources they described by a simple wave equations which admits a solution of the form:

$$\phi_{\mu\nu} = A_{\mu\nu} \cos \left( k_\alpha x^\alpha \right),$$

where $A_{\mu\nu}$ is a symmetric tensor called polarization tensor including information of the amplitude and the polarization properties of the GWs. $k_\alpha \equiv (k^0 = \omega/c, \vec{k})$ is the wave-vector.

For a single plane wave with given wave-vector $k_\alpha$ ($\hat{n} = k/|k|$), from eqn (26) we see that the non-zero components of $h_{ij}$ are in a plane transverse to $\hat{n}$ since the condition $h_{ij} \cdot \hat{n} = 0$ becomes $n^i h_{ij} = 0$. 
This solution (27) satisfies Hilbert’s gauge condition, that is:

\[ 0 = \phi_{\mu\nu}^{\cdot\nu} = -A_{\mu\nu} k^\nu \sin(k_\alpha x^\alpha) \]

which lead to the orthogonality condition

\[ A_{\mu\nu} k^\nu = 0. \] (28)

explains the definition of TRANSVERSE gauge.

From the wave equation (14) we get

\[ 0 = \phi_{\mu\nu}^{\cdot\alpha,\alpha} = -A_{\mu\nu} k^\alpha k_\alpha \cos(k_\alpha x^\alpha) \Rightarrow k^\alpha k_\alpha = 0. \] (29)

This relation suggests that the wave vector \( k^\alpha \) is null i.e. gravitational waves are propagating with the speed of light. But, (29) implies that \( \omega^2 = c^2 |\vec{k}|^2 \) i.e. both group and phase velocity of GWs are equal to the speed of light.

\[ v_{\text{group}} = \frac{\partial \omega}{\partial k} \quad \text{and} \quad v_{\text{phase}} = \frac{\lambda}{T} = \frac{\omega}{k} \] (30)
Based on the gauge freedom which allows to choose $\xi^\mu$ we derived the following relations

$$h^{0\mu} = 0, \quad h^i_i = 0, \quad h^{ij},_i = 0 \quad (31)$$

which define the so-called **Transverse - Traceless (TT) Gauge**.

Then for a GW propagating in the $z$ direction i.e. it has a wave vector of the form $k_\mu = (\omega/c, 0, 0, -\omega/c)$ where $k_0 = \omega/c$ is the frequency of the wave that:

$$h^{\mu\nu} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - z/c)] \quad (32)$$

While $h_+$ and $h_\times$, are the amplitudes of the gravitational waves in the two polarizations.

The GWs described in this specific gauge are **Transverse** and **Traceless**, and we will use the notation $h^{TT}_{\mu\nu}$.
GW: The Transverse - Traceless (TT) Gauge

The line element will be written as (where $\varphi = \omega(t - z/c)$):

$$ds^2 = -c^2 dt^2 + [1 + h_+ \cos(\varphi)] \, dx^2 + [1 - h_+ \cos(\varphi)] \, dy^2 + 2h_\times \cos(\varphi) \, dx \, dy$$  \hspace{1cm} (33)

**Projection onto the TT gauge**

If a plane wave $h^{\mu\nu}(x)$ propagates in the direction $\hat{n}$ and it is described in the Lorentz gauge but not in the TT gauge, we can find the form of the wave in the TT-gauge as follows.

Introduce the **symmetric and transverse** ($n^i P_{ij} = 0$) **projector** ($P_{ik} P_{kj} = P_{ij}$) tensor with trace $P_{ij} = 2$

$$P_{ij}(\hat{n}) = \delta_{ij} - n_i n_j$$  \hspace{1cm} (34)

With the help of $P_{ij}$ we can construct

$$\Lambda_{ij,kl}(\hat{n}) = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}$$  \hspace{1cm} (35)

It can be shown (how?) that for a plane wave $h^{\mu\nu}(x)$ in the Lorentz gauge, but not in the TT gauge, the GW in the TT gauge is given via the **Lambda tensor**

$$h_{ij}^{TT} = \Lambda_{ij,kl} h_{kl}$$  \hspace{1cm} (36)
**GW: Effects on particles**

- A static or slowly moving particle has velocity vector $u^\mu \approx (1, 0, 0, 0)$ and one can assume that $\tau \approx t$. Then in linearized gravity the geodesic equation will be written as:

$$\frac{du^\mu}{dt} = -\frac{1}{2} \left( h_{\mu\alpha,\beta} + h_{\beta \mu,\alpha} - h_{\alpha \beta,\mu} \right) u^\alpha u^\beta \quad (37)$$

leading to

$$\frac{du^\mu}{dt} = \left. \left( h_{\mu 0,0} - \frac{1}{2} h_{00,\mu} \right) \right. \quad (38)$$

If we now use the T-T gauge ($h_{00} = h_{\mu 0} = 0$) we conclude that GWs do not affect isolated particles (at linear order).

- If instead we consider a pair of test particles on the cartesian axis $Ox$ being at distances $x_0$ and $-x_0$ from the origin and we assume a GW traveling in the $z$-direction then their distance will be given by the relation:

$$d\ell^2 = g_{\mu \nu} dx^\mu dx^\nu = \ldots$$

$$= -g_{11}(dx)^2 = (1 - h_{11})(2x_0)^2 = (1 - h_+ \cos \omega t)(2x_0)^2 \quad (39)$$

or approximately

$$\Delta \ell \approx \left( 1 - \frac{1}{2} h_+ \cos \omega t \right) (2x_0) \quad (40)$$
In a similar way we can show for two particles on the $Oy$ axis that:

$$\Delta \ell \approx \left( 1 + \frac{1}{2} h_+ \cos \omega t \right) (2y_0).$$  \hspace{1cm} (41)

In other words the coordinate distance of two particles is varying periodically with the time.

**Figure 1:** The effect of a travelling GW on a ring of particles
Figure 2: The effect of a travelling GW on a ring of particles