Finite-Elements Method [2]

January 29, 2014

[2] From *Applied Numerical Analysis* Gerald-Wheatley (2004), Chapter 9.

Introduction

Finite-element methods (FEM) are based on some mathematical physics techniques and the most fundamental of them is the so-called **Rayleigh-Ritz method** which is used for the solution of boundary value problems.

Two other methods which are more appropriate for the implementation of the FEM will be discussed, these are **the collocation method** and **the Galerkin method**.

Rayleigh-Ritz method

In the Rayleigh-Ritz (RR) method we solve a boundary-value problem by approximating the solution with a linear approximation of basis functions. The method is based on a part of mathematics called **calculus of variations**. In this method we try to minimize a special class of functions called **functionals**.

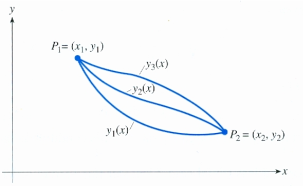
The usual form for functional in problems with one variable is

$$I[y] = \int_a^b F(x, y, y')dx \tag{1}$$

The argument of $I[y]$ is not a simple variable but a function $y = y(x)$. The value of $I[y]$ is varying with $y(x)$, but for fixed $y(x)$ is a scalar quantity.

** We seek the $y(x)$ that **minimizes** $I[y]$.

Let's find the function $y(x)$ that minimizes the distance between two points. Although, we know the answer, i.e. that it is a straight line, we will pretend that we don't, and that we will choose among the set of curves $y_i(x)$ as in the figure.



The functional is the integral of the distance along any of these curves:

$$I[y] = \int_{x_1}^{x_2} \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1 + (dy/dx)^2}dx \qquad (2)$$

To minimize $I[y]$, we set its derivatives to zero.

Each curve must pass through the points (x_1, y_1) and (x_2, y_2).

In addition, for the optimal trajectory, the **Euler-Lagrange equation** must be satisfied:

$$\frac{d}{dx}\left[\frac{\partial}{\partial y'}F(x,y,y')\right]-\frac{\partial}{\partial y}F(x,y,y')=0 \tag{3}$$

For the functional of equation (2) we get:

$$\begin{aligned} F(x,y,y') &= \left(1+(y')^2\right)^{1/2} \\ F_y &= 0 \\ F_{y'} &= y'\left(1+(y')^2\right)^{-1/2} \end{aligned} \tag{4}$$

Finally, from the Euler-Lagrange equation (3) we get

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right)=\frac{\partial F}{\partial y}=0 \tag{5}$$

From which it follows that :

$$\frac{y'}{\sqrt{1+(y')^2}}=c \quad\rightarrow\quad y'=\sqrt{\frac{c^2}{1-c^2}}=b \quad\rightarrow\quad y=bx+a \tag{6}$$

Rayleigh-Ritz method: an example II

Let's try a more complicated case: Apply the RR method to the 2nd order boundary value problem

$$y'' + Q(x)y - F(x) = 0\,, \quad y(a) = y_0\,, \quad y(b) = y_n \tag{7}$$

The above boundary condition are known as **Dirichlet conditions**. The functional that corresponds to eqn (7) is:

$$I[u] = \int_a^b \left[(u')^2 - Qu^2 + 2Fu\right] dx \tag{8}$$

Note I: when the above functional is optimized (via Euler-Lagrange equation) leads to the original equation (7).
Note II: we now have only 1st order instead of 2nd order derivatives.

- If we know the solution to the ODE, substituting it for u in eqn (8) will make $I[u]$ a minimum.
- If the solution is not known, we may try to approximate it by some arbitrary function and see whether we can minimize the functional by a suitable choice of the parameters of the approximations.
- **The Rayleigh-Ritz method is based on this idea.**

We let $u(x)$, which is the approximation to $y(x)$ (the exact solution), be a sum:

$$u(x) = c_0 v_0(x) + c_1 v_1(x) + \cdots + c_n v_n(x) = \sum_{i=0}^{n} c_i v_i(x) \qquad (9)$$

There are two conditions on the trial functions $v_i(x)$:

- They must be chosen such that $u(x)$ meets the boundary conditions
- The individual $v_i(x)$ should be linearly independent

The $v_i(x)$ and c_i are to be chosen to make $u(x)$ a good approximation to the true solution of eqn (7).
Since we have no prior knowledge of the true function $y(x)$ we have no chance to guess the $v_i(x)$ that will provide a solution to closely resemble $y(x)$.
Thus we go for the usual choice that is the use of polynomials! and we will try to find a way to get the values of the c_i.

These requirements can be fulfilled by using the functional of eqn (8). If we substitute $u(x)$ as defined by equation (9) into the functional of eqn (8) we get:

$$I(c_0, c_1, \ldots, c_n) = \int_a^b \left[\left(\frac{d}{dx} \sum c_i v_i \right)^2 - Q \left(\sum c_i v_i \right)^2 + 2F \sum c_i v_i \right] dx$$

Thus I is a function of the unknown c_i. To minimize I we take the partial derivatives with respect to each unknown c_i and set zero.

$$\frac{\partial I}{\partial c_i} = 2 \int_a^b u' \frac{\partial u'}{\partial c_i} dx - \int_a^b 2Qu \frac{\partial u}{\partial c_i} dx + \int_a^b 2F \frac{\partial u}{\partial c_i} dx \qquad (10)$$

Thus we led to a system of n equations to solve. This will define the $u(x)$ of equation (9).

EXAMPLE: Solve the ODE $y'' + y - 3x^2 = 0$ with the boundary conditions (0,0) and (2,3.5).
We will use polynomials up to 3rd order, we can define $u(x)$ as:

$$u(x) = \frac{7}{4}x + c_2 x(x-2) + c_3 x^2(x-2) \tag{11}$$

note that the functions v are linearly independent and that $u(x)$ satisfies the boundary conditions.
The following terms are needed for the evaluation of equations (10):

$$\begin{aligned} u' &= \frac{7}{4} + 2c_2(x-1) + c_3(3x^2 - 4x) \\ \frac{\partial u'}{\partial c_2} &= 2x - 2 \quad \text{and} \quad \frac{\partial u'}{\partial c_3} = 3x^2 - 4x \\ \frac{\partial u}{\partial c_2} &= x(x-2) \quad \text{and} \quad \frac{\partial u}{\partial c_3} = x^2(x-2) \end{aligned} \tag{12}$$

By substituting the above equations into equation (10) we get two equations for the two constants c_2 and c_3.

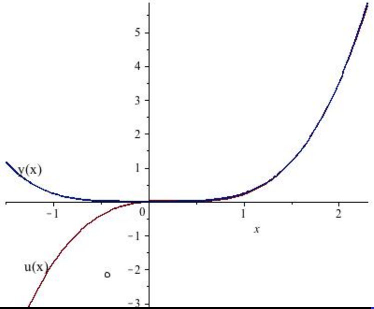
The results which come from trivial integrations are:

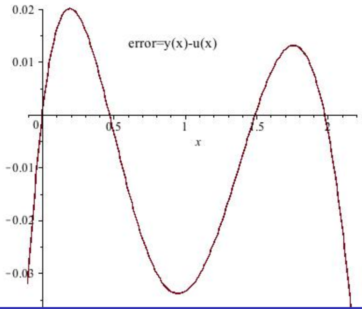
$$\frac{\partial I}{\partial c_2} : \quad \frac{16}{5} c_2 + \frac{16}{5} c_3 = \frac{74}{15}$$
$$\frac{\partial I}{\partial c_3} : \quad \frac{16}{5} c_2 + \frac{128}{21} c_3 = \frac{36}{15}$$

From their solution we get the values of c_2 and c_3 and we come to the approximate solution:

$$u(x) = \frac{119}{152}x^3 - \frac{46}{57}x^2 + \frac{53}{228}x \tag{13}$$

NOTE: The exact solution is: $y(x) = 6\cos x + 3(x^2 - 2)$.





The Collocation Method

The **collocation method** uses an alternative method in approximating the solution $y(x)$ of the BVP defined in (7).
We define the residual function $R(x)$ as:

$$R(x) = y'' + Qy - F \tag{14}$$

We approximate again $y(x)$ with $u(x)$ equal to a sum of trial functions (linearly independent polynomials) as with the RR method, and we try to make $R(x) = 0$ by suitable choice of the coefficients.
Of course this is not possible to be achieved everywhere in the interval and thus we may choose arbitrarily to make $R(x) = 0$ at a number of points inside the interval. The number of interior points should be the same as the number of the unknowns.

The Collocation Method : Example

We will solve the previous example using the collocation method.

We take again

$$u(x) = \frac{7}{4}x + c_2 x(x-2) + c_3 x^2 (x-2) \tag{15}$$

The residual is defined as:

$$R(x) = u'' + u - 3x^2 \tag{16}$$

and when we differentiate $u(x)$ we get:

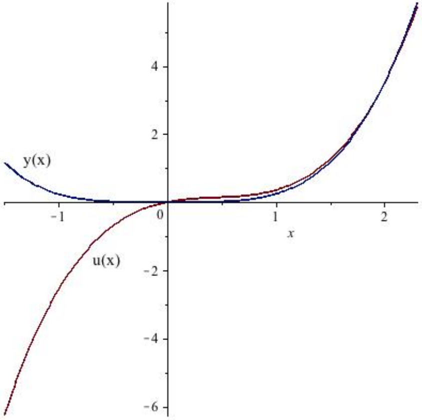
$$R(x) = 2c_2 + 2c_3(x-2) + 4c_3 x + (7/4)x + c_2 x(x-2) + c_3 x^2 (x-2) - 3x^2$$

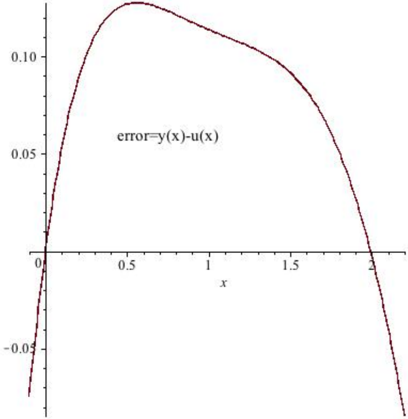
Since we have 2 unknown constants we will use 2 points in the interval $[0, 2]$, e.g $x = 0.7$ and $x = 1.3$.
Then by setting $R(x) = 0$ for these two choices we get:

$$\begin{aligned} x = 0.7 \ : &\qquad 1090c_2 - 437c_3 - 245 = 0 \\ x = 1.3 \ : &\qquad 1090c_2 + 2617c_3 - 2795 = 0 \end{aligned}$$

and by solving for c_2 and c_3 we get

$$u(x) = \frac{425}{509}x^3 - \frac{61607}{55481}x^2 + \frac{140023}{221924}x \tag{17}$$





The Galerkin Method

- This method can be considered as a variation of the collocation method i.e. is a "*residual method*" that use the function $R(x)$ defined in (14). The difference is that here we multiply with weighting functions $W_i(x)$ which can be chosen in many ways.
- Galerkin showed that the individual trial functions $v_i(x)$ used in (9) are a good choice.
- Once we have selected the $v_i(x)$ from eqn (9) we compute the unknown coefficients by setting the integral of the weighted residual to zero:

$$\int_a^b W_i(x)R(x)dx = 0 \;, i = 0, 1, \ldots, n \quad \text{where} \quad W_i(x) = v_i(x) \tag{18}$$

- Notice that using the Dirac delta functions for the $W_i(x)$ we reduce to the collocation method.

The Galerkin Method: Example

We will solve the previous example using the Galerkin method.

We take again

$$u(x) = \frac{7}{4}x + c_2 x(x-2) + c_3 x^2(x-2) \tag{19}$$

so that $v_2 = x(x-2)$ and $v_3 = x^2(x-2)$.
The residual is

$$R(x) = u'' + u - 3x^2 \tag{20}$$

and after substituting u'' and u we get

$$R(x) = 2c_2 + c_3(6x-4) + 7x/4 + c_2 x(x-2) + c_3 x^2(x-2) - 3x^2 \tag{21}$$

We carry out two integrations:

$$\text{If} \quad v_2 \to W_i : \qquad \int_0^2 \left[x(x-2)\right] R(x) dx = 0$$

$$\text{If} \quad v_3 \to W_i : \qquad \int_0^2 \left[x^2(x-2)\right] R(x) dx = 0$$

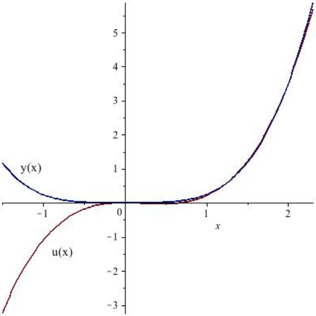
These intergrations give two equations for the c_i:

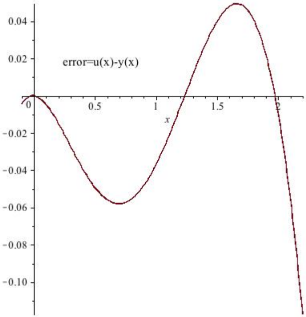
$$\begin{aligned} 24c_2 + 24c_3 - 37 &= 0 \\ 21c_2 + 40c_3 - 45 &= 0 \end{aligned}$$

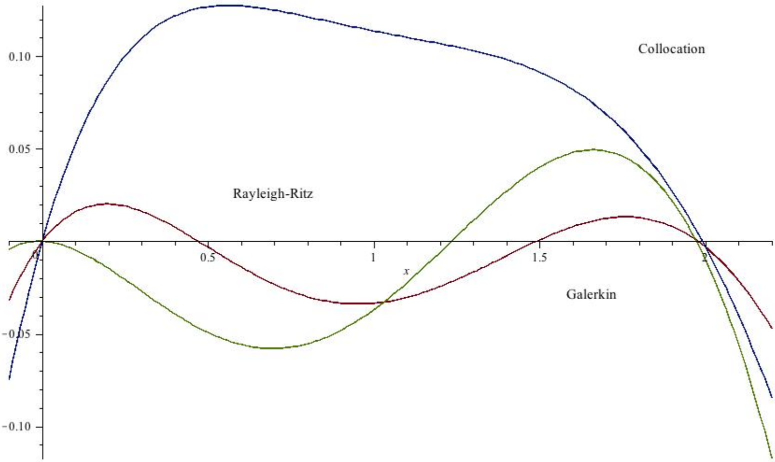
and by solving for c_2 and c_3 we get

$$u(x) = \frac{101}{152}x^3 - \frac{103}{228}x^2 - \frac{1}{128}x \tag{22}$$

COMMENT: Although the RR method is more accurate than the others the Galerkin method is much easier to be implemented since we don't need to find the variational form.






Collocation
Rayleigh-Ritz
Galerkin
0.10
0.05
-0.05
-0.10
0
0.5
1
1.5
2
x

Finite Elements for ODEs

In order to improve the accuracy and also to be able to treat longer intervals $[a, b]$, we follow the steps bellow:

- Subdivide $[a = x_0, b = x_n]$into n subintervals, the **elements**, that join at the **nodes** x_1, x_2,..., x_{n-1}
- Apply the Galerkin method to each element separately to interpolate between the end point nodal values $u(x_{i-1})$ and $u(x_i)$
- Use a low-degree polynomial for $u(x)$, e.g. even a 1st degree can do the work (higher order polynomials are better but too complicated to be implemented)
- When Galerkin's method is applied to element (i) we get a pair of eqns with unknowns the nodal values at the ends of the element (i), the c_i.
- If we do it for each element we end up with a system of equations involving all the nodal values
- The equations adjusted to take into account the boundary conditions and the solution is an approximation to $y(x)$ at the nodes; intermediate values can be taken by interpolation.

We will describe step by step the procedure for solving the following boundary value problem:

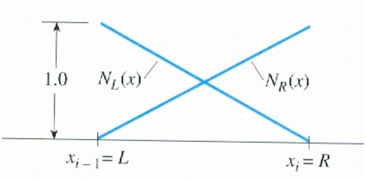
$$y'' + Q(x)y = F(x) \quad \text{with BC at } x = a \text{ and } x = b \tag{23}$$

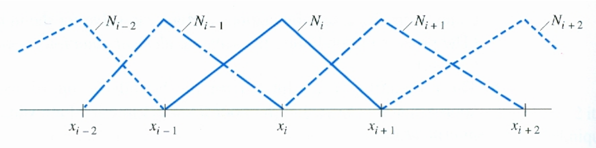
STEP 1 : Subdivide the interval $[a, b]$ into n elements, and for a specific element between x_{i-1} and x_i name the left node as L and the right as R.

STEP 2 : Write $u(x)$ for the element (i):

$$u(x) = c_L N_L + c_R N_R = c_L \frac{x - R}{L - R} + c_R \frac{x - L}{R - L} = c_L \frac{x - R}{-h_i} + c_R \frac{x - L}{-h_i} \tag{24}$$

Notice that the N's are 1st-degree Lagrange polynomials usually called **hat functions**. The following figure sketches the N_L and N_R for the (i) element.





STEP 3 : Apply the Galerkin method to element (i), the residual is

$$R(x) = y'' + Qy - F = u'' + Qu - F \tag{25}$$

The Galerkinn method sets the integral of R weighted with each of the N

$$\begin{aligned} \int_L^R N_L R(x) dx &= 0 \\ \int_L^R N_R R(x) dx &= 0 \end{aligned} \tag{26}$$

These lead to the integrals

$$\begin{aligned} \int_L^R u'' N_L dx + \int_L^R Q u N_L dx - \int_L^R F N_L &= 0 \\ \int_L^R u'' N_R dx + \int_L^R Q u N_R dx - \int_L^R F N_R &= 0 \end{aligned}$$

STEP 4 : Transform the previous equations by applying integration by parts (how?) to the first integral. In the 2nd integral take Q out from the integrant as Q_{av} (an average value within the element). We do the same also for the 3rd integral.

$$\int_L^R u' N_L' dx - Q_{av} \int_L^R u N_L dx + F_{av} \int_L^R N_L dx - N_L u'|_{x=R} + N_L u'|_{x=L} = 0 \tag{27}$$

Remember that $N_L = 1$ at L and 0 at R. Thus the equation can be written as:

$$\int_L^R u' N_L' dx - Q_{av} \int_L^R u N_L dx + F_{av} \int_L^R N_L dx - u'|_{x=L} = 0 \tag{28}$$

and similarly the other equation:

$$\int_L^R u' N_R' dx - Q_{av} \int_L^R u N_R dx + F_{av} \int_L^R N_R dx + u'|_{x=R} = 0 \tag{29}$$

STEP 5 : Substitute in the previous integrals u, u', N_L' and N_R' and carry out the integrations.

$$\int_L^R u' N_L' dx = \cdots = (c_L - c_R) / h_i \quad (30)$$

$$-Q_{av} \int_L^R u N_L dx = \cdots = -Q_{av} h_i (2c_L - c_R) / 6 \quad (31)$$

$$F_{av} \int_L^R N_L dx = \cdots = F_{av} h_i / 2 \quad (32)$$

and we do the same for equation (29)

STEP 6 : Substitute the results of the previous steps into equations (28) and (29) and by rearrangement we get 2 equations of the unknowns c_L and c_R:

$$\left(\frac{1}{h_i} - \frac{Q_{av} h_i}{3}\right) c_L - \left(\frac{1}{h_i} + \frac{Q_{av} h_i}{6}\right) c_R = -\frac{F_{av} h_i}{2} - u'|_{x=L} \quad (33)$$

$$\left(\frac{-1}{h_i} - \frac{Q_{av} h_i}{3}\right) c_L + \left(\frac{1}{h_i} - \frac{Q_{av} h_i}{6}\right) c_R = -\frac{F_{av} h_i}{2} + u'|_{x=L} \quad (34)$$

These are the so called **element equations**.

** We do the same for each element to get n such pairs. **

STEP 7 :

- We assemble all the element equations to form a system of linear equations for the problem.
- Notice that the point R in the element (i) is the same as the point L in the element $(i+1)$.
- Renumber the c as $c_0,\ c_1,\ \dots c_n$
- Notice that the gradient u' must be the same on either side of the elements i.e. $u'_{x=R}$ in the element (i) equals $u'_{x=L}$ in element $(i+1)$. Thus these terms will cancel when we assemble except in the first and last equations.
- The results is a system of $n+1$ equations of the form

$$\mathbf{K}\vec{c} = \vec{b} \tag{35}$$

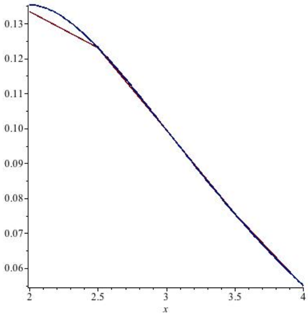
The matrix $\mathbf{K}$ contains combination of the quantities $Q_{av,i}$ and h_i.
The matrix $\mathbf{K}$ is tridiagonal
The vector $\vec{c}$ contains the $c_0,\ c_1,\ \dots c_n$
The vector $\vec{b}$ contains combinations of $F_{av,i}$ and h_i.

STEP 8 : Adjust the set of the equations for the boundary conditions which can be either **Dirichlet** or **Neumann**.

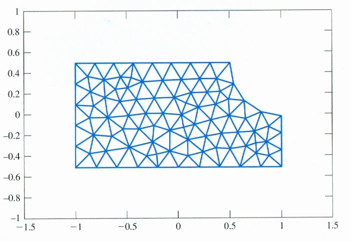
STEP 9 : Solve the system and find the c_0, c_1, $\ldots c_n$.

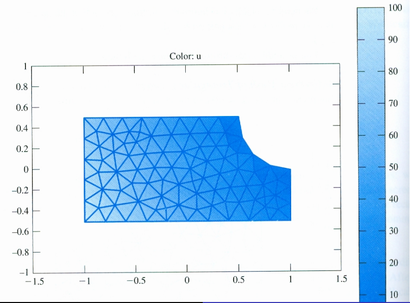
*** Then relax :-)

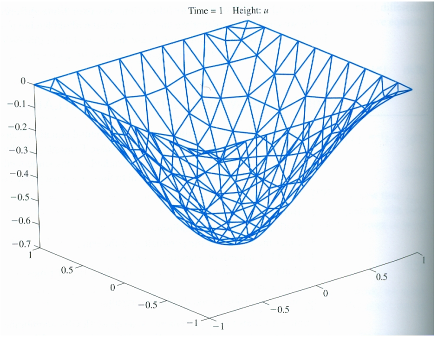
EXAMPLE: Solve $y'' - (x+1)y = e^{-x}(x^2 - x + 2)$ with the Neumann conditions $y'(2) = 0$ and $y'(4) = -0.036631$. (The exact solution is $y(x) = e^{-x}(x-2)$).



Quite complicated.







Solving the wave equation

Finite Elements for PDEs