Fast Rotating Relativistic Stars: Spectra and Stability without Approximation

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14 Nov 2019

Motivation and Aim

- Knowledge of NS oscillation frequencies and damping times are essential for asteroseismology.
- Previous studies always relied on simplifications due to complexity of the problem such as
    - Newtonian gravitation,
    - slow-rotation approximation.
    - Cowling approximation,
- Provide universal relations to help solve the *inverse problem*, i.e. turn NS frequencies and/or damping times into NS bulk parameters (mass, radius, ...).

Equations to solve

- Einstein equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu}\,, \qquad \nabla_\mu T^{\mu\nu} = 0$$

- Neutron star modelled as perfect fluid

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu}$$

- Metric describing a rotating star can be written as

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}r^2\sin^2\theta(d\phi - \omega dt)^2 + e^{2\mu}(dr^2 + r^2 d\theta^2)$$

- Use the rns-code to solve these equations and construct equilibrium models.

Linear Perturbation Theory

- Introduce small, time-dependent perturbations away from equilibrium:

$$q(t, r, \theta, \phi) = \bar{q}(r, \theta) + \delta q(t, r, \theta, \phi)$$

- Expand the Einstein equations and throw away all terms that are non-linear in the perturbations.
  $\rightarrow$ Resulting equations are somewhat lengthy...
- We pick the *Hilbert gauge* which is the analogue to the Lorenz gauge in E/M:

$$\nabla_\mu h^\mu_\nu = 0$$

($h_{\mu\nu}$ are the metric perturbations.)

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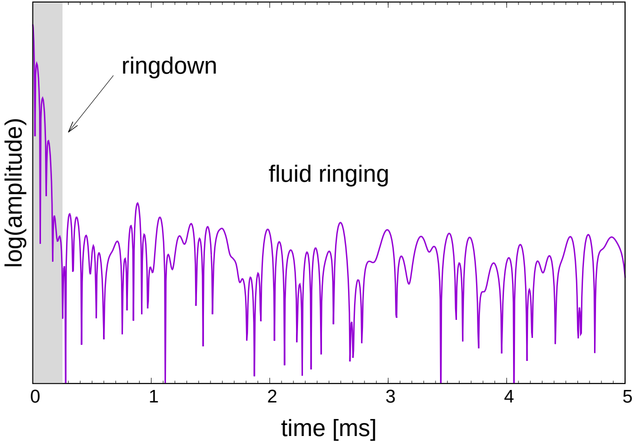
Exemplary: The wave equation for $h_{tt} \sim L$

$$
\begin{aligned}
&\frac{(e^{\mu})^2 \left(\frac{\partial^2}{\partial t^2} LL\right)}{(e^{\nu})^2} = 8 (e^{\nu})^2 (e^{\mu})^4 \pi\, Q3 + \Bigg(8 \pi\, u\_t^2 (\rho + p) (-\Omega + \omega) (e^{\psi})^2 (e^{\nu})^2 (e^{\mu})^2 \\
&\quad - \frac{\left(2 \mu_r \omega_r r^2 - 8 \psi_r \omega_r r^2 - 2 \omega_{r,r} r^2 - 6 r \omega_r - 3 \psi_\theta \omega_\theta + \nu_\theta \omega_\theta - \omega_\theta \cot(\theta) - \omega_{\theta,\theta}\right) (e^{\psi})^2}{r^2} \Bigg) AA + 2 (e^{\psi})^2 \omega_r \left(\frac{\partial}{\partial r} AA\right) + \frac{(e^{\psi})^2 \left(\frac{\partial}{\partial \theta} AA\right) \omega_\theta}{r^2} \\
&\quad + \left(3 \omega_r \psi_\theta + \omega_r \nu_\theta + \omega_r \cot(\theta) - 2 \mu_r \omega_\theta + 2 \psi_r \omega_\theta + \omega_{r,\theta}\right) (e^{\psi})^2 BB + (e^{\psi})^2 \omega_r \left(\frac{\partial}{\partial \theta} BB\right) + \Bigg(\Bigg(8 \pi\, u\_t^2 (\rho + p) (e^{\nu})^2 - 8 \pi p - \frac{m^2}{\sin(\theta)^2 (e^{\psi})^2 r^2} \\
&\quad + \frac{\omega^2 m^2}{(e^{\nu})^2} \Bigg) (e^{\mu})^2 - \frac{1}{r^2} \Big(2 \mu_r \psi_r r^2 + 2 \mu_r \nu_r r^2 + 2 \psi_r^2 r^2 - \nu_{r,r} r^2 + 2 \mu_{r,r} r^2 + \psi_{r,r} r^2 + 6 \mu_r r + 6 \psi_r r - 4 \mu_\theta \nu_\theta + 2 \psi_\theta^2 + 2 \psi_\theta \nu_\theta \\
&\quad + 4 \psi_\theta \cot(\theta) + 2 \nu_\theta^2 + 2 \nu_\theta \cot(\theta) + 2 \nu_{\theta,\theta} + 2 \mu_{\theta,\theta} + 2 \psi_{\theta,\theta} + 2 \Big) - \frac{1}{2} \frac{\sin(\theta)^2 (r \omega_r - \omega_\theta)(r \omega_r + \omega_\theta)(e^{\psi})^2}{(e^{\nu})^2} \Bigg) LL \\
&\quad - \frac{(2 \mu_r r - \psi_r r - \nu_r r - 2) \left(\frac{\partial}{\partial r} LL\right)}{r} - \frac{(2 \mu_\theta + \nu_\theta - \psi_\theta - \cot(\theta)) \left(\frac{\partial}{\partial \theta} LL\right)}{r^2} + \Bigg(\\
&\quad - \frac{2 \mu_r \psi_\theta r + 2 \mu_r \nu_\theta r + 2 \mu_r \cot(\theta) r + 4 \mu_\theta \nu_r r - 2 \nu_r \psi_\theta r - 2 \nu_r \nu_\theta r - 2 \nu_r \cot(\theta) r - \psi_{r,\theta} r - 3 \nu_{r,\theta} r + 2 \psi_\theta + 2 \nu_\theta + 2 \cot(\theta)}{r^3} \\
&\quad - \frac{\sin(\theta)^2 (e^{\psi})^2 \omega_r \omega_\theta}{(e^{\nu})^2} \Bigg) MM + \frac{2 \mu_\theta \left(\frac{\partial}{\partial r} MM\right)}{r^2} - \frac{2 (\mu_r r - \nu_r r + 1) \left(\frac{\partial}{\partial \theta} MM\right)}{r^3} + \frac{\partial^2}{\partial r^2} LL + \frac{\frac{\partial^2}{\partial \theta^2} LL}{r^2} + 4 \nu_r (e^{\mu})^2 \left(\frac{\partial}{\partial t} KK\right) \\
&\quad + \frac{2 \nu_\theta (e^{\mu})^2 \left(\frac{\partial}{\partial t} QQ\right)}{r^2} + \Bigg(\frac{2 I\, YY\, m (\psi_r r - \nu_r r + 1)(e^{\nu})^2}{\sin(\theta)^2 r^3} \\
&\quad - \frac{I\, m \left(-4 KK \nu_r \omega r^2 + 2 \omega_r KK r^2 - 2 \omega_r HH r^2 + 2 \omega_r WW r^2 - 2 QQ \nu_\theta \omega + QQ \omega_\theta\right)}{r^2} - \frac{2 I\, \omega\, m \left(\frac{\partial}{\partial t} LL\right)}{(e^{\nu})^2} \Bigg) (e^{\mu})^2
\end{aligned}
$$

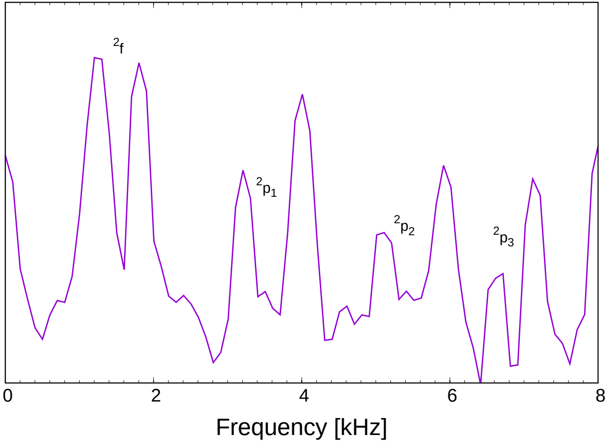
Have 10 such equations for the spacetime and 4 equations for the fluid.

Echo from a rotating neutron star



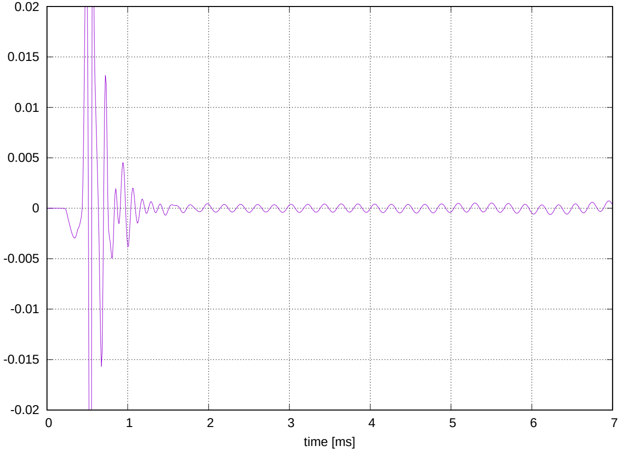
Observe ringdown ($t \in [0.0\,\mathrm{ms}, 0.25\,\mathrm{ms}]$), followed by fluid ringing ($t > 0.25\,\mathrm{ms}$).

Spectral density of fluid ringing



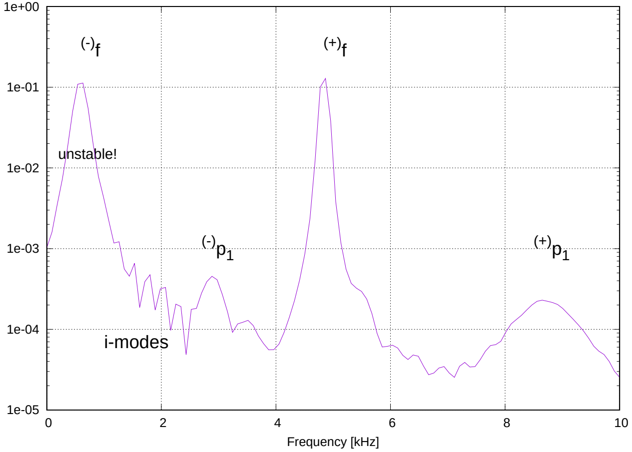
"Zeeman" splitting of the fundamental mode and its overtones clearly visible.

Echo from a fast rotating star



EoS SLy, $M = 2.0\,M_\odot$, $\Omega = 1.3\,\text{kHz}$, $\Omega/\Omega_K = 0.98$.

Spectral density of fast rotating star



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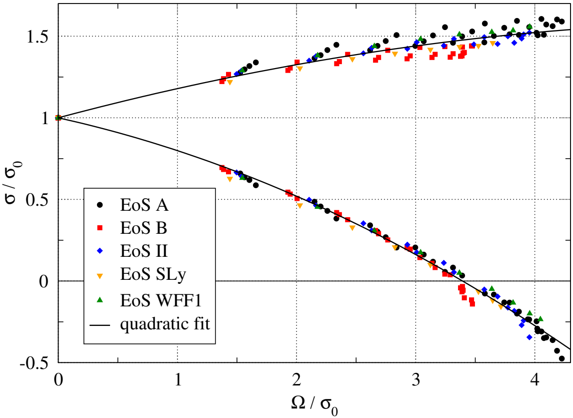
How the f-mode frequency changes with NS rotation



- Curves show universal behaviour.
- f-mode becomes unstable when $\Omega \gtrsim 3.4\sigma_0$.

A different model for the f-mode frequency[1]

- Define parameter "*effective compactness*"

$$\eta := \sqrt{\left(\frac{M}{M_\odot}\right)^3 \left(\frac{I}{10^{45}\,\mathrm{g\,cm^2}}\right)^{-1}}.$$

- Turns out to be useful parameter for fitting formula.
- Typical values for uniformly rotating NS are $\eta \in [1.0, 2.3]$.

[1] inspired by Doneva, Kokkotas (2015) PRD 92, 124004

- Modeling f-mode frequency, σ, via[2]

$$M\sigma = \left(a_1 + a_2\hat{\Omega} + a_3\hat{\Omega}^2\right) + \left(b_1 + \cancel{b_2\hat{\Omega}} + b_3\hat{\Omega}^2\right)\eta,$$

with $\hat{\Omega} = M\Omega$.

- 3 parameters: mass, rotation rate, moment of inertia.
- Fitting values from 126 simulations yields

$$\begin{aligned} a_1 &= -2.1 \quad (2.5\%), & b_1 &= 3.4 \quad (1.0\%), \\ a_2 &= -0.20 \quad (0.9\%), & b_2 &= 0.0, \\ a_3 &= -8.5 \cdot 10^{-3} \quad (5.7\%), & b_3 &= -2.2 \cdot 10^{-3} \quad (10\%). \end{aligned}$$

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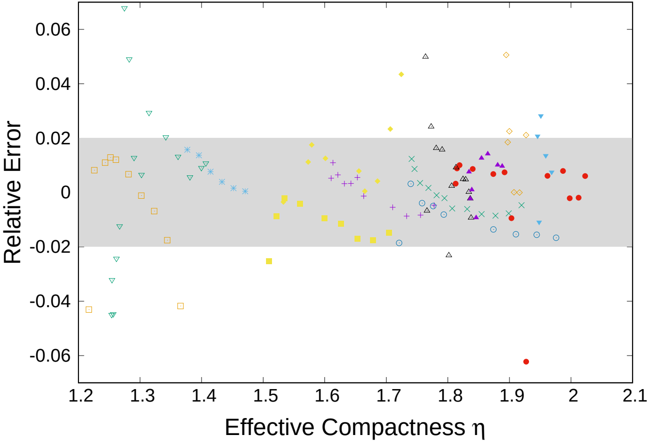
Relative Error of model for f-mode frequency



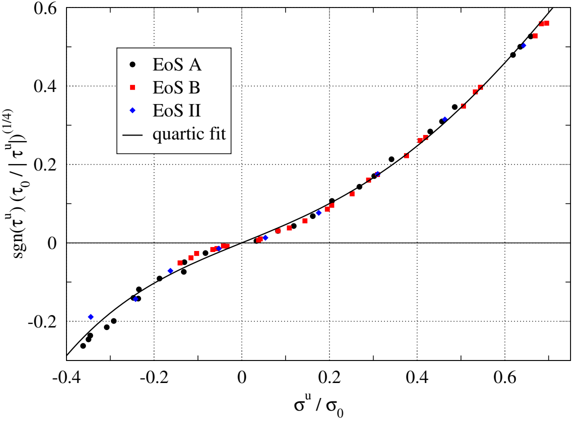
The model deviates less than 2% for the majority of simulations.

Damping time of the f-mode

- Difficult to determine from the simulation
- Approximate using Quadrupole formula:

$$\tau = -\frac{2E_{kin}}{< dE/dt >_{GW}}$$

Damping time of the f-mode



Conclusions and Outlook

- First fitting formulae for f-mode frequency in full GR.
- Extract w-mode frequencies from ringdown and damping times of various modes.
- Find different universal relations.
- Do asteroseismology.