A Thesis for the degree
of
Bachelor of Science

NORMAL MODES OF ROTATING CYLINDRICAL POLYTROPES

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INTRODUCTION

-Stellar oscillations provide information about:

- Stellar structure
- System stability

→ Stellar Evolution

- Evolution of physical events (i.e. NS merger)
- Physical mechanism (i.e. effect of rotation)
- Unique events might excite more modes

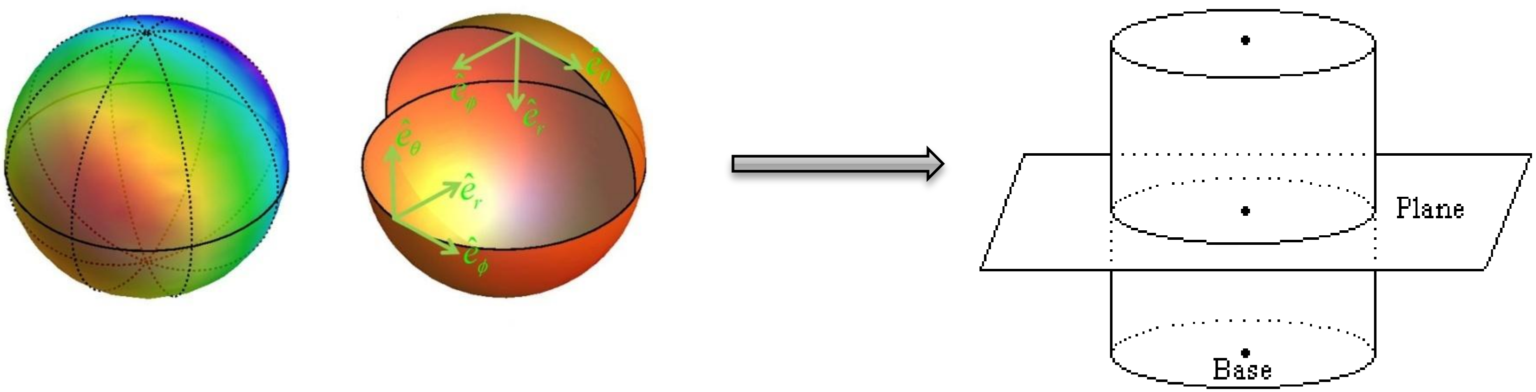
- De Pietri et al (2018)

- Post-merger remnant oscillates in radial and non-radial modes (quadrupole m=2)
- convective instability in the post-merger remnant
- probe the rotational and thermal state

INTRODUCTION

<u>Considerations & Approximations:</u>

- Polytropic fluid
- Uniform rotation Ω
- Small *adiabatic* oscillations
- Spherical symmetry → Cylindrical symmetry(ϖ, φ, z)



- Newtonian limit

HYDROSTATIC EQUILIBRIUM

- **Equation of HE:**

$$\frac{1}{\rho}\frac{dP}{d\varpi} = -\frac{d\Phi}{d\varpi} + \Omega^2(\varpi)\varpi$$

- **Polytropic fluid:**

$$P = K\rho^\Gamma$$

Γ=1+1/n
n-> polytropic index
Γ-> polytropic exponent

polytropic constant

- **Dimensionless formulation (Lane-Emden):**

$$\beta = \frac{\Omega^2}{2\pi G\rho_c}$$

$$\rho = \rho_c\theta^n$$

$$a = [\frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}]^{1/2}$$

$$\varpi = a\xi$$

$$\frac{d^2\theta}{d\xi^2} + \frac{1}{\xi}\frac{d\theta}{d\xi} = -\theta^n + \beta$$

Test(n=1, β)

- Present work

n	$\beta_{critical}$
0.5	0.5
1	0.2871
1.5	0.1783

polytropic index	β	ξ_1
n=0.5	0	2.18966
	0.15	2.42696
	0.35	2.93422
	0.49	3.76126
	0.50	3.99999
n=1	0	2.40482
	0.15	2.77926
	0.25	3.25242
	0.28	3.57159
	0.2871	3.7976
n=1.5	0	2.64777
	0.09	2.97852
	0.15	3.38497
	0.178	3.98154
	0.1783	4.04101

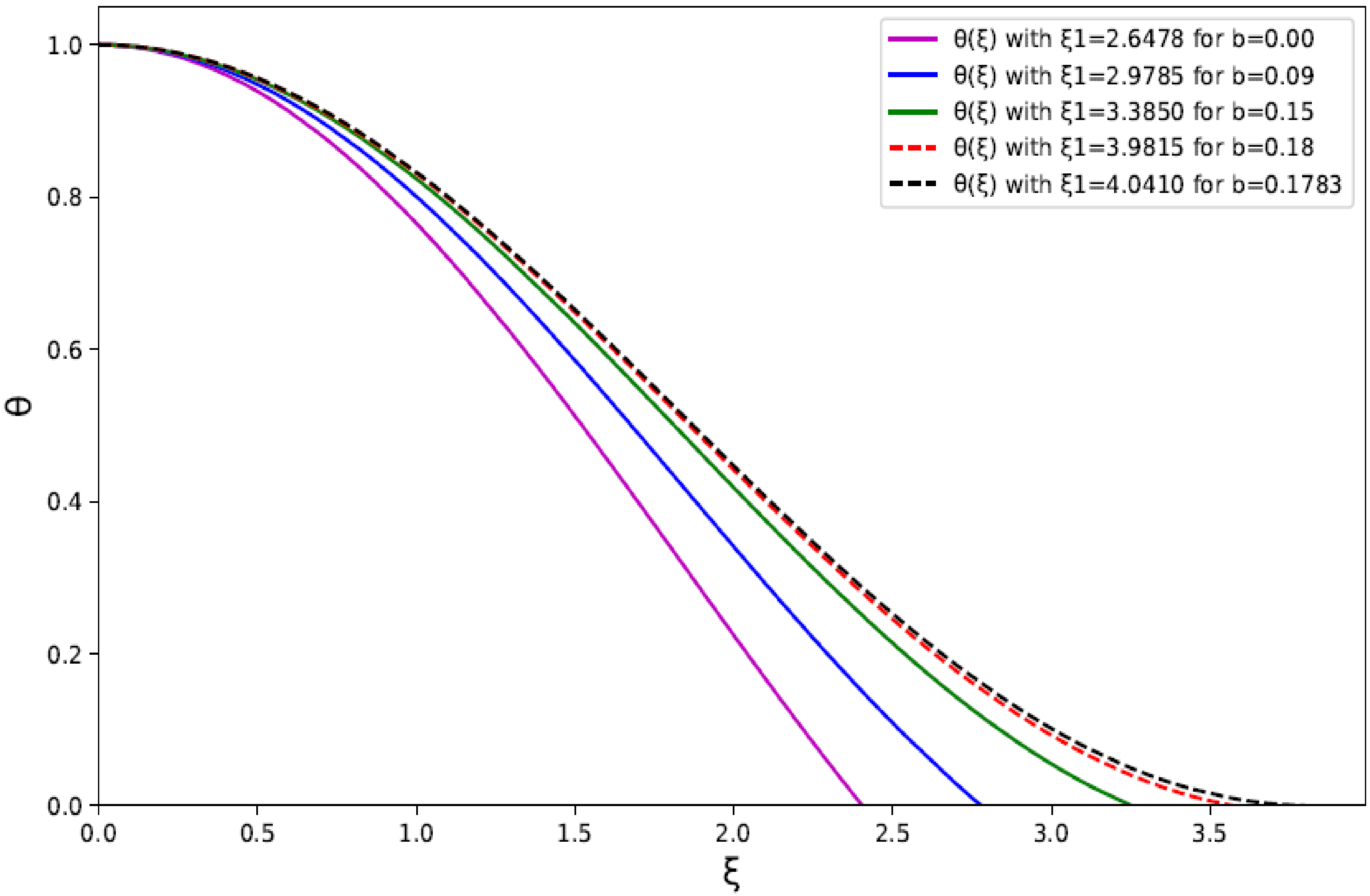
- Robe (1968)

n	β_c
0	1.00
1	2.87−1
3	5.47−2
6	8.21−3

n	ξ_1
−0.9	1.71782384
−0.8	1.74556088
−0.5	1.83413266
−0.2	1.93087184
0	2.00000000
0.5	2.18966219
1	2.40482556
1.5	2.64777677
2	2.92132072
3	3.57390098
4	4.39526586
5	5.42757459
6	6.72452797
10	1.62227407E+1
20	1.60596473E+2

Present work

- n=1.5



STELLAR OSCILLATIONS

- Small Eulerian perturbation: $\delta Q(\varpi, \phi, z, t) = \delta Q(\varpi) e^{i(\sigma t + m\phi + kz)}$

Eulerian variation $\delta Q = Q(x,t) - Q_0(x,t)$

Lagrangian variation $\Delta Q = Q[x + \Delta r(x,t), t] - Q_0(x,t)$

- Perturbed equations

Continuity Equation $\frac{D\rho}{Dt} + \rho \nabla \vec{v} = 0 \longrightarrow \delta\rho = -div(\rho_0 \Delta \vec{r})$

Adiabaticity $\Delta P = \Gamma_1 \frac{P_0}{\rho_0} \Delta\rho = -\Gamma_1 P_0 div \Delta \vec{r}$

Equation of motion $\frac{D\vec{v}}{Dt} = -\frac{1}{\rho} grad P - grad \Phi$

Poisson's equation $\nabla^2 \delta\Phi = 4\pi G \delta\rho$

STELLAR OSCILLATIONS

- **Small Eulerian perturbation:** $\delta Q(\varpi,\phi,z,t) = \delta Q(\varpi)e^{i(\sigma t+m\phi+kz)}$ (with $kz \to 0$)

- **Perturbed equations**

$$Y = \frac{\delta P/\rho_0 + \delta\Phi}{\varpi\sigma^2}$$

$$\varpi\frac{dY}{d\varpi} = [1 - \frac{4\Omega^2}{\sigma^2} - \frac{A}{\sigma^2\rho_0}\frac{dP_0}{d\varpi}]\Delta\varpi - [1 + \frac{2m\Omega}{\sigma\varpi}]Y - A[Y\varpi - \frac{\delta\Phi}{\sigma^2}]$$

$$A = [n - \frac{(n+1)}{\Gamma_1}]\frac{1}{a\theta}\frac{d\theta}{d\xi}$$

$$\varpi\frac{d\Delta\varpi}{d\varpi} = [\frac{2m\Omega}{\sigma\varpi} - \frac{1}{\Gamma_1 P_0}\frac{dP_0}{d\varpi} - \frac{1}{\varpi}]\varpi\Delta\varpi + m^2 Y - \frac{\varpi\rho_0}{\Gamma_1 P_0}(\varpi\sigma^2 Y - \delta\Phi)$$

$$\frac{1}{\varpi}\frac{d}{d\varpi}(\varpi\frac{d\delta\Phi}{d\varpi}) - \frac{m^2}{\varpi^2}\delta\Phi = 4\pi G\rho_0[-A\varpi + \frac{\rho_0}{\Gamma_1 P_0}(\varpi\sigma^2 Y - \delta\Phi)]$$

STELLAR OSCILLATIONS

Numerical Method

1. Shooting Method (i.e. Runge-Kutta)
2. Fitting Point r_m (i.e. $\xi_1/2$)
3. Shooting-1 (center → fitting point)
4. Shooting-2 (surface → fitting point)
5. At the fitting point is required

$$\Delta = det \begin{pmatrix} X^{(0,1)} & X^{(0,2)} & X^{(s,1)} & X^{(s,2)} \\ Y^{(0,1)} & Y^{(0,2)} & Y^{(s,1)} & Y^{(s,2)} \\ F_1^{(0,1)} & F_1^{(0,2)} & F_1^{(s,1)} & F_1^{(s,2)} \\ F_2^{(0,1)} & F_2^{(0,2)} & F_2^{(s,1)} & F_2^{(s,2)} \end{pmatrix}_{r=r_m} = 0$$

→ Eigenfrequency!!!

STELLAR OSCILLATIONS

Constructing the Solution (Eigenfaunction)

$$C^{(0,1)}y^{(0,1)}(zm)_j + C^{(0,2)}y^{(0,2)}(zm)_j = C^{(s,1)}y^{(s,1)}(zm)_j + C^{(s,2)}y^{(s,2)}(zm)_j$$

$y_j = X; Y; F_1; F_2$ → homogeneous linear system 4x4

Solution of homogeneous system 4x4 of linear equations

$$A = \begin{pmatrix} X^{(0,1)} & X^{(0,2)} & -X^{(s,1)} & -X^{(s,2)} \\ Y^{(0,1)} & Y^{(0,2)} & -Y^{(s,1)} & -Y^{(s,2)} \\ F_1^{(0,1)} & F_1^{(0,2)} & -F_1^{(s,1)} & -F_1^{(s,2)} \\ F_2^{(0,1)} & F_2^{(0,2)} & -F_2^{(s,1)} & -F_2^{(s,2)} \end{pmatrix}$$

→

$$C^{(0,1)} = 1$$

$$C^{(0,2)} = V_2/V_1$$

$$C^{(s,1)} = V_3/V_1$$

$$C^{(s,2)} = V_4/V_1$$

RADIAL OSCILLATIONS

Linear Adiabatic Wave Equation (LAWE)

$\Delta\varpi / \varpi$

$$\frac{d^2x}{d\xi^2} + [\frac{3}{\xi} + \frac{(n+1)}{\theta}\frac{d\theta}{d\xi}]\frac{dx}{d\xi} + [\omega^2 - \frac{(n+1)\beta}{\Gamma_1} - \frac{(n+1)}{\xi}\frac{(2-2\Gamma_1)}{\Gamma_1}\frac{d\theta}{d\xi}]\frac{x}{\theta} = 0$$

where $\omega^2 = \frac{(n+1)}{4\pi G\rho_c\Gamma_1}\sigma^2$

Boundary Conditions($\xi \longrightarrow z$, $\theta \longrightarrow w$)

- Inner boundary (Center) - regularity

$$x = a_0 + a_1 z + a_2 z^2$$

$a_1 \rightarrow 0$

$$a_2 = -\frac{a_0}{8}[\omega^2 - (2-\Gamma_1)\frac{n+1}{\Gamma_1}\beta + (1-\Gamma_1)\frac{n+1}{\Gamma_1}]$$

RADIAL OSCILLATIONS

<u>Boundary Conditions</u>($\xi \longrightarrow$ z, $\theta \longrightarrow$ w)

- Outer boundary (Surface) $\frac{1}{w(z)} = \frac{1}{\frac{dw}{dz}|_{z=z_n}(z - z_n)} - \frac{\frac{d^2w}{dz^2}}{2(\frac{dw}{dz})^2}|_{z=z_n}$

$$x = 1 + a_{n-1}(z - z_n) + a_{n-2}(z - z_n)^2$$

$$a_{n-2} = \frac{1}{(n+1)(n+2)}\left[G'\left(\frac{-3}{2z} - \frac{\Omega'}{\frac{dw}{dz}} + \frac{G'}{2}\right) + \frac{\Omega'}{2\frac{dw}{dz}}\left[\left[\frac{3}{z_n} + \frac{\Omega'}{\frac{dw}{dz}} + \frac{(n+1)\frac{d^2w}{dz^2}}{\frac{dw}{dz}}\right]\right]\right]|_{z=z_n}$$

$$a_{n-1} = -\frac{1}{n+1}\left[\frac{\omega^2 - \frac{n+1}{\Gamma_1}\beta}{\frac{dw}{dz}|_{z=z_n}} - G'\right]$$

where $G' = \frac{2-2\Gamma_1}{\Gamma_1 z_n}(n+1)$ $\quad \Omega' = \omega^2 - \frac{(n+1)\beta}{\Gamma_1}$

RADIAL OSCILLATIONS

Test 1 ($n=1$, $\Gamma_1=5/3$, $\beta=0$)

- Robe (1968)

n	ω_0^2	ω_1^2	ω_2^2	ω_3^2
0	6.67−1	5.67+0	1.40+1	2.57+1
1	4.63−1	2.27+0	4.99+0	8.62+0
3	2.94−1	7.02−1	1.26+0	1.95+0
6	1.38−1	1.97−1	2.71−1	3.63−1

- Present Work

β	ω_0^2	ω_1^2	ω_2^2
0	0.555464	2.72823	5.986513
0.15	0.597872	2.035402	4.189147
0.25	0.609836	1.415298	2.615437
0.28	0.5996322	1.066983	1.75
0.2871	0.586931	0.749701	0.936736

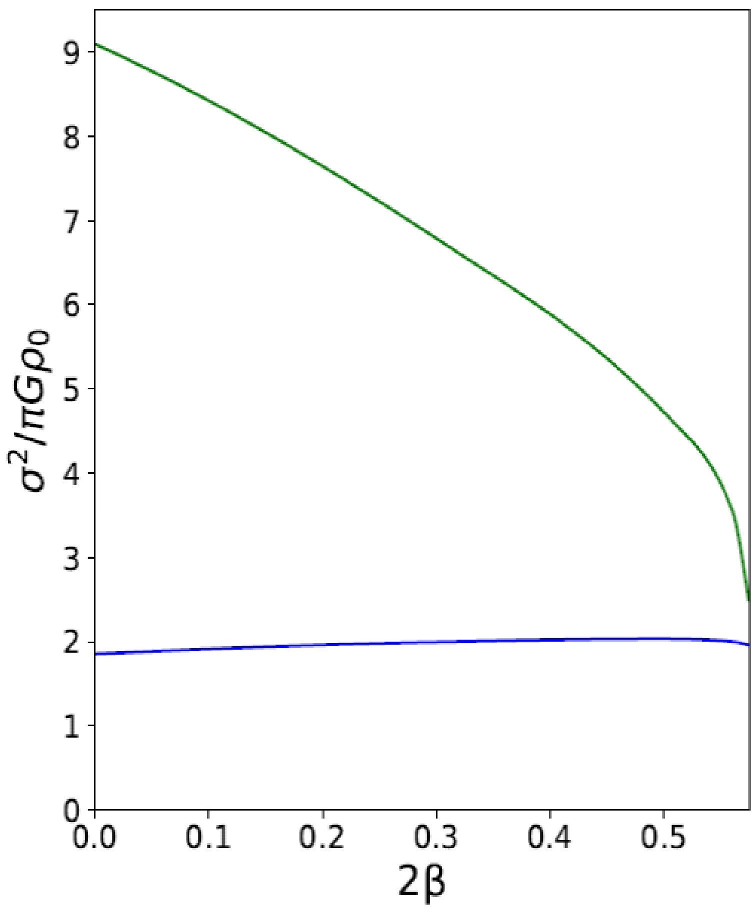
⟹ 0.463 , 2.27, 4.99

× $\Gamma_1/(n+1)$

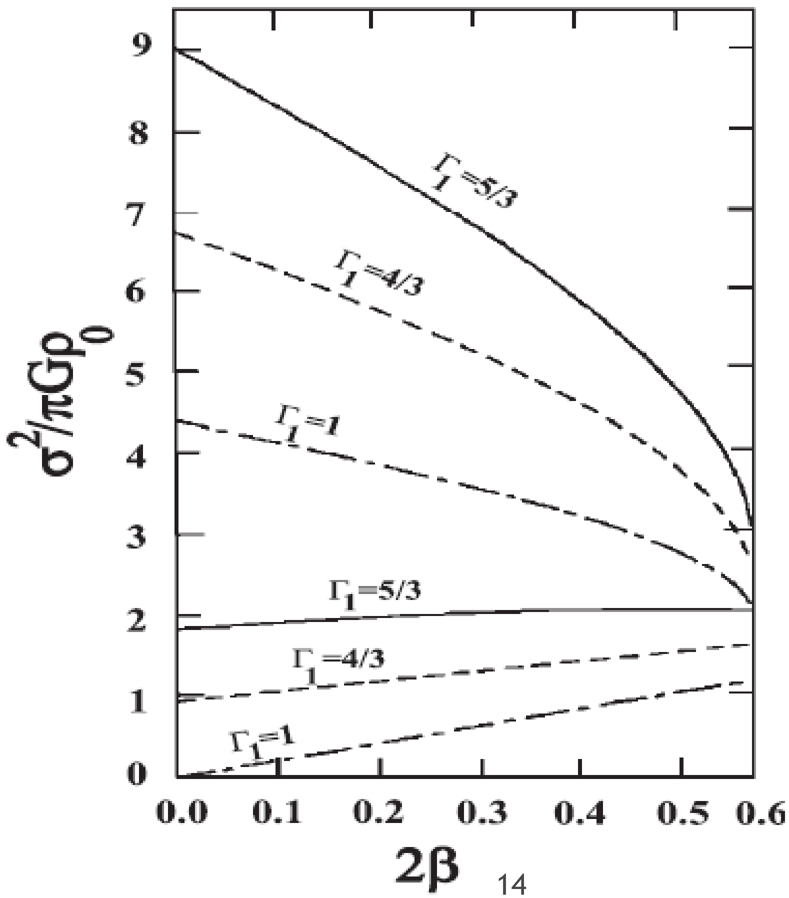
RADIAL OSCILLATIONS

Test 2($n=1$, $\Gamma_1=5/3$, β)

- Present work



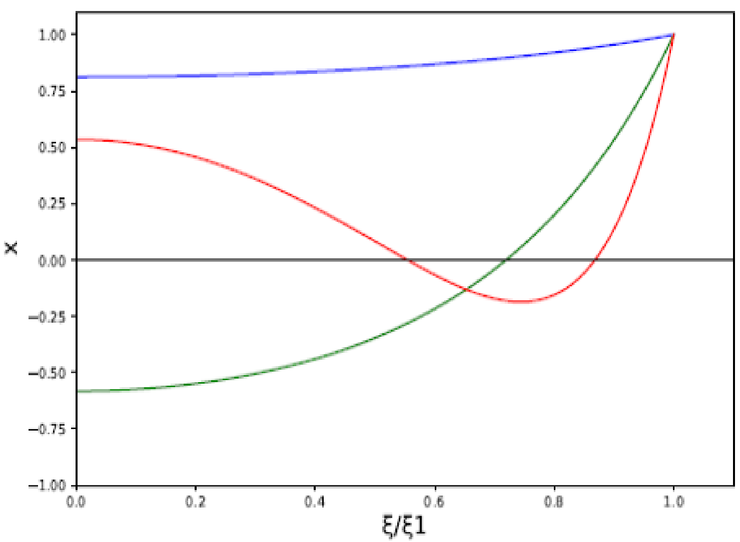
- Robe (1968)

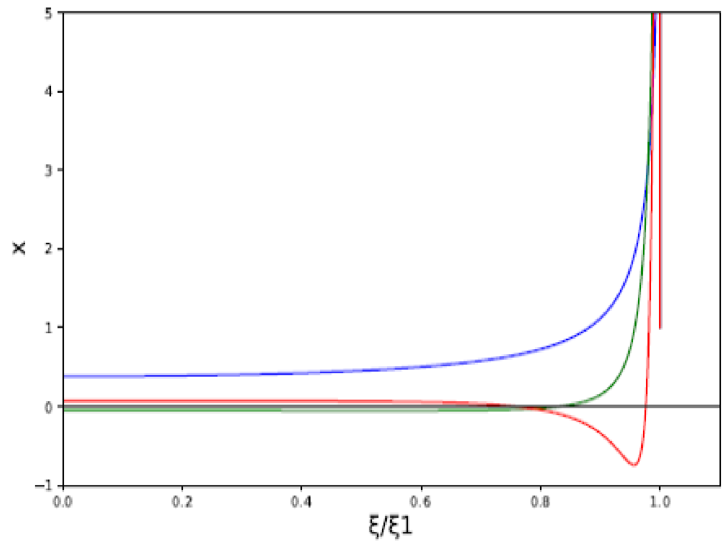


RADIAL OSCILLATIONS

<u>Present work</u>

- $n=1$, $\Gamma_1=5/3$





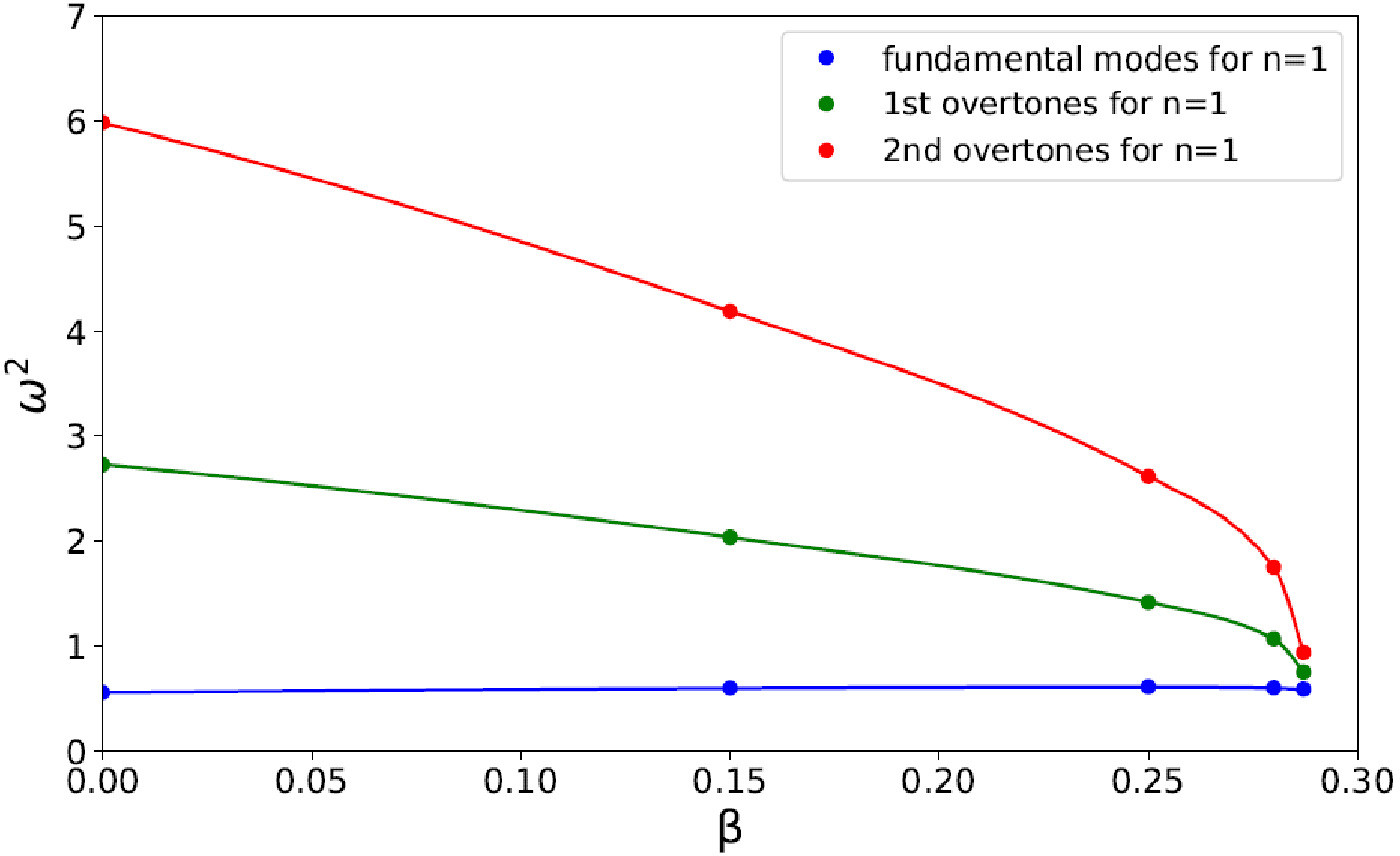
$\beta=0$

$\beta=\beta_{max}$

RADIAL OSCILLATIONS

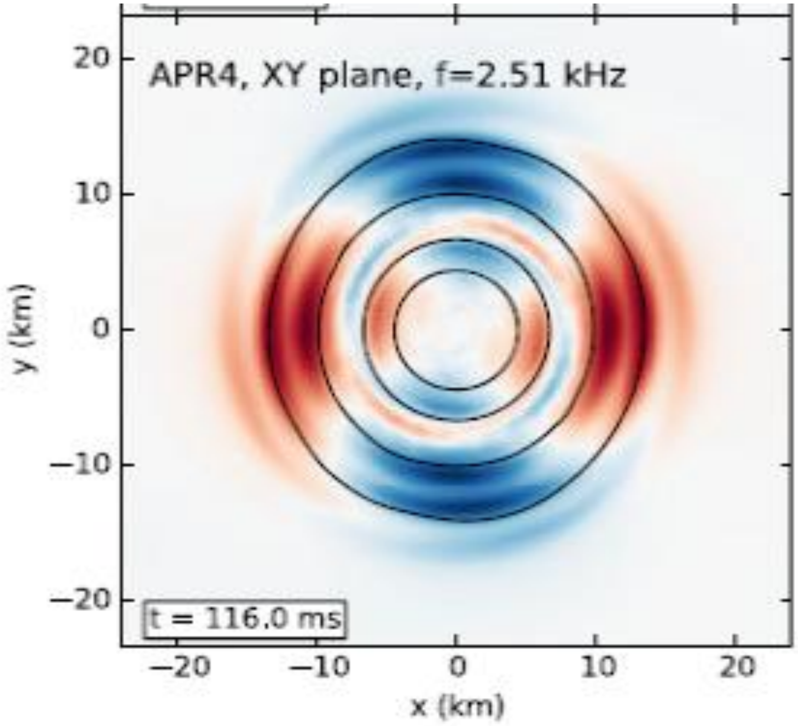
Present work

- $n=1.5$, $\Gamma_1=5/3$



NON-RADIAL OSCILLATIONS

- Simulation De Pietri et al. (2018)



During the merging of Binary Neutron Star

m=2 Quadrupole Oscillations

NON-RADIAL OSCILLATIONS

Boundary Conditions

- Inner boundary (Center)

$$\Delta\varpi = \varpi^c \sum_{\nu=0}^{\infty} X_\nu \varpi^\nu \qquad Y = \varpi^d \sum_{\nu=0}^{\infty} Y_\nu \varpi^\nu \longrightarrow Y(0) = \frac{1}{m}\left[1 - \frac{(2\beta)^{1/2}}{\tilde{\omega}}\right]\Delta\varpi(0)$$

$$F_1 = -m^2 \frac{\delta\tilde{\Phi}}{\xi^2} \qquad F_2 = -\xi^{-1}\frac{d\delta\tilde{\Phi}}{d\xi} \longrightarrow F_2(0) = \frac{F_1(0)}{m}$$

- Outer boundary (Surface)

$$F_2(\xi_1) = -\frac{F_1(\xi_1)}{m}$$

Pressure vanishes during oscillations (ΔP=0)

$$\frac{\delta P}{\rho_0}\Big|_{\xi=\xi_1} = -\Delta\varpi(\xi_1) \longrightarrow Y(\xi_1) = \frac{-1}{\xi_1\tilde{\omega}^2}\left[\left(\frac{\xi}{m}\right)^2 F_1 + \Delta\varpi(\xi_1)\right]$$

NON-RADIAL OSCILLATIONS

Test 1($m=2$, $\Gamma_1=5/3$, $\beta=0$)

- Robe (1968)

n	g_3	g_2	g_1	f	p_1	p_2	p_3
1	−9.10 − 3	−1.58 − 2	−3.50 − 2	3.46−1	1.88+0	4.52+0	8.11+0
3	1.44−2	2.31−2	4.39−2	2.22−1	6.39−1	1.18+0	1.87+0
6	2.31−2	3.61−2	6.46−2	1.30−1	1.78−1	2.54−1	3.49−1

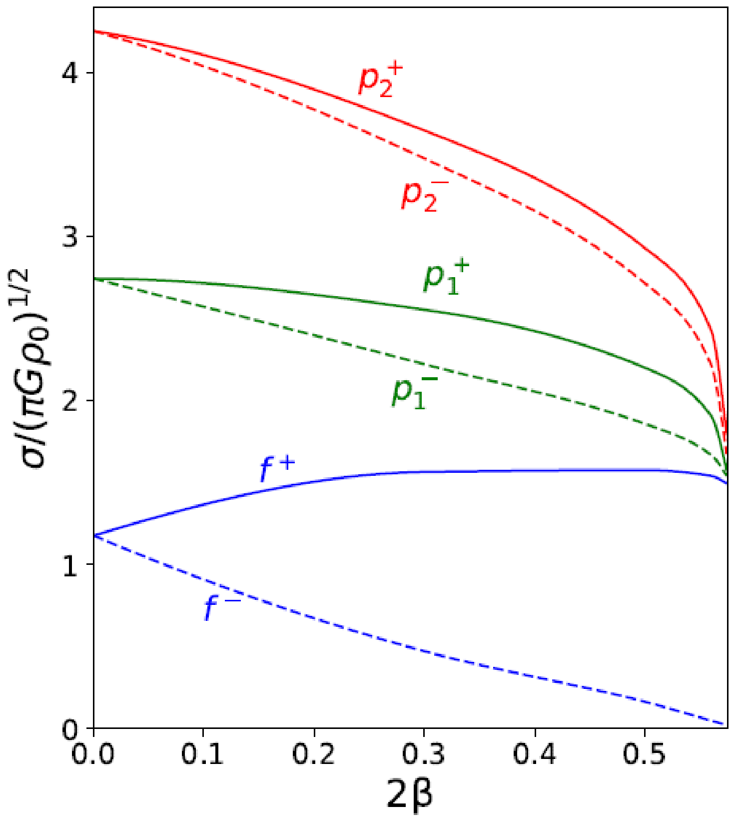
- Present work

n	g_3	g_2	g_1	f	p_1	p_2	p_3
1	-0.009099	-0.015822	-0.034961	0.346129	1.881	4.52	8.1201
3	0.0144	0.0231	0.043869	0.2219	0.6392	1.181	1.8701
6	0.02313	0.03608	0.06464	0.13	0.178	0.254	0.349

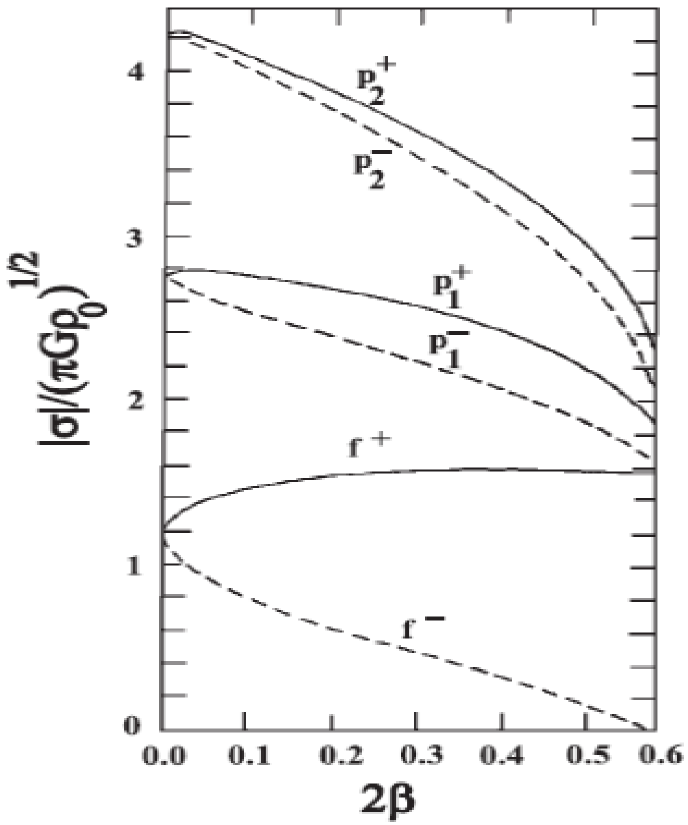
NON-RADIAL OSCILLATIONS

Test 2($n=1$, $\Gamma_1=5/3$, $m=2$, β)

- Present Code

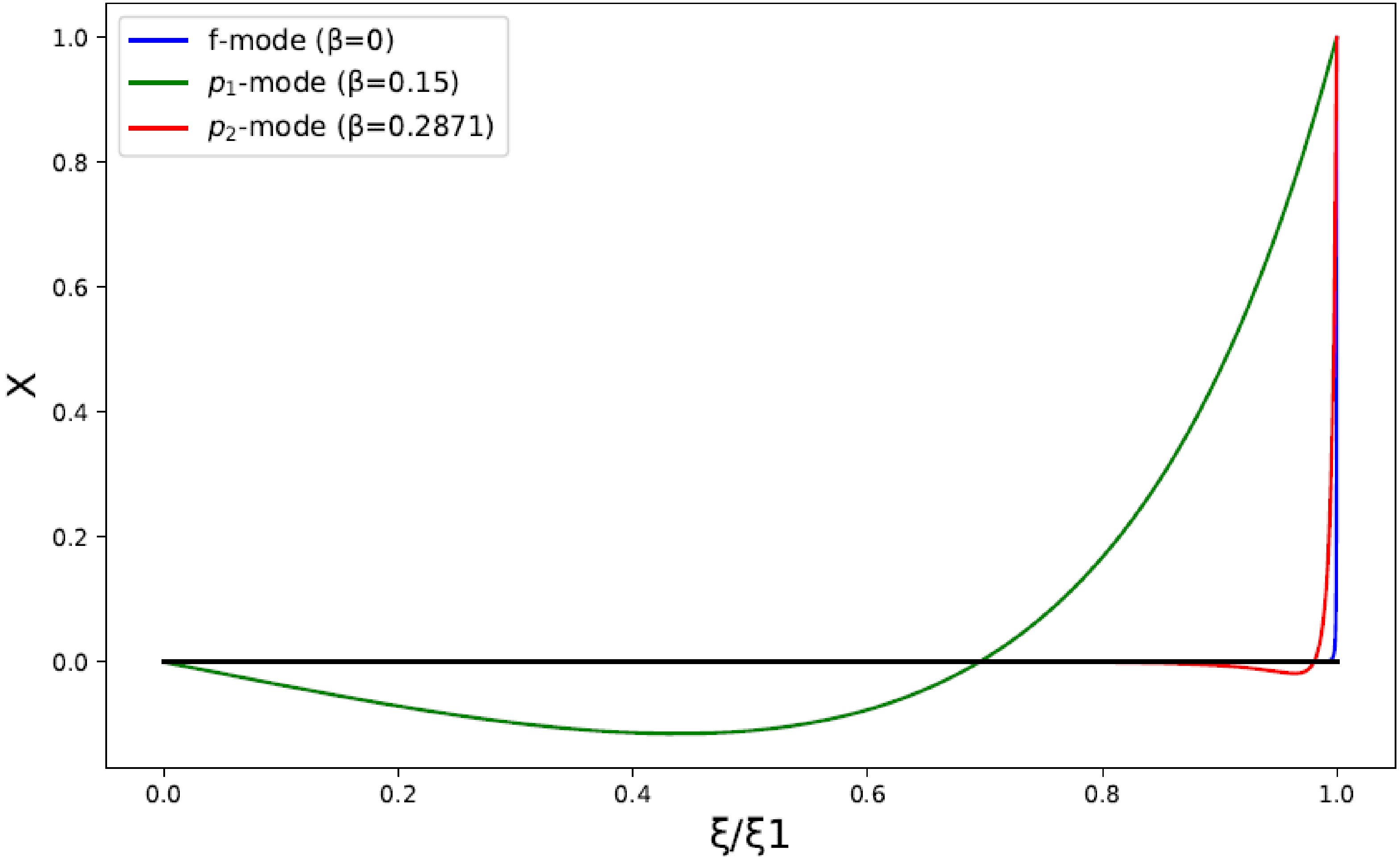


- Robe (1968)



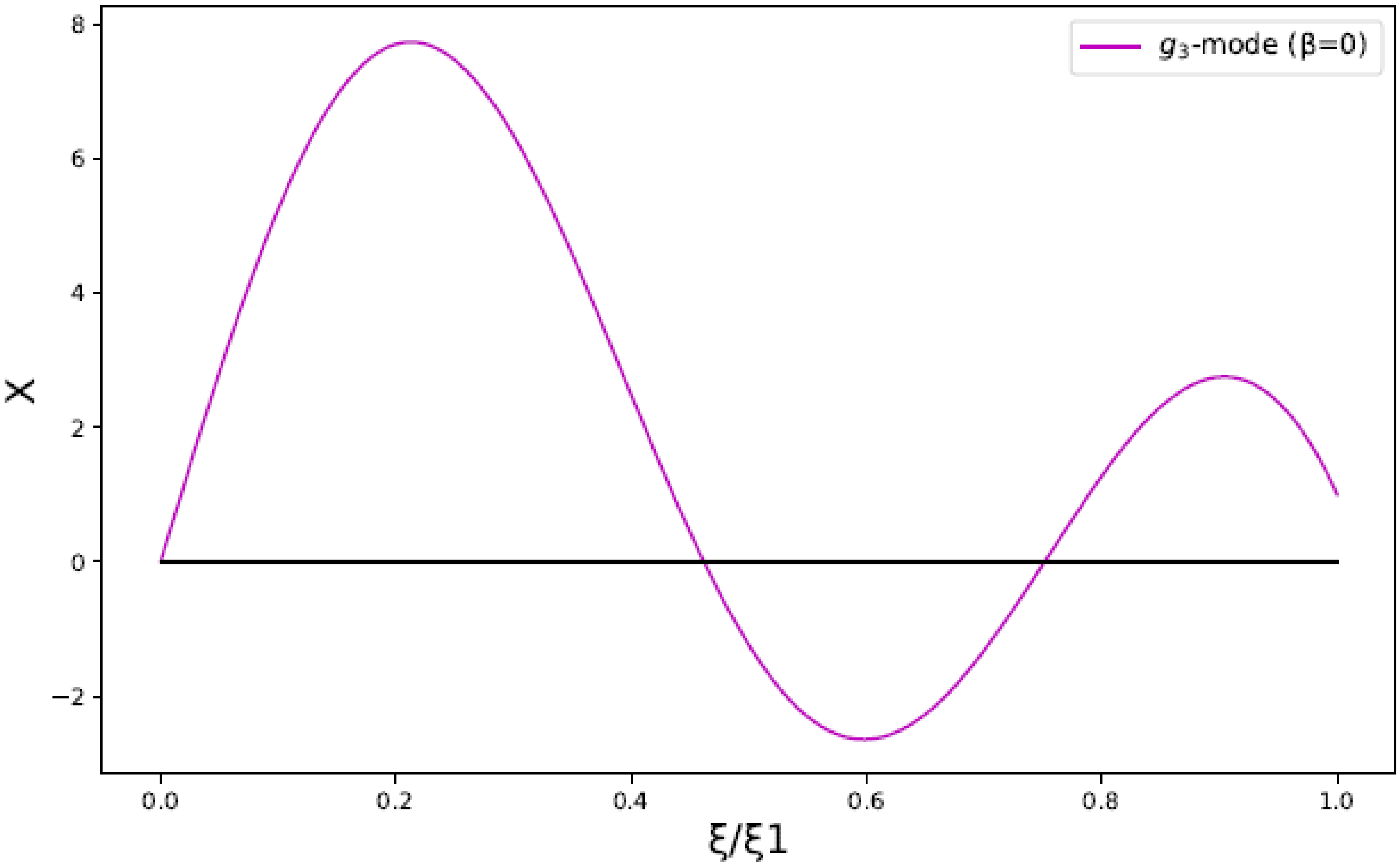
NON-RADIAL OSCILLATIONS

($n=1$, $\Gamma_1=5/3$, $m=2$, β)



NON-RADIAL OSCILLATIONS

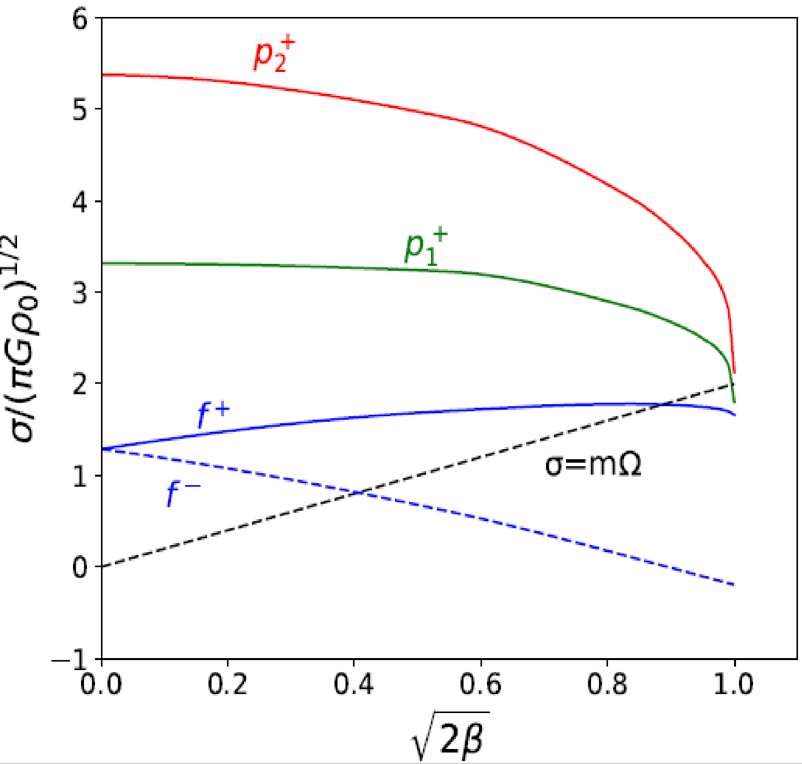
($n=1$, $\Gamma_1=5/3$, $m=2$, β)



NON-RADIAL OSCILLATIONS

Present work ($m=2$, $\Gamma_1=5/3$, β)

- n=0.5



◉ n=0.5 → p_2-mode≈$m\Omega_c$
p_1-mode≈ $m\Omega_{c\text{-}}$
f-mode≈$0.8m\Omega_c$

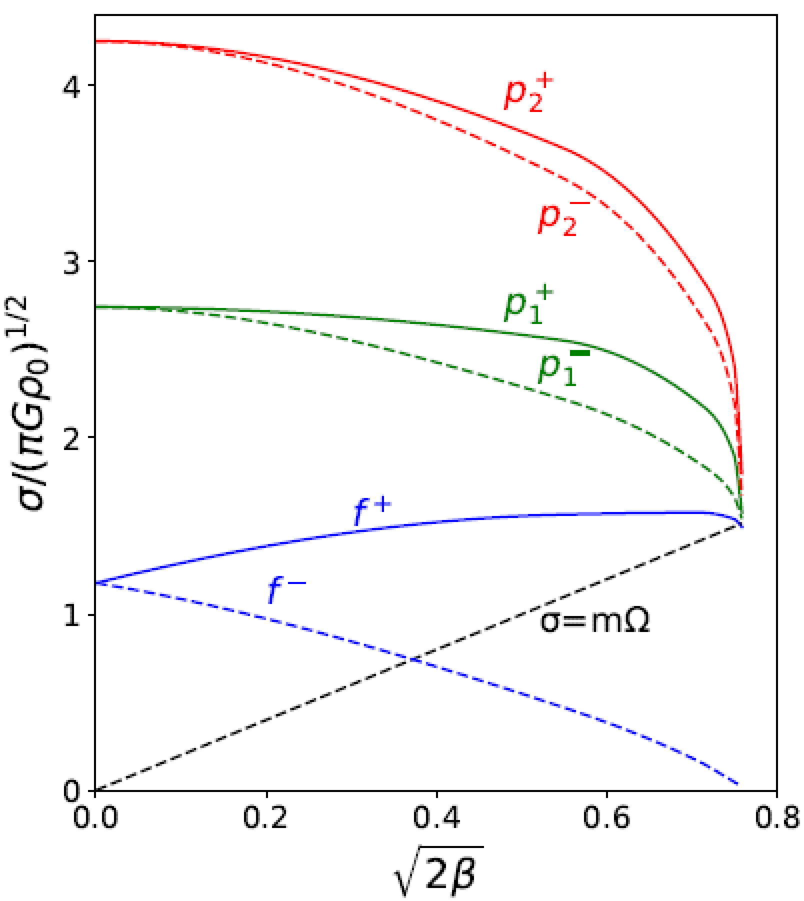
1. Co-rotating mode viscosity driven instability
   $\sigma_r=0 \Rightarrow \sigma_i=2\Omega$
2. Counter-rotating mode instability
   $\sigma_i=0$
   secular gravitational-wave driven instability (CFS)

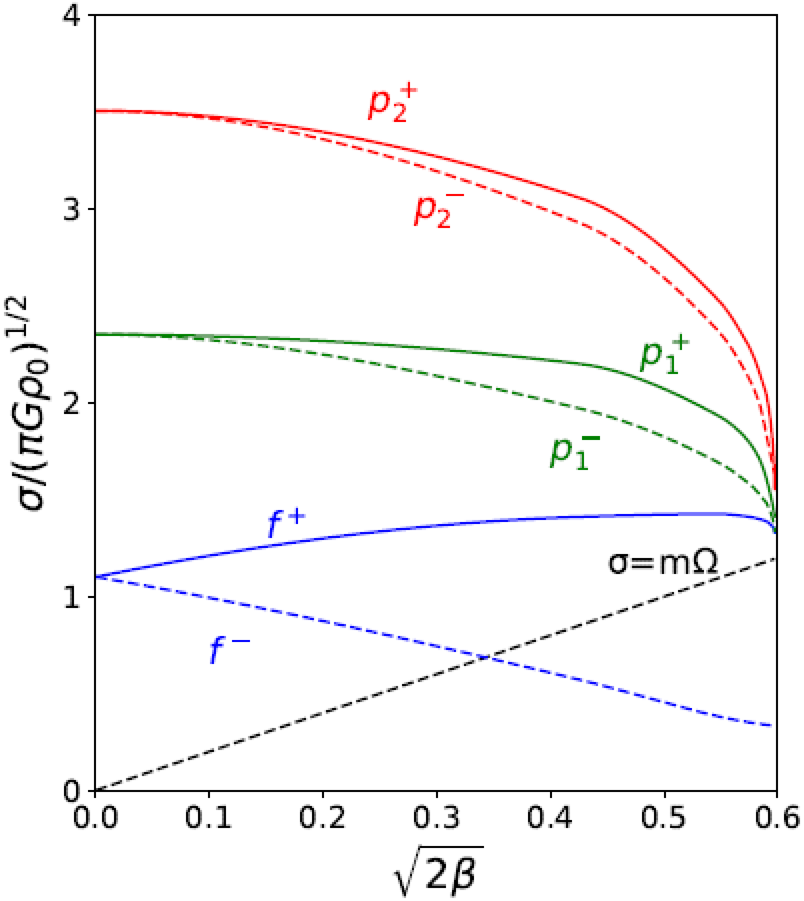
[see Friedman J.L. and Stergioulas N. (2013)]

NON-RADIAL OSCILLATIONS

Present work ($m=2$, $\Gamma_1=5/3$, β)

- $n=1$ ⇨ f-mode $\approx m\Omega c$
- $n=1.5$





GOALS FOR THE FUTURE

- Differential rotation and comparison
- Inertial modes
- Simulations
- Effect of rotation to co-rotating modes and counter-rotating modes
- Comparison between numerical methods
- Extension to spherical stars (Spherical Harmonics) +Relativity

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THANK YOU!