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MATHEMATISCH-
NATURWISSENSCHAFTLICHE
FAKULTÄT

Practical Course in Astronomy

Chaos in planetary systems

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Contents

Note

The goal of the labwork at hand for the student is to learn and implement some basic integrators to solve the initial value problem. The first section is identical to the first section of the other computational astrophysics labwork "N-Body Simulations with REBOUND".
If you have no or only scarce experience in programming and computer science, we advise you to favour the other labwork. In this labwork you will have to implement a N-Body integrator which requires some basic programming skills.

1 Introduction

The classical N-body problem in Astrophysics is important to understand the evolution and stability of planetary systems, star clusters and galactic nuclei. The dynamics of these many body systems is dominated by pairwise gravitational interactions between single bodies. Hence, this two-body interaction has to be accounted with high accuracy in the numerical treatment.

Typical numbers of the aforementioned systems are ~ 10 for planetary systems, 10^4 to 10^6 for star clusters and more than over 10^8 for galactic nuclei.

For particle numbers of this magnitude, the methods of statistical mechanics can only be applied partially, and one has to rely on the direct integration of each individual particle trajectory with regard of the mutual gravitational forces.

In this practical course, different numerical schemes for the solution of ordinary differential equations (ODEs) will be implemented and tested on the application to the integration of these trajectories. The main validation and test problem is the two-body problem and its analytical solution. More simulations include the restricted three-body problem.

2 The Classical N-Body-Problem

The motion of N point masses in their mutual gravitational field is the classical N-body problem. Each particle i with mass m_i has the location $\mathbf{r}_i$ and the velocity $\mathbf{v}_i$ at time t. The Hamiltonian of this system reads

$$H = \sum_{i=1}^{N} \frac{{\mathbf{p}_i}^2}{2m_i} - \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{G m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|} \tag{1}$$

with the canonical coordinates momentum $\mathbf{p}_i = m_i \mathbf{v}_i$ and location $\mathbf{q}_i = \mathbf{r}_i$ for all $i = 1 \dots N$ point masses. The Hamiltonian Equations yield the equations of motion for particle i

$$\dot{\mathbf{q}}_i(t) = \frac{\partial H}{\partial \mathbf{p}_i} \qquad \Longrightarrow \qquad \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \mathbf{v}_i \tag{2}$$

$$\dot{\mathbf{p}}_i(t) = -\frac{\partial H}{\partial \mathbf{q}_i} \qquad \Longrightarrow \qquad \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = \mathbf{a}_i \tag{3}$$

with the acceleration

$$\mathbf{a}_i(t) = \sum_{j \neq i}^{N} G m_j \frac{\mathbf{r}_{ij}}{r_{ij}^3} \quad . \tag{4}$$

Later, we will also need the time derivative of the acceleration, the so-called jerk

$$\dot{\mathbf{a}}_i(t) = \sum_{j \neq i}^{N} G m_j \left(\frac{\mathbf{v}_{ij}}{r_{ij}^3} - \frac{3(\mathbf{v}_{ij} \cdot \mathbf{r}_{ij})}{r_{ij}^5} \mathbf{r}_{ij} \right) \tag{5}$$

where $\mathbf{r}_{ij} := \mathbf{r}_j(t) - \mathbf{r}_i(t)$, $r_{ij} := |\mathbf{r}_{ij}|$, $\mathbf{v}_{ij} := \mathbf{v}_j(t) - \mathbf{v}_i(t)$, $v_{ij} := |\mathbf{v}_{ij}|$.

To calculate all accelerations at a certain time, one needs to evaluate $N\,(N-1)/2$ terms (using symmetry), that is the computing time for large numbers of N is asymptotically $\sim N^2$.

The Two-Body Problem

The Hamiltonian (1) of the two-body problem can be separated into the motion of the center of mass and the relative motion, that is $H = H(\mathbf{r}_1, \mathbf{r}_2, \mathbf{p}_1, \mathbf{p}_2) \rightarrow H(\mathbf{r}, \mathbf{p}, \mathbf{p}_{cm})$ with

$$H = H_{\mathrm{cm}} + H_{\mathrm{rel}} = \frac{\mathbf{p}_{\mathrm{cm}}^2}{2M} + \frac{\mathbf{p}^{\,2}}{2\mu} - \frac{G m_1 m_2}{r} \tag{6}$$

with the total mass $M := m_1 + m_2$, the reduced mass $\mu := m_1 m_2 / M$, the distance vector between the two bodies $\mathbf{r} := \mathbf{r}_1 - \mathbf{r}_2$, the distance $r := |\mathbf{r}\,|$, the relative momentum $\mathbf{p} := \mathbf{p}_1 - \mathbf{p}_2$ and the momentum of the center of mass $\mathbf{p}_{\mathrm{cm}}$.

The Hamiltonian of the center of mass motion H_{cm} (the first term on the right hand side of equation (6)) has trivial solutions. This follows from the face, that H_{cm} depends only on the momemtum of the center of mass $\mathbf{p}_{\mathrm{cm}}$ and not from the conjugate variable $\mathbf{r}_{\mathrm{cm}}$, the location of the center of mass. It follows a steady motion of the center of mass.

The Hamiltonian of the relative motion H_{rel} yields the equation of motion of the classical two-body problem

$$\dot{\mathbf{p}} = \mu \dot{\mathbf{v}} = -\frac{G m_1 m_2}{r^3} \mathbf{r} \tag{7}$$

with $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 = \mathbf{p}/\mu$, and finally in the familiar notation

$$\ddot{\mathbf{r}} = \dot{\mathbf{v}} = -\frac{GM}{r^3} \mathbf{r} \quad . \tag{8}$$

The solution of the Kepler-problem are conic sections with the orbital plane, the motion of $\mathbf{r}$ is characterised by an ellipse, a parabola or a hyperbola. The energy E, the specific angular momentum

$$\mathbf{j} = \mathbf{r} \times \mathbf{v} \tag{9}$$

and the Runge-Lenz vector

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{j}}{GM} - \frac{\mathbf{r}}{r} \tag{10}$$

are conserved values of the two-body motion. Using eq. (8), we find for the evolution of the angular momentum

$$\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t} = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \dot{\mathbf{v}} = -\frac{GM}{r^3} (\mathbf{r} \times \mathbf{r}) = 0 \quad . \tag{11}$$

To prove the conservation of $\mathbf{e}$, we need some calculus with the use of the vector identity $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$. It follows

$$\frac{\mathbf{j}\times\mathbf{r}}{r^3} = \frac{(\mathbf{r}\times\mathbf{v})\times\mathbf{r}}{r^3} = \frac{\mathbf{v}}{r} - \mathbf{r}\frac{(\mathbf{r}\cdot\mathbf{v})}{r^3} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) \quad . \tag{12}$$

With this term and the help of eq. (8) and (11), we find

$$\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} = -\frac{GM}{r^3}\frac{\mathbf{r}\times\mathbf{j}}{GM} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) = \frac{\mathbf{j}\times\mathbf{r}}{r^3} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) = 0 \quad . \tag{13}$$

Additionally, we regard the following term

$$\mathbf{r}\cdot\mathbf{e} + r = \frac{\mathbf{r}\cdot(\mathbf{v}\times\mathbf{j})}{GM} = \frac{(\mathbf{r}\times\mathbf{v})\cdot\mathbf{j}}{GM} = \frac{j^2}{GM} \quad . \tag{14}$$

The scalar product $\mathbf{r}\cdot\mathbf{e}$ is expressed with the angle $\phi - \phi_0$ between $\mathbf{e}$ and the location vector $\mathbf{r}$, and $re\cos(\phi-\phi_0) + r = j^2/(GM)$, it follows

$$r(\phi) = \frac{j^2/(GM)}{1 + e\cos(\phi-\phi_0)}. \tag{15}$$

This is the well-known equation of a conic section. For a constrained motion, the absolute value of the Runge-Lenz vector $e = |\mathbf{e}|$ specifies the eccentricity of the ellipse. Then, the maximum and minimum distance of the two point masses is given by

$$r_{\max/\min} = \frac{j^2/(GM)}{1 \pm e}, \tag{16}$$

the semi-major axis of the ellipse is given by

$$a = \frac{1}{2}(r_{\min} + r_{\max}) = \frac{j^2/(GM)}{1 - e^2}, \tag{17}$$

and the semi-minor axis is given by the geometric mean

$$b = \sqrt{r_{\min} r_{\max}}. \tag{18}$$

Because of Kepler's third law ("The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit."), it follows for the orbital period

$$\omega = \sqrt{\frac{GM}{a^3}}. \tag{19}$$

Please see fig. 1 for an illustrative sketch of the properties of a bound Kepler orbit.

3 Numerical Solution of the Equation of Motion: Time integrators

In order to calculate the trajectories of the point masses, one has to solve the corresponding equation of motion for each particle i, whic is the system of ODEs given by eqs. (2) and (3) with appropriate initial values for locations and velocites. In general, this is only possible by the help of numerical methods, which leads to the fact, that the particle distribution can only be determined at discrete time values. Starting from the current point in time $t = t_n$, the location

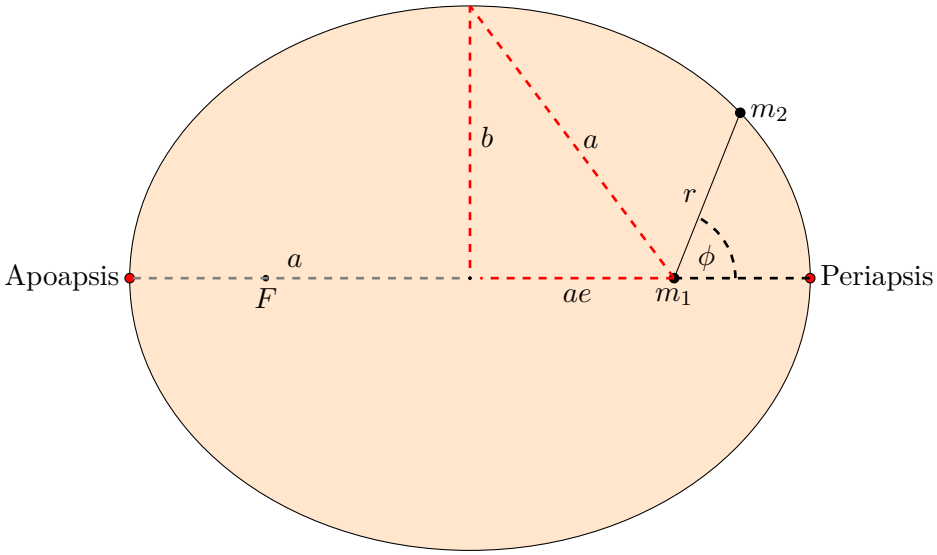


Figure 1: Bound Two-Body orbit. The mass m_1 is in the focus of the ellipse with semi-major and semin-minor axes a and b, and eccentricity e. The distance of m_1 to apoapsis is $r_{\max}$ and to periapsis $r_{\min}$. The equation 15 denotes the orbit with the choice of the coordinate system so that $\phi_0 = 0$.

and the velocity at a later point in time $t_{n+1} = t_n + \Delta t$ are calculated. Then, the new point in time becomes the current one and the algorithm restarts at the new time.

In the following, different numerical schemes will be presented which can be used to solve a system of ODEs of the form $\mathrm{d}y/\mathrm{d}t = f(y,t)$ with initial values. These schemes are called time integrators for obvious reasons in this context. They can, however, be applied for any initial value problem. The indices in the following refer always to the equivalent point in time, that is $r_n = r(t_n)$, and so forth. The index i for the different point masses will be left in the following.

3.1 Single-Step Methods

Basic idea: Consider the differentials $\mathrm{d}y$ and $\mathrm{d}t$ as finite intervals Δy and Δt

$$\frac{\mathrm{d}y}{\mathrm{d}t} \Rightarrow \frac{\Delta y}{\Delta t} = f(t,y)$$

and discretise with $\Delta t = h$

$$\Delta y = y_{n+1} - y_n = \Delta t f = hf.$$

Generally, single-step methods can be written in the following form

$$y_{n+1} = y_n + h\Phi(t_n, y_n, y_{n+1}, h),$$

where Φ is called the evolution function. The scheme is called explicit if $\Phi = \Phi(t_n, y_n, h)$ and implicit for $\Phi = \Phi(t_n, y_n, y_{n+1}, h)$. The step is from t_n to t_{n+1} and from y_n to y_{n+1}.

Single-step methods may be constructed by Taylor expansion. At first, we will discuss the simple explicit one-step method, the explicit Euler-method with $\Phi(t_n, y_n, h) = f(t_n, y_n)$.

3.2 Simple time integrators: The Euler-method and the Euler-Cromer-method

The simplest way for a discrete time integration scheme is achieved with the help of the Taylor expansion of location and velocity

$$\begin{aligned} v(t_n + \Delta t) &= v(t_n) + \frac{\mathrm{d}v}{\mathrm{d}t}(t_n)\Delta t + \mathcal{O}(\Delta t^2), && (20)\\ &= v(t_n) + a(t_n)\Delta t + \mathcal{O}(\Delta t^2), && (21)\\ r(t_n + \Delta t) &= r(t_n) + \frac{\mathrm{d}r}{\mathrm{d}t}(t_n)\Delta t + \mathcal{O}(\Delta t^2), && (22)\\ &= r(t_n) + v(t_n)\Delta t + \mathcal{O}(\Delta t^2), && (23) \end{aligned}$$

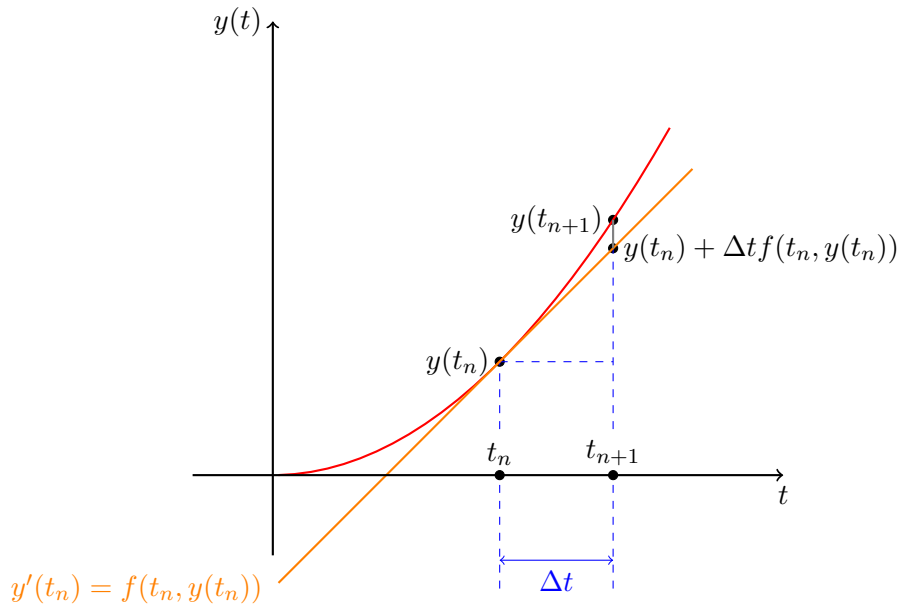
where the acceleration a is given by eq. (4).

These equations directly yield an algorithm, the so-called Euler-method, where one calculates

$$\begin{aligned} v_{n+1} &= v_n + a_n \Delta t, && (24)\\ r_{n+1} &= r_n + v_n \Delta t && (25) \end{aligned}$$

for every time step.

The Euler-method uses the simple slope triangle to perform one forward step



For each point in time t_n, the evolution of the function $y(t)$ from time t_n to $t_{n+1} = t_n + \Delta t$ is approximated by $\Delta t y'(t_n) = \Delta t f(t_n, y(t_n))$. The discretization error is given by the difference $y(t_{n+1}) - [y(t_n) + \Delta t f(t_n, y(t_n))]$.

As an alternative to the Euler-method where the velocity is calculated first, there is the so-called Euler-Cromer-method. Here, the new locations are calculated with the velocities at the new time step t_{n+1} instead of t_n like in the standard Euler-method.

$$\begin{aligned} v_{n+1} &= v_n + a_n \Delta t, && (26)\\ r_{n+1} &= r_n + v_{n+1} \Delta t. && (27) \end{aligned}$$

Averaging both methods yields the scheme

$$v_{n+1} = v_n + a_n \Delta t, \tag{28}$$

$$r_{n+1} = r_n + \frac{1}{2}\left(v_n + v_{n+1}\right)\Delta t. \tag{29}$$

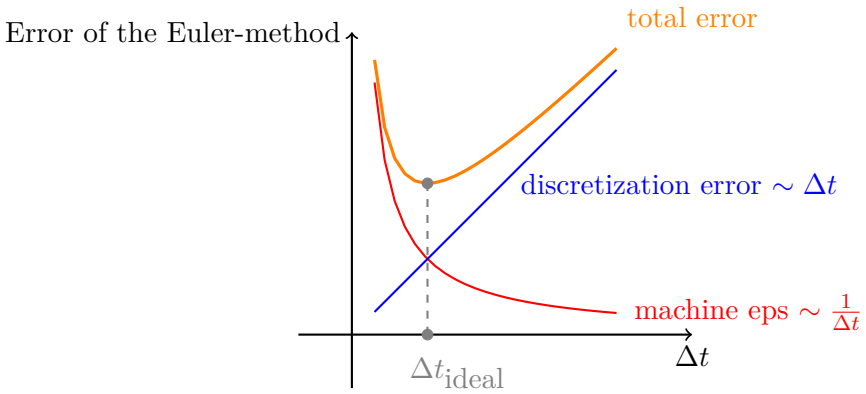
3.2.1 Accuracy of the scheme

The main disadvantage of these methods is the low accuracy. If the trajectories of particles have to be determined until the space in time $t_{\max}$ and each time step is of length Δt, we need $N_t := t_{\max}/\Delta t$ time steps. The error per time step is of the order $O(\Delta t^2)$ (see Taylor expansion), leading to a total error of the simulation of the order $O(N_t \Delta t^2) = O(\Delta t)$. And the computational accuracy was not even accounted, yet.

3.2.2 Computational accuracy

Let ϵ be the machine accuracy, the relative error due to rounding in floating point arithmetic. The error caused by ϵ grows with $O(N_t \epsilon)$. In order to improve the accuracy of the numerical scheme, one can choose smaller time steps Δt, which consequently yields a higher error by ϵ, since we need a higher number of time steps N_t. Hence, global values of a N-body simulation like the total energy E have an error of the order $O(N_t N \epsilon)$.

Some numbers for clarification: Depending on the numerical scheme, approximately 100 to 1000 steps per orbit are required for the two-body problem to achieve an error for the energy in the order of the machine accuracy, that is $\Delta E/E \sim 10^{-13}$. The error in the energy adds up to $\Delta E/E \sim 10^{-9}$ for a complete orbit. To simulate the evolution of a star cluster with about 10^6 stars and for the time of about 10^4 typical orbital periods, one has to evaluate 10^{10} orbits with 1000 time steps each, in total 10^{13} time steps, eventually leading to an error of the total energy of $\Delta E \sim 100$ %.



3.3 The Leap-Frog-method

One improvement of the Euler-method is to use the velocity for the calculation of the new locations at the point in time $t_{n+1/2} = t_n + \Delta t/2$ and not on the times t_n or t_{n+1}. The scheme

of this method is

$$
\begin{aligned}
&\text{First step:}\\
&\qquad \mathbf{r}_{1/2} = \mathbf{r}_0 + \mathbf{v}_0 \frac{\Delta t}{2} \quad ,\\
&\qquad\qquad \text{Calculate } \mathbf{a}_{1/2} = \mathbf{a}(t_{1/2}, \mathbf{r}_{1/2}) \quad ,\\
&\text{Regular steps:}\\
&\qquad \mathbf{v}_{n+1} = \mathbf{v}_n + \mathbf{a}_{n+1/2}\, \Delta t \quad ,\\
&\qquad \mathbf{r}_{n+3/2} = \mathbf{r}_{n+1/2} + \mathbf{v}_{n+1}\, \Delta t \quad ,\\
&\text{Last step:}\\
&\qquad \mathbf{r}_{n+1} = \mathbf{r}_{n+1/2} + \mathbf{v}_{n+1} \frac{\Delta t}{2} \quad .
\end{aligned}
$$

The name of the this scheme "Leap Frog" originates from the fact that the positions and velocities are updated at interleaved time points separated by $\Delta t/2$, staggered in such a way, that they leapfrog over each other. Hence, locations and velocities are not known at the same point in time In the limit of $\Delta t \to 0$, one gets the usual canonical equations. Moreover, for finite Δt, the scheme is symplectic, which means, there is a Hamiltonian $\hat{H} = H + \Delta t^2 H_2 + \Delta t^4 H_4 + \ldots$, which is solved exactly by the scheme. This has a positive impact on the consistency of conserved quantities.

3.4 The Verlet-method

The Leap-Frog-method can be written as the so-called kick-drift-kick algorithm

$$\tilde{\mathbf{v}}(\Delta t) = \mathbf{v}(t) + \frac{1}{2}\Delta t\, \mathbf{a}(t) \tag{30}$$

$$\mathbf{r}(t + \Delta t) = \mathbf{r}(t) + \Delta t\, \tilde{\mathbf{v}}(t) \tag{31}$$

$$\mathbf{v}(t + \Delta t) = \tilde{\mathbf{v}}(t) + \frac{1}{2}\Delta t\, \mathbf{a}(t + \Delta t) \tag{32}$$

The intermediate value $\tilde{\mathbf{v}}(\Delta t)$ denotes the velocity at time $t + 1/2\Delta t$.

The advantage of this velocity Verlet-method is the higher accuracy in the calculation of the locations. The Verlet-method is identical to the Leap-frog-method. Note, that also in this method, only one evaluation of the force term is required per time step.

3.5 Runge-Kutta-Integrators

The error of one single step in the Euler method can be analyzed by looking at the Taylor expansion. The Taylor expansion of $y(t + h)$ at t yields with $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y, t)$

$$y(t + h) = y(t) + h\frac{\mathrm{d}y}{\mathrm{d}t} + \mathcal{O}(h^2) \tag{33}$$

$$= y(t) + hf(y, t) + \mathcal{O}(h^2) \tag{34}$$

Hence, we could get better accuracy if we also include higher terms.

3.5.1 Basic idea of RK schemes

Now, let's expand up to third order

$$y(t+h) = y(t) + h\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{h^2}{2}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \mathcal{O}(h^3) \tag{35}$$

$$= y(t) + hf(y,t) + \frac{h^2}{2}\frac{\mathrm{d}f}{\mathrm{d}t} + \mathcal{O}(h^3) \tag{36}$$

The derivative of f with respect to t is given by

$$\frac{\mathrm{d}f(y(t),t)}{\mathrm{d}t} = \frac{f(y(t+h),t+h) - f(y(t),t)}{h} + \mathcal{O}(h). \tag{37}$$

Unfortunately, we do not know $y(t+h)$. By using the Euler scheme to estimate $y(t+h)$, we get

$$\frac{\mathrm{d}f(y(t),t)}{\mathrm{d}t} = \frac{f(y(t)+hf(y(t),t),t+h) - f(y(t),t)}{h} + \mathcal{O}(h). \tag{38}$$

Inserting eq. (38) into (36) yields

$$y(t+h) = y(t) + hf(y(t),t) + \frac{h}{2}\left\{f(y(t)+hf(y(t),t),t+h) - f(y(t),t)\right\} + \mathcal{O}(h^3) \tag{39}$$

$$= y(t) + \frac{h}{2}\left\{f(y(t),t) + f(y(t)+hf(y(t),t),t+h)\right\} + \mathcal{O}(h^3). \tag{40}$$

We have improved the accuracy for the cost of an additional calculation of f at substep $y(t) + hf(y,t)$. This is the basic idea of any Runge-Kutta method. In fact, we have just derived a RK 2nd order scheme, the Heun method.

The general form of an explicit RK method is

$$y_{n+1} = y_n + h\sum_i b_i k_i, \tag{41}$$

where the k_i denote the substeps and b_i are some weights. Please see a textbook for further details. Here, we provide only the terms for the RK4 method.

3.5.2 The classical Runge-Kutta scheme

A system of first order ODEs can be written in vector notation

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}(t)), \quad \mathbf{y}(t_0) = \mathbf{y}_0 \tag{42}$$

with the vector for the initial values $\mathbf{y}_0$. In our N-body problem, $\mathbf{y}$ will be $(\mathbf{r}, \mathbf{v})$ with initial values $(\mathbf{r}(0), \mathbf{v}(0))$. A very common integration scheme is the so-called 4th order Runge-Kutta-method (RK4). In this scheme, estimated values for the derivative at single substeps in the interval $t_n, t_n + h$ are calculated. In vector notation, it reads

$$\mathbf{K}_1 = \mathbf{f}(t_n, \mathbf{y}_n) \tag{43}$$

$$\mathbf{K}_2 = \mathbf{f}\left(t_n + \frac{1}{2}h, \mathbf{y}_n + \frac{h}{2}\mathbf{K}_1\right) \tag{44}$$

$$\mathbf{K}_3 = \mathbf{f}\left(t_n + \frac{1}{2}h, \mathbf{y}_n + \frac{h}{2}\mathbf{K}_2\right) \tag{45}$$

$$\mathbf{K}_4 = \mathbf{f}\left(t_{n+1}, \mathbf{y}_n + h\mathbf{K}_3\right) \tag{46}$$

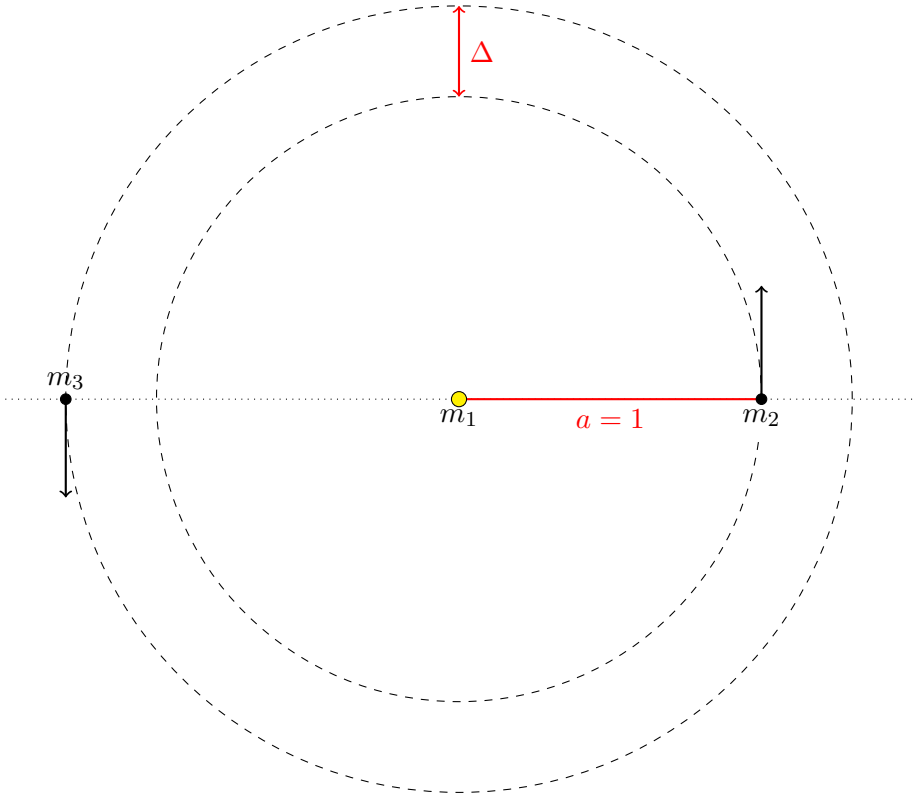


Figure 2: Schematic representation of a three body problem, where two smaller test masses m_2 and m_3 orbit a larger central mass m_1, it holds $m_2 + m_3 \ll m_1$. The initial semi-major axis of mass m_2 is normed to 1 in respect to the central mass m_1, and the semi-major axis of m_3 to $1 + \Delta$. The initial locations of the smaller bodies are opposed, $\delta\phi = \pi$, (after Gladman, 1993).

where $\mathbf{y}_n, \mathbf{y}_{n+1}, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4$ and $\mathbf{f}$ are vectors in the vector space R^m. The value of the function $\mathbf{y}$ at the new time is calculated by

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{6}\left(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4\right). \tag{47}$$

The error of the scheme is of the order $\mathcal{O}(h^5)$. The price for this high accuracy is the required calculation of the function which is four times per time step. However, the stability is also improved compared to the Euler-method. For a more detailed description of the RK4-method, we refer to the book "Numerical Recipes"[1] and wikipedia. We will apply the RK4-method in this course on the system of equations (2) and (3).

4 The co-planar Three-Body-Problem

In the last part of the practical course, we want to investigate the stability of the restricted three-body problem, schematically shown in Fig. 2. The setup of this part is from a publication by Gladman (1993).

[1] `http://apps.nrbook.com/c/index.html`, p.710

The initial setup is as follows: A central mass m_1, which is initially orbited by two smaller bodies (planets) on circular orbits, which start in opposition ($\delta\phi = \pi$). The mutual gravitational perturbations will change the shape of the orbits, the eccentricities e and semi-major axes a. Large changes are expected when the two planets are close in conjunction. A system is called Hill-stable (or simply stable), if close encounters are excluded for all times.

The stability of the system can be examined in dependence of the initial conditions and the masses. We will investigate the conditions described in Fig. 2.

For small initial separations Δ of the two given masses m_2, m_3, we will expect larger gravitational perturbations of the two masses and hence higher changes in e and a. For a fixed initial separation Δ, we expect instability if the masses get larger.

For details concerning the setup, analytical estimations and results, please see the paper by Gladman (1993).

Topological analysis of stability

The stability of the general co-planar three-body problem can be investigated also using algebraic methods. Here, the dynamics of the system is determined by the function c^2h, where c denotes the total angular momentum and h the total energy of the system. Both are conserved quantities, and so is c^2. As a function of the masses, one can find a critical value $(c^2h)_{\rm crit}$. For higher values as this critical value, the dynamics of the system shows a bifurcation. For

$$(c^2h) > (c^2h)_{crit} \tag{48}$$

the system is Hill-stable. Is the relation already given for the initial values, it is valid for all times, since c and h are both conserved quantities.

Gladman (1993) finds for small masses m_2 and m_3 and initially circular orbits the following criterion for stability

$$\Delta_c \simeq 2.40\,(\mu_2 + \mu_3)^{1/3}. \tag{49}$$

For non-circular orbits, more general criteria can be derived, but this is off the scope of this experiment.

5 Description of the practical course

In the following exercises, we will first implement some of the integrators and validate their functionality. Afterwards, we will investigate the limits for stability in the above-mentioned co-planar three-body problem numerically.

The scaling in the following is $G = 1$ and the mass of the central object $m_1 = 1$. Hence, the typical time scale is given by $\approx 2\pi$, since the period of a massless testparticle with semi-major axis $a = 1$ in these units is given by 2π. Please use the center-of-mass system for all initial positions and velocities.

In order to study the stability, we start with a given Δ and perform the integration of the trajectory for 10^3 orbits. The time evolution of the orbital elements have to be plotted.

Instability will be detected by close-encounters. A close-encounter is given, when the distance between the two planets is lower than the Hill radius of the more massive one of the two bodies.

The Hill radius is given by

$$R_{in} \approx a\sqrt[3]{\mu/3}, \tag{50}$$

where the distance of the planet to the star is denoted by a and μ denotes the mass ratio of the smaller object to the more massive one.

In the exercises, we determine numerically the critical distance Δ_c. For values lower than Δ_c, the system is unstable. The precise value of Δ_c might depend on the applied numerical integrator.

Exercise 1: The code

Develop a N-body simulation program to integrate the trajectories of various bodies under their mutual gravitational influence. **Use the center-of-mass coordinate system.** The code should additionally calculate the total energy E and the specific angular momentum $\mathbf{j}$,

Implement the following time integrators: Euler, Euler-Cromer, Velocity Verlet, RK4.

Determination of the time step Δt

The time step can be chosen to be fixed for the whole simulation

$$\Delta t = \eta. \tag{51}$$

However, to improve the accuracy of the simulation, the time step can also be adaptive and changes during the simulation. The time step may for example depend on the current curvature of the trajectory In this case, the time step is calculated from the accelerations and the jerks

$$\Delta t = \tilde{\eta} \min_{i=1\ldots N} \left(\frac{|\mathbf{a}_i(t_n)|}{|\dot{\mathbf{a}}_i(t_n)|} \right). \tag{52}$$

At first, your code should use a fixed time step size according to eq. (51). Later, you will also use an adaptive time step size according to eq. (52). If an adaptive time step cannot be determined (e.g., for the first time step when all velocities vanish, or so), the algorithm should fallback on the fixed time step.

Exercise 2: The two-body problem

In this exercise, the properties of the various time integrators are studied with the help of the two-body problem.

1. Integrate the two-body problem until $t_{max} = 10\mathrm{P}$ (10 orbits of the system) with 500 fixed time steps per orbit ($\eta = \Delta t = \frac{1}{500}\mathrm{P}$).

   The initial values are

   $$m_2 = 10^{-3} \qquad e_2 = 0.5 \qquad a_2 = 1.$$

   The initial velocity of the more massive body (star) is zero, and the less massive body (planet) is located at its apoapsis (the point of greatest distance to the star).

There, the location and velocity is given by the vis-viva-equation

$$x = a(1+e) \qquad y = 0$$
$$v_x = 0 \qquad v_y = \sqrt{\frac{G(m_1+m_2)}{a}\frac{1-e}{1+e}}.$$

The code units are scaled in a way, that one period is 2π. Use these formulae to calculate the initial position and velocity and transform them to the center-of-mass system before you start the integration.

Apply the Euler-, Euler-Cromer-, Verlet- and RK4-method.

2. Plot the trajectories of the second body (in the center-of-mass system) for 10 orbits and the four different integration schemes.
3. Additionally, calculate for each time step the specific angular momentum $\mathbf{j}$ (eq. 9), the Runge-Lenz-Vector $\mathbf{e}$ (eq. 10) and the semi-major axis a (eq. 17).
4. The quality of the calculation and especially the quality of the time integrator can be checked by various criterions. Examine
   - the conservation of E, $|\mathbf{j}|$, $|\mathbf{e}\,|$, and a by plotting the time evolution of the values.
   - the accuracy by plotting the time evolution of $\log|(E - E^{\text{start}})/E^{\text{start}}|$, $\log|(e - e^{\text{start}})/e^{\text{start}}|$ and $\log|(a - a^{\text{start}})/a^{\text{start}}|$, for different time step sizes η (e.g., $\frac{1}{10}$P, $\frac{1}{50}$P, $\frac{1}{100}$P, $\frac{1}{500}$P, $\frac{1}{1000}$P).

   Please use the logarithmic scale, since the differences between the values are rather small, especially at the start of the simulation.
5. Re-run the simulations with an adaptive time step size and plot evolution of the adaptive time step.

Exercise 3: Stability of circular orbits

1. Test the prediction of the topological stability criterion for the co-planar three-body problem.

   The initial values are

   $$m_2 = 10^{-5} \qquad e_2 = 0 \qquad a_2 = 1$$
   $$m_3 = 10^{-5} \qquad e_3 = 0 \qquad a_3 = 1 + \Delta,$$

   where Δ denotes the separation between the two planets, see figure 2.

   The initial locations of the two smaller bodies should be opposite ($\delta\phi = \pi$).

   Determine for each time step the orbital elements a and e by treating each trajectory of the planet like a two-body problem (sun-planet). Integrate the system for 1000 orbits of the inner planet with RK4 and plot the time evolution of a and e. Use an adaptive time step. What do you observe? What is the influence of the time step size?
2. Reduce the initial distance between the planets Δ stepwise and repeat the integration (note, that you have to adapt the velocity of the outer planet if you change Δ).

   For which values of Δ do you expect instability? Which value Δ_c do you find numerically?[2]

[2] Interesting values for Δ are in the range ≤ 0.066 according to eq. 49.

Exercise 4: Chaotic Dynamics

To determine the value of the stability limit Δ_c, we have stopped the evolution of the system at the first encounter of the two planets. However, the further evolution of the system is also interesting, since it shows complex dynamic behaviour, which is chaotic. In this last exercise, we want to investigate this chaotic phenomenon empirically.

The probability for close encounters is especially high if the planetary orbits overlap initially. The circular ring around the central star, in which the outer planet has to start initially in order to experience a close encounter later on, is called crossing zone. The outer rim of the crossing zone is the Hill limit Δ_c.

1. The system

   $$m_2 = 10^{-6} \qquad e_2 = 0 \qquad a_2 = 1 \qquad m_3 = 10^{-6} \qquad e_3 = 0 \qquad a_3 = 1 + \Delta$$

   is stable for $\Delta_c = 0.0302$ (eq. 49).

   Calculate the time evolution of the system for the initial values of Δ in the crossing zone, for instance

   $$\Delta = 0.026, \quad 0.028, \quad 0.030, \quad 0.0302, \quad 0.0305$$

   Count the number of close encounters during the first 1000 periods of the inner planet.

2. Plot the time evolution of both eccentricities e_2 and e_3, and both semi-major axes a_2 and a_3. What do you find?

Hints for the experiment protocol

During the course of this experiment and especially for the protocol, you will produce a lot of data that you want to visualize and present. Before you add an enormous number of figures or data plots to your protocol, first think about what you really have to plot (see the exercises) and how you can combine several plots into one figure or one figure with several subfigures. For example, when you plot the time evolution of the energy for different time integrators and different time step sizes (exercise 2.4), you can essentially create one figure with a panel for each time step size (Period/x, x=10,50,100,500,1000) which shows the three different integrators Euler, Verlet, RK4 in one plot, instead of creating 3x5=15 figures. I will not accept protocols with more than 30 pages.

Although a nice diagram and a figure speaks more than one million words, please add your results and findings in text form also to your protocol. Additionally, keep in mind that the protocol should be readable and understandable without the experiment instructions! Write down your initial conditions, explain what you do and why you do it. Do not copy and paste from the instructions, write your own words in own sentences. Do not add code listings to the protocol, send your source files separately, including instructions on how to run your code.

Moreover, grammar does not really matter as long as you get the formulae right in Physics, but you mihgt agrea that it impruves the readbillitty. No one wants to read a protocol with a lot of typos.

See the guidelines of the labwork for more information.

Enjoy the experiment! Astrophysics is fun 

References and further reading

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