EBERHARD KARLS
UNIVERSITÄT
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MATHEMATISCH-
NATURWISSENSCHAFTLICHE
FAKULTÄT

Practical Course in Astronomy

N-Body Simulations with REBOUND

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Contents

Note

The goal of the labwork at hand for the student is to learn and apply the widely-used N-Body integrator software package REBOUND. The first section is identical to the first section of the other computational astrophysics labwork "Chaos in the planetary system".

If you have no or scarce experience in programming and computer science, we advise you to favour this labwork. You will start with the installation of a scientific python programming environment on your Laptop/PC/Workstation, e.g., anaconda[1], followed by the installation of REBOUND and with a little effort, you will be able to solve the exercises from section 4.

You need a computer running either Linux or macOS or Windows 10 with the Windows Subsystem for Linux to run REBOUND. We strongly suggest to use the python interface. Make sure to install REBOUND before the labwork class. If you face any problems during the installation, please mail to Christoph Schaefer (ch.schaefer@uni-tuebingen.de).

Introduction

The classical N-body problem in Astrophysics is important to understand the evolution and stability of planetary systems, star clusters and galactic nuclei. The dynamics of these many body systems is dominated by pairwise gravitational interactions between single bodies. Hence, this two-body interaction has to be accounted with high accuracy in the numerical treatment.
Typical numbers of the aforementioned systems are ~ 10 for planetary systems, 10^4 to 10^6 for star clusters and more than over 10^8 for galactic nuclei.
For particle numbers of this magnitude, the methods of statistical mechanics can only be applied partially, and one has to rely on the direct integration of each individual particle trajectory with regard of the mutual gravitational forces.
In this practical course, we will use the existing REBOUND software package to address several small N-body problems. In the first section, the classical N-body problem and the numerical methods to solve it are presented. In the second section, the software package REBOUND and its installation is described. The third section contains the exercises.

1 The Classical N-Body-Problem

The motion of N point masses in their mutual gravitational field is the classical N-body problem. Each particle i with mass m_i has the location $\mathbf{r}_i$ and the velocity $\mathbf{v}_i$ at time t. The Hamiltonian of this system reads

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i{}^2}{2m_i} - \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{G m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|} \tag{1}$$

[1] `https://www.anaconda.com/distribution/`

with the canonical coordinates momentum $\mathbf{p}_i = m_i \mathbf{v}_i$ and location $\mathbf{q}_i = \mathbf{r}_i$ for all $i = 1 \dots N$ point masses. The Hamiltonian Equations yield the equations of motion for particle i

$$\dot{\mathbf{q}}_i(t) = \frac{\partial H}{\partial \mathbf{p}_i} \qquad \Longrightarrow \qquad \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \mathbf{v}_i \tag{2}$$

$$\dot{\mathbf{p}}_i(t) = -\frac{\partial H}{\partial \mathbf{q}_i} \qquad \Longrightarrow \qquad \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = \mathbf{a}_i \tag{3}$$

with the acceleration

$$\mathbf{a}_i(t) = \sum_{j \neq i}^{N} G m_j \frac{\mathbf{r}_{ij}}{r_{ij}^3} \quad . \tag{4}$$

Later, we will also need the time derivative of the acceleration, the so-called jerk

$$\dot{\mathbf{a}}_i(t) = \sum_{j \neq i}^{N} G m_j \left(\frac{\mathbf{v}_{ij}}{r_{ij}^3} - \frac{3(\mathbf{v}_{ij} \cdot \mathbf{r}_{ij})}{r_{ij}^5} \mathbf{r}_{ij} \right) \tag{5}$$

where $\mathbf{r}_{ij} := \mathbf{r}_j(t) - \mathbf{r}_i(t)$, $r_{ij} := |\mathbf{r}_{ij}|$, $\mathbf{v}_{ij} := \mathbf{v}_j(t) - \mathbf{v}_i(t)$, $v_{ij} := |\mathbf{v}_{ij}|$.
To calculate all accelerations at a certain time, one needs to evaluate $N(N-1)/2$ terms (using symmetry), that is the computing time for large numbers of N is asymptotically $\sim N^2$.

The Two-Body Problem

The Hamiltonian (1) of the two-body problem can be separated into the motion of the center of mass and the relative motion, that is $H = H(\mathbf{r}_1, \mathbf{r}_2, \mathbf{p}_1, \mathbf{p}_2) \to H(\mathbf{r}, \mathbf{p}, \mathbf{p}_{cm})$ with

$$H = H_{\mathrm{cm}} + H_{\mathrm{rel}} = \frac{\mathbf{p}_{\mathrm{cm}}^2}{2M} + \frac{\mathbf{p}^{\,2}}{2\mu} - \frac{G m_1 m_2}{r} \tag{6}$$

with the total mass $M := m_1 + m_2$, the reduced mass $\mu := m_1 m_2 / M$, the distance vector between the two bodies $\mathbf{r} := \mathbf{r}_1 - \mathbf{r}_2$, the distance $r := |\mathbf{r}\,|$, the relative momentum $\mathbf{p} := \mathbf{p}_1 - \mathbf{p}_2$ and the momentum of the center of mass $\mathbf{p}_{\mathrm{cm}}$.
The Hamiltonian of the center of mass motion H_{cm} (the first term on the right hand side of equation (6)) has trivial solutions. This follows from the face, that H_{cm} depends only on the momemtum of the center of mass $\mathbf{p}_{\mathrm{cm}}$ and not from the conjugate variable $\mathbf{r}_{\mathrm{cm}}$, the location of the center of mass. It follows a steady motion of the center of mass.
The Hamiltonian of the relative motion H_{rel} yields the equation of motion of the classical two-body problem

$$\dot{\mathbf{p}} = \mu \dot{\mathbf{v}} = -\frac{G m_1 m_2}{r^3} \mathbf{r} \tag{7}$$

with $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 = \mathbf{p}/\mu$, and finally in the familiar notation

$$\ddot{\mathbf{r}} = \dot{\mathbf{v}} = -\frac{GM}{r^3} \mathbf{r} \quad . \tag{8}$$

The solution of the Kepler-problem are conic sections with the orbital plane, the motion of $\mathbf{r}$ is characterised by an ellipse, a parabola or a hyperbola. The energy E, the specific angular momentum

$$\mathbf{j} = \mathbf{r} \times \mathbf{v} \tag{9}$$

and the Runge-Lenz vector

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{j}}{GM} - \frac{\mathbf{r}}{r} \tag{10}$$

are conserved values of the two-body motion. Using eq. (8), we find for the evolution of the angular momentum

$$\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t} = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \dot{\mathbf{v}} = -\frac{GM}{r^3}(\mathbf{r} \times \mathbf{r}) = 0 \quad . \tag{11}$$

To prove the conservation of $\mathbf{e}$, we need some calculus with the use of the vector identity $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C})$. It follows

$$\frac{\mathbf{j} \times \mathbf{r}}{r^3} = \frac{(\mathbf{r} \times \mathbf{v}) \times \mathbf{r}}{r^3} = \frac{\mathbf{v}}{r} - \mathbf{r}\frac{(\mathbf{r} \cdot \mathbf{v})}{r^3} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) \quad . \tag{12}$$

With this term and the help of eq. (8) and (11), we find

$$\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} = -\frac{GM}{r^3}\frac{\mathbf{r} \times \mathbf{j}}{GM} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) = \frac{\mathbf{j} \times \mathbf{r}}{r^3} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) = 0 \quad . \tag{13}$$

Additionally, we regard the following term

$$\mathbf{r} \cdot \mathbf{e} + r = \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{j})}{GM} = \frac{(\mathbf{r} \times \mathbf{v}) \cdot \mathbf{j}}{GM} = \frac{j^2}{GM} \quad . \tag{14}$$

The scalar product $\mathbf{r} \cdot \mathbf{e}$ is expressed with the angle $\phi - \phi_0$ between $\mathbf{e}$ and the location vector $\mathbf{r}$, and $re\cos(\phi - \phi_0) + r = j^2/(GM)$, it follows

$$r(\phi) = \frac{j^2/(GM)}{1 + e\cos(\phi - \phi_0)}. \tag{15}$$

This is the well-known equation of a conic section. For a constrained motion, the absolute value of the Runge-Lenz vector $e = |\mathbf{e}|$ specifies the eccentricity of the ellipse. Then, the maximum and minimum distance of the two point masses is given by

$$r_{\mathrm{max/min}} = \frac{j^2/(GM)}{1 \pm e}, \tag{16}$$

the semi-major axis of the ellipse is given by

$$a = \frac{1}{2}(r_{\mathrm{min}} + r_{\mathrm{max}}) = \frac{j^2/(GM)}{1 - e^2}, \tag{17}$$

and the semi-minor axis is given by the geometric mean

$$b = \sqrt{r_{\mathrm{min}} r_{\mathrm{max}}}. \tag{18}$$

Because of Kepler's third law ("The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit."), it follows for the orbital period

$$\omega = \sqrt{\frac{GM}{a^3}}. \tag{19}$$

Please see fig. 1 for an illustrative sketch of the properties of a bound Kepler orbit.

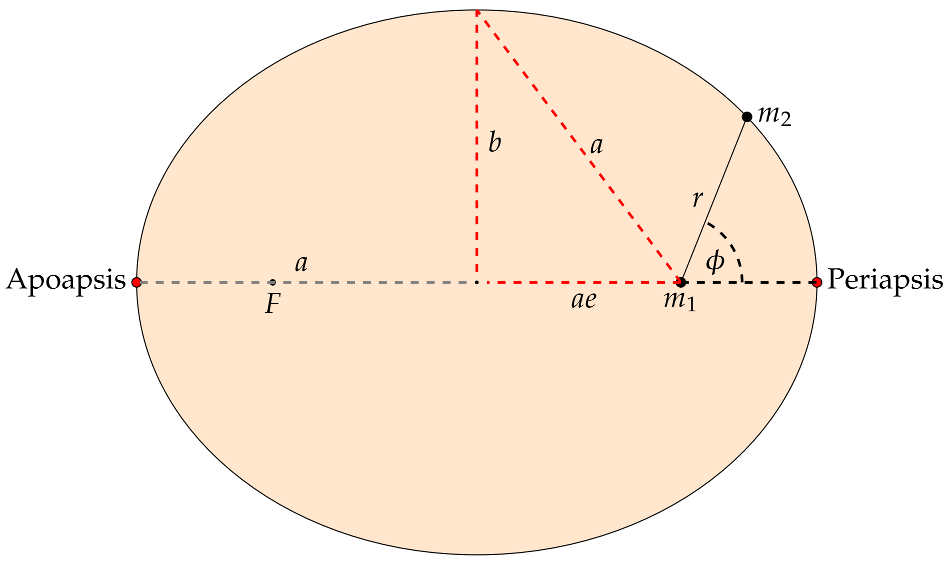


Figure 1: Bound Two-Body orbit. The mass m_1 is in the focus of the ellipse with semi-major and semin-minor axes a and b, and eccentricity e. The distance of m_1 to apoapsis is r_{max} and to periapsis r_{min}. The equation 15 denotes the orbit with the choice of the coordinate system so that $\phi_0 = 0$.

2 Numerical Solution of the Equation of Motion: Time integrators

In order to calculate the trajectories of the point masses, one has to solve the corresponding equation of motion for each particle i, whic is the system of ODEs given by eqs. (2) and (3) with appropriate initial values for locations and velocites. In general, this is only possible by the help of numerical methods, which leads to the fact, that the particle distribution can only be determined at discrete time values. Starting from the current point in time $t = t_n$, the location and the velocity at a later point in time $t_{n+1} = t_n + \Delta t$ are calculated. Then, the new point in time becomes the current one and the algorithm restarts at the new time.
In the following, different numerical schemes will be presented which can be used to solve a system of ODEs of the form $\mathrm{d}y/\mathrm{d}t = f(y,t)$ with initial values. These schemes are called time integrators for obvious reasons in this context. They can, however, be applied for any initial value problem. The indices in the following refer always to the equivalent point in time, that is $r_n = r(t_n)$, and so forth. The index i for the different point masses will be left in the following.

2.1 Single-Step Methods

Basic idea: Consider the differentials $\mathrm{d}y$ and $\mathrm{d}t$ as finite intervals Δy and Δt

$$\frac{\mathrm{d}y}{\mathrm{d}t} \Rightarrow \frac{\Delta y}{\Delta t} = f(t,y)$$

and discretise with $\Delta t = h$

$$\Delta y = y_{n+1} - y_n = \Delta t f = hf.$$

Generally, single-step methods can be written in the following form

$$y_{n+1} = y_n + h\Phi(t_n, y_n, y_{n+1}, h),$$

where Φ is called the evolution function. The scheme is called explicit if $\Phi = \Phi(t_n, y_n, h)$ and implicit for $\Phi = \Phi(t_n, y_n, y_{n+1}, h)$. The step is from t_n to t_{n+1} and from y_n to y_{n+1}.
Single-step methods may be constructed by Taylor expansion. At first, we will discuss the simple explicit one-step method, the explicit Euler-method with $\Phi(t_n, y_n, h) = f(t_n, y_n)$.

2.2 Simple time integrators: The Euler-method and the Euler-Cromer-method

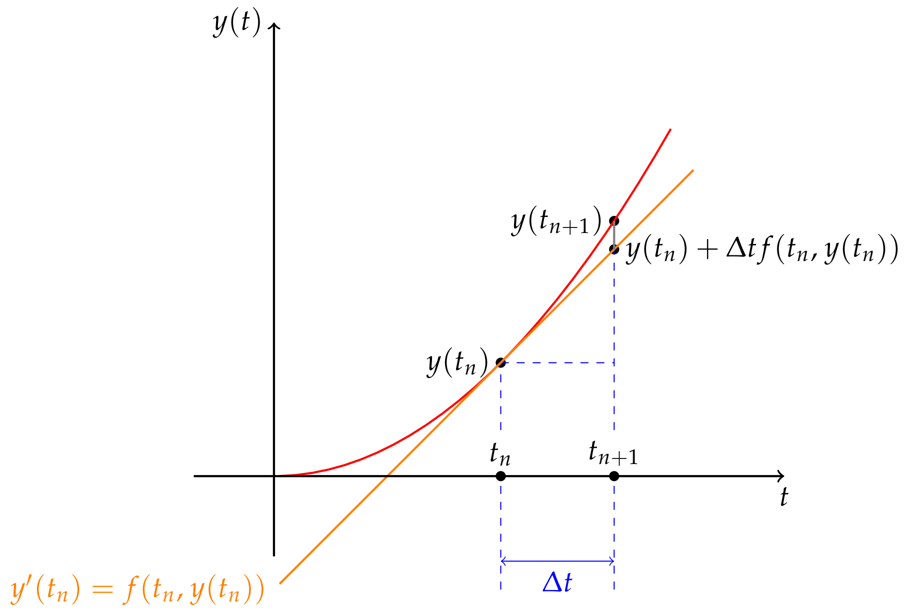
The simplest way for a discrete time integration scheme is achieved with the help of the Taylor expansion of location and velocity

$$\begin{aligned}
v(t_n + \Delta t) &= v(t_n) + \frac{\mathrm{d}v}{\mathrm{d}t}(t_n)\Delta t + \mathcal{O}(\Delta t^2), && (20)\\
&= v(t_n) + a(t_n)\Delta t + \mathcal{O}(\Delta t^2), && (21)\\
r(t_n + \Delta t) &= r(t_n) + \frac{\mathrm{d}r}{\mathrm{d}t}(t_n)\Delta t + \mathcal{O}(\Delta t^2), && (22)\\
&= r(t_n) + v(t_n)\Delta t + \mathcal{O}(\Delta t^2), && (23)
\end{aligned}$$

where the acceleration a is given by eq. (4).
These equations directly yield an algorithm, the so-called Euler-method, where one calculates

$$\begin{aligned}
v_{n+1} &= v_n + a_n\Delta t, && (24)\\
r_{n+1} &= r_n + v_n\Delta t && (25)
\end{aligned}$$

for every time step.
The Euler-method uses the simple slope triangle to perform one forward step


$y(t)$
$y(t_{n+1})$
$y(t_n) + \Delta t f(t_n, y(t_n))$
$y(t_n)$
t_n
t_{n+1}
t
Δt
$y'(t_n) = f(t_n, y(t_n))$

For each point in time t_n, the evolution of the function $y(t)$ from time t_n to $t_{n+1} = t_n + \Delta t$ is approximated by $\Delta t y'(t_n) = \Delta t f(t_n, y(t_n))$. The discretization error is given by the difference $y(t_{n+1}) - [y(t_n) + \Delta t f(t_n, y(t_n))]$.
As an alternative to the Euler-method where the velocity is calculated first, there is the so-called Euler-Cromer-method. Here, the new locations are calculated with the velocities at the new time step t_{n+1} instead of t_n like in the standard Euler-method.

$$v_{n+1} = v_n + a_n \Delta t, \tag{26}$$
$$r_{n+1} = r_n + v_{n+1} \Delta t. \tag{27}$$

Averaging both methods yields the scheme

$$v_{n+1} = v_n + a_n \Delta t, \tag{28}$$
$$r_{n+1} = r_n + \frac{1}{2} (v_n + v_{n+1}) \Delta t. \tag{29}$$

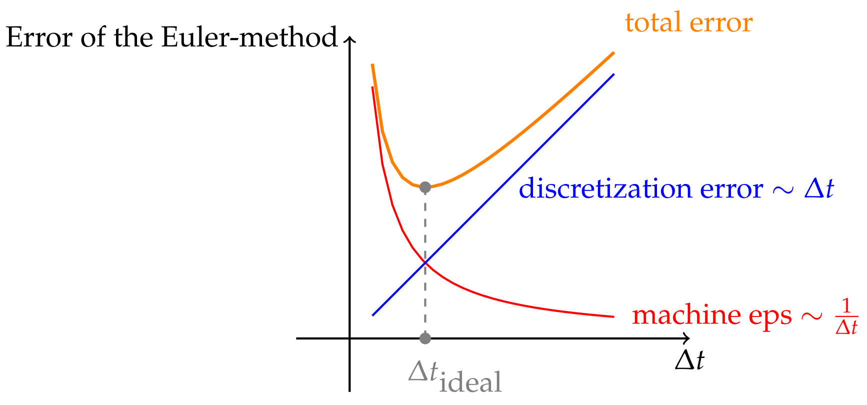
2.2.1 Accuracy of the scheme

The main disadvantage of these methods is the low accuracy. If the trajectories of particles have to be determined until the space in time $t_{\max}$ and each time step is of length Δt, we need $N_t := t_{\max}/\Delta t$ time steps. The error per time step is of the order $O(\Delta t^2)$ (see Taylor expansion), leading to a total error of the simulation of the order $O(N_t \Delta t^2) = O(\Delta t)$. And the computational accuracy was not even accounted, yet.

2.2.2 Computational accuracy

Let ϵ be the machine accuracy, the relative error due to rounding in floating point arithmetic. The error caused by ϵ grows with $O(N_t \epsilon)$. In order to improve the accuracy of the numerical scheme, one can choose smaller time steps Δt, which consequently yields a higher error

by ϵ, since we need a higher number of time steps N_t. Hence, global values of a N-body simulation like the total energy E have an error of the order $O(N_t N \epsilon)$.
Some numbers for clarification: Depending on the numerical scheme, approximately 100 to 1000 steps per orbit are required for the two-body problem to achieve an error for the energy in the order of the machine accuracy, that is $\Delta E/E \sim 10^{-13}$. The error in the energy adds up to $\Delta E/E \sim 10^{-9}$ for a complete orbit. To simulate the evolution of a star cluster with about 10^6 stars and for the time of about 10^4 typical orbital periods, one has to evaluate 10^{10} orbits with 1000 time steps each, in total 10^{13} time steps, eventually leading to an error of the total energy of $\Delta E \sim 100$ %.



2.3 The Leap-Frog-method

One improvement of the Euler-method is to use the velocity for the calculation of the new locations at the point in time $t_{n+1/2} = t_n + \Delta t/2$ and not on the times t_n or t_{n+1}. The scheme of this method is

$$\begin{aligned}
&\text{First step:} \\
&\mathbf{r}_{1/2} = \mathbf{r}_0 + \mathbf{v}_0 \frac{\Delta t}{2} \quad , \\
&\text{Calculate } \mathbf{a}_{1/2} = \mathbf{a}(t_{1/2}, \mathbf{r}_{1/2}) \quad , \\
&\text{Regular steps:} \\
&\mathbf{v}_{n+1} = \mathbf{v}_n + \mathbf{a}_{n+1/2}\, \Delta t \quad , \\
&\mathbf{r}_{n+3/2} = \mathbf{r}_{n+1/2} + \mathbf{v}_{n+1}\, \Delta t \quad , \\
&\text{Last step:} \\
&\mathbf{r}_{n+1} = \mathbf{r}_{n+1/2} + \mathbf{v}_{n+1} \frac{\Delta t}{2} \quad .
\end{aligned}$$

The name of the this scheme “Leap Frog” originates from the fact that the positions and velocities are updated at interleaved time points separated by $\Delta t/2$, staggered in such a way, that they leapfrog over each other. Hence, locations and velocities are not known at the same point in time In the limit of $\Delta t \to 0$, one gets the usual canonical equations. Moreover, for finite Δt, the scheme is symplectic, which means, there is a Hamiltonian $\hat{H} = H + \Delta t^2 H_2 + \Delta t^4 H_4 + \ldots$, which is solved exactly by the scheme. This has a positive impact on the consistency of conserved quantities.

2.4 The Verlet-method

The Leap-Frog-method can be written as the so-called kick-drift-kick algorithm

$$\tilde{\mathbf{v}}(\Delta t) = \mathbf{v}(t) + \frac{1}{2}\Delta t\, \mathbf{a}(t) \tag{30}$$

$$\mathbf{r}(t+\Delta t) = \mathbf{r}(t) + \Delta t\, \tilde{\mathbf{v}}(t) \tag{31}$$

$$\mathbf{v}(t+\Delta t) = \tilde{\mathbf{v}}(t) + \frac{1}{2}\Delta t\, \mathbf{a}(t+\Delta t) \tag{32}$$

The intermediate value $\tilde{\mathbf{v}}(\Delta t)$ denotes the velocity at time $t + 1/2\Delta t$.
The advantage of this velocity Verlet-method is the higher accuracy in the calculation of the locations. The Verlet-method is identical to the Leap-frog-method. Note, that also in this method, only one evaluation of the force term is required per time step.

2.5 Runge-Kutta-Integrators

The error of one single step in the Euler method can be analyzed by looking at the Taylor expansion. The Taylor expansion of $y(t+h)$ at t yields with $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t)$

$$y(t+h) = y(t) + h\frac{\mathrm{d}y}{\mathrm{d}t} + \mathcal{O}(h^2) \tag{33}$$

$$= y(t) + hf(y,t) + \mathcal{O}(h^2) \tag{34}$$

Hence, we could get better accuracy if we also include higher terms.

2.5.1 Basic idea of RK schemes

Now, let's expand up to third order

$$y(t+h) = y(t) + h\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{h^2}{2}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \mathcal{O}(h^3) \tag{35}$$

$$= y(t) + hf(y,t) + \frac{h^2}{2}\frac{\mathrm{d}f}{\mathrm{d}t} + \mathcal{O}(h^3) \tag{36}$$

The derivative of f with respect to t is given by

$$\frac{\mathrm{d}f(y(t),t)}{\mathrm{d}t} = \frac{f(y(t+h),t+h) - f(y(t),t)}{h} + \mathcal{O}(h). \tag{37}$$

Unfortunately, we do not know $y(t+h)$. By using the Euler scheme to estimate $y(t+h)$, we get

$$\frac{\mathrm{d}f(y(t),t)}{\mathrm{d}t} = \frac{f(y(t)+hf(y(t),t),t+h) - f(y(t),t)}{h} + \mathcal{O}(h). \tag{38}$$

Inserting eq. (38) into (36) yields

$$y(t+h) = y(t) + hf(y(t),t) + \frac{h}{2}\left\{f(y(t)+hf(y(t),t),t+h) - f(y(t),t)\right\} + \mathcal{O}(h^3) \tag{39}$$

$$= y(t) + \frac{h}{2}\left\{f(y(t),t) + f(y(t)+hf(y(t),t),t+h)\right\} + \mathcal{O}(h^3). \tag{40}$$

We have improved the accuracy for the cost of an additional calculation of f at substep $y(t) + hf(y,t)$. This is the basic idea of any Runge-Kutta method. In fact, we have just derived a RK 2nd order scheme, the Heun method.
The general form of an explicit RK method is

$$y_{n+1} = y_n + h \sum_i b_i k_i, \tag{41}$$

where the k_i denote the substeps and b_i are some weights. Please see a textbook for further details. Here, we provide only the terms for the RK4 method.

2.5.2 The classical Runge-Kutta scheme

A system of first order ODEs can be written in vector notation

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}(t)), \quad \mathbf{y}(t_0) = \mathbf{y}_0 \tag{42}$$

with the vector for the initial values $\mathbf{y}_0$. In our N-body problem, $\mathbf{y}$ will be $(\mathbf{r}, \mathbf{v})$ with initial values $(\mathbf{r}(0), \mathbf{v}(0))$. A very common integration scheme is the so-called 4th order Runge-Kutta-method (RK4). In this scheme, estimated values for the derivative at single substeps in the interval $t_n, t_n + h$ are calculated. In vector notation, it reads

$$\mathbf{K}_1 = \mathbf{f}(t_n, \mathbf{y}_n) \tag{43}$$

$$\mathbf{K}_2 = \mathbf{f}\left(t_n + \frac{1}{2}h, \mathbf{y}_n + \frac{h}{2}\mathbf{K}_1\right) \tag{44}$$

$$\mathbf{K}_3 = \mathbf{f}\left(t_n + \frac{1}{2}h, \mathbf{y}_n + \frac{h}{2}\mathbf{K}_2\right) \tag{45}$$

$$\mathbf{K}_4 = \mathbf{f}\left(t_{n+1}, \mathbf{y}_n + h\mathbf{K}_3\right) \tag{46}$$

where $\mathbf{y}_n, \mathbf{y}_{n+1}, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4$ and $\mathbf{f}$ are vectors in the vector space R^m. The value of the function $\mathbf{y}$ at the new time is calculated by

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{6}\left(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4\right). \tag{47}$$

The error of the scheme is of the order $\mathcal{O}(h^5)$. The price for this high accuracy is the required calculation of the function which is four times per time step. However, the stability is also improved compared to the Euler-method. For a more detailed description of the RK4-method, we refer to the book "Numerical Recipes"[2] and wikipedia. We will apply the RK4-method in this course on the system of equations (2) and (3).

2.5.3 Special integrators for N-Body problems

Although the presented integrators from the previous sections may yield profound and solid results for ODEs, special integrators for the N-Body problem have been developed in the recent years. The N-Body code that you will use in this labwork features several symplectic integrators (WHFast, WHFastHelio, SEI, LEAPFROG) and a high accuracy non-symplectic integrator with adaptive timestepping (IAS15). These integrators are described nicely in the following publications by Hanno Rein

[2] `http://apps.nrbook.com/c/index.html`, p.710

1. Rein & Liu 2012 (Astronomy and Astrophysics, Volume 537, A128) describe the code structure and the main feature including the gravity and collision routines for many particle systems. `http://adsabs.harvard.edu/abs/2012A%26A...537A.128R`
2. Rein & Spiegel 2015 (Monthly Notices of the Royal Astronomical Society, Volume 446, Issue 2, p.1424-1437) describe the versatile high order integrator IAS15 which is now part of REBOUND. `http://adsabs.harvard.edu/abs/2015MNRAS.446.1424R`
3. Rein & Tamayo 2015 (Monthly Notices of the Royal Astronomical Society, Volume 452, Issue 1, p.376-388) describe WHFast, the fast and unbiased implementation of a symplectic Wisdom-Holman integrator for long term gravitational simulations. `http://adsabs.harvard.edu/abs/2015MNRAS.452..376R`
4. Rein & Tamayo 2016 (Monthly Notices of the Royal Astronomical Society, Volume 459, Issue 3, p.2275-2285) develop the framework for second order variational equations. `http://arxiv.org/abs/1603.03424`
5. Rein & Tamayo 2017 (Monthly Notices of the Royal Astronomical Society, Volume 467, Issue 2, p.2377-2383) describes the Simulation Archive for exact reproducibility of N-body simulations. `https://arxiv.org/abs/1701.07423`
6. Rein & Tamayo 2018 (Monthly Notices of the Royal Astronomical Society, Volume 473, Issue 3, p.3351–3357) describes the integer based JANUS integrator. `https://arxiv.org/abs/1704.07715`
7. Rein, Hernandez, Tamayo, Brown, Eckels, Holmes, Lau, Leblanc & Silburt 2019 (Monthly Notices of the Royal Astronomical Society, Volume 485, Issue 4, p.5490-5497) describes the hybrid symplectic intergator MERCURIUS. `https://arxiv.org/abs/1903.04972`
8. Rein, Tamayo & Brown 2019 (submitted) describes the implementation of the high order symplectic intergators SABA, SABAC, SABACL, WHCKL, WHCKM, and WHCKC. `https://arxiv.org/abs/1907.11335`

3 The N-Body Code REBOUND

REBOUND is a software package that can integrate the motion of particles under the influence of gravity. The particles can represent stars, planets, moons, ring or dust particles. REBOUND is free software and published under the terms of the GNU General Public License as published by the Free Software Foundation, either version 3 of the License, or (at your option) any later version[3]. REBOUND's maintainer is Professor Hanno Rein (`https://rein.utsc.utoronto.ca/`). The source code can be obtained via github `https://github.com/hannorein/rebound` and the complete documentation is online via `https://rebound.readthedocs.io/en/latest/`.
REBOUND's core is written in the programming language C, but it offers a very nice and user-friendly python interface. **We highly recommend to use the python interface for the exercises.**

[3]`http://www.gnu.org/licenses/`

3.1 Download and Installation

Depending on your choice of programming language, you have to install REBOUND in the following way. Please consult also the documentation about the installation if you face any problems.

3.1.1 Python

To install the python module for REBOUND, you can use `pip`. Make sure to use python version 3 only.

```
pip install rebound
```

To verify the installation, try to import the module in the commandline interpreter and load the help for the module

```
> python
Python 3.7.3 (default, Apr  3 2019, 05:39:12)
[GCC 8.3.0] on linux
Type "help", "copyright", "credits" or "license" for more information.
>>> import rebound
>>> help(rebound)
```

3.1.2 C

To clone the source code from the github repository, use the following command, which also runs a first example to test if the compilation was successfull

```
git clone http://github.com/hannorein/rebound && cd rebound/examples/
    shearing_sheet && make && ./rebound
```

For your own exercises, you can simply copy one of the example files in rebound/examples and modify it and use the Makefile provided there, or you use the following Makefile template (Linux, macOS) to compile source file `foo.c` and link to rebound (make sure to set `REBOUND_SRC` accordingly)

```
OPT+= -Wall -g -Wno-unused-result
OPT+= -std=c99 -Wpointer-arith -D_GNU_SOURCE -O3
LIB+= -lm -lrt

# your preferred compiler
CC = gcc
# path to the source directory of your rebound installation
REBOUND_SRC=../../src/rebound/src/

all: librebound
        @echo ""
        @echo "Compiling problem file ..."
        $(CC) -I$(REBOUND_SRC) -Wl,-rpath,./ $(OPT) $(PREDEF) foo.c -L. -
    lrebound $(LIB) -o foo
        @echo ""
        @echo "REBOUND compiled successfully."

librebound:
        @echo "Compiling shared library librebound.so ..."
        $(MAKE) -C $(REBOUND_SRC)
        @-rm -f librebound.so
        @ln -s $(REBOUND_SRC)/librebound.so .

clean:
        @echo "Cleaning up shared library librebound.so ..."
        @-rm -f librebound.so
        $(MAKE) -C $(REBOUND_SRC) clean
        @echo "Cleaning up local directory ..."
        @-rm -vf rebound
```

3.2 Documentation and Tutorial

REBOUND is a well documented software package and the webpage offers a large number of examples both in C and python. Hence, we do not give a more detailed description in this script. Please see `https://rebound.readthedocs.io/en/latest/examples.html` for examples and `https://rebound.readthedocs.io/en/latest/quickstart.html` for the quick-start guide.
If you run into problems during the installation and fail to run an example, please contact your tutor!

4 Exercises

In the following subsections you find the exercises that you have to solve during this lab-work. Be sure to add your findings in your minutes and if important to the final protocol. Think about meaningful plots to state your results and add them to the protocol.

4.1 The Two-Body Problem

Consider the two-body problem with the following setup: eccentricity $e = 0.3$, semi-major axis $a = 1$, $m_1 = 1$, $m_2 = 10^{-3}$. With the gravitational constant G set to 1, the orbital period is $\approx 2\pi$. Energy and angular momentum should be conserved, they are given by

$$E = -\mu \frac{GM}{2a},$$
$$L = \mu \sqrt{(1-e^2)GMa},$$

where μ denotes the reduced mass and M the total mass

$$\mu = \frac{m_1 m_2}{M},$$
$$M = m_1 + m_2.$$

Choose the z-axis as the direction of the angular momentum $L_z = L$. Choose the x-axis as the direction of the periapsis to body 2. At time $t_0 = t(0) = 0$, both bodies are at closest distance $d_p = a(1-e)$. At time t_0, the positions and the velocities of the two bodies are given by $v_{y1}(0) = -L/(d_p m_1)$, $x_1(0) = -d_p m_2/M$ and $v_{y2}(0) = L/(d_p m_2)$, $x_2(0) = d_p m_1/M$. The motion of the two bodies is solely in the xy-plane.

We want to test the quality of the various integrators from the REBOUND software package. Integrate the system for 10^3 orbital periods at first with the Leap-Frog integrator with different fixed time-steps 1 to 10^{-6} and observe the energy, angular momentum, and the orbital elements. Try also another integrator!

4.1.1 Solution with C

```c
// note: this is not the complete source code, just a snippet
// add the missing lines

#define TWOPI 6.283185307179586

struct reb_simulation* r = reb_create_simulation();

/* setup of the simulation
 * create two particles via
 * struct reb_particle p1 and p2
 * add the two particles to the simulation via reb_add(r, p)
 *
 */

 // choose Leap-frog integrator
r->integrator = REB_INTEGRATOR_LEAPFROG;
// fixed time step
r->dt = 1e-4; // and 1 and 1e-1 and 1e-2 and 1e-3 and ...

// function pointer to the heartbeat function.
// the function is called after each dt
r->heartbeat = heartbeat;

// move all to center of mass frame
reb_move_to_com(r);

// now integrate for 1000 orbits, 2pi is one orbit
reb_integrate(r, 1000*TWOPI);
```

Define a heartbeat function `void heartbeat(struct reb_simulation *r)` to write the orbits, eccentricities and energy with the help of the builtin functions `reb_output_ascii` and/or `reb_output_orbits` to files and plot the values (using gnuplot or python or your preferred plotting tool).

4.1.2 Solution with Python

```python
#!/usr/bin/env python3
# integrate two body problem, awful formatting to fit on a single page
import sys
import numpy as np
import matplotlib.pyplot as plt
import rebound

# create a sim object
sim = rebound.Simulation()

# set the integrator to type REB_INTEGRATOR_LEAPFROG
```

```python
sim.integrator = "leapfrog"
# and use a fixed time step
sim.dt = 1e-4
# here add your particles....
sim.add(m=1.0)
sim.add(m=1e-3, a=1.0, e=0.3)
# do not forget to move to the center of mass
sim.move_to_com()
# create time array, let's say 1 orbit, plot 250 times per orbit
Norbits = 1
Nsteps = Norbits*250
times = np.linspace(0, Norbits*2*np.pi, Nsteps)
x = np.zeros((sim.N, Nsteps)) # coordinates for both particles
y = np.zeros_like(x)
energy = np.zeros(Nsteps) # energy of the system
# now integrate
for i, t in enumerate(times):
    print(t, end="\r")
    sim.integrate(t, exact_finish_time=0)
    energy[i] = sim.calculate_energy()
    for j in range(sim.N):
        x[j,i] = sim.particles[j].x
        y[j,i] = sim.particles[j].y
print("Done, now plotting...")
# plot the orbit
fig, ax = plt.subplots()
ax.scatter(x,y, s=2)
ax.set_title("Orbit with step size %g" % sim.dt)
ax.set_aspect("equal")
ax.set_xlabel("x-coordinate")
ax.set_ylabel("y-coordinate")
plt.grid(True)
fig.savefig("orbit"+str(sim.dt)+".pdf")
# plot the energy
fig, ax = plt.subplots()
ax.scatter(times, np.abs(energy-energy[0])/np.abs(energy[0]), s=2)
print(energy)
ax.set_title("Energy with step size %g" % sim.dt)
ax.set_xlabel("time")
ax.set_yscale("log")
ax.set_ylabel("energy")
plt.grid(True)
fig.savefig("energy"+str(sim.dt)+".pdf")
print("Done.")
```

The orbital elements are also stored in `sim.particles`, e.g., the eccentricity of the second particle is given by `sim.particles[1].e`. Plot the orbital elements for different fixed time steps and test if the conservation of energy and angular momentum is given.

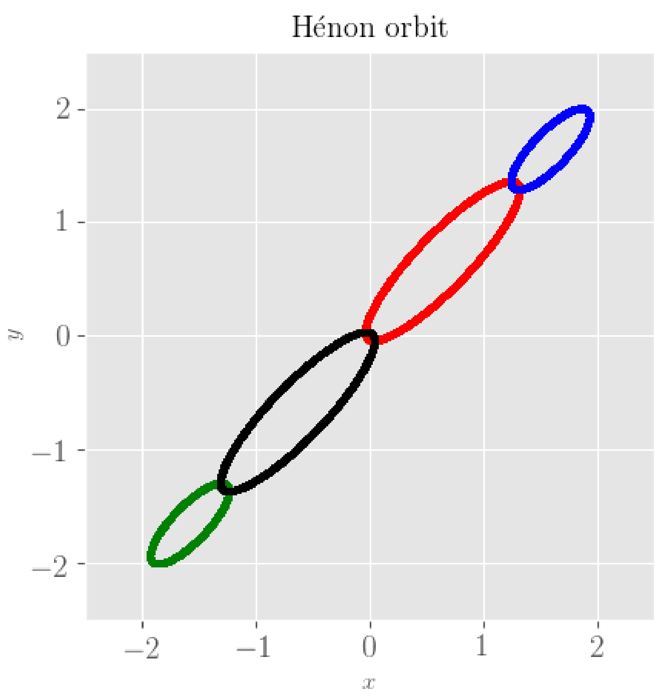


Figure 2: Hénon choreography.

4.2 Hénon orbit

Here, we investigate a four-body problem. Consider four point masses with mass $m = 1$, coordinates and velocities given by

$$\begin{aligned}
x_0 &= 0.504457024898985 & y_0 &= 1.058520304479640 \\
x_1 &= 1.581323394863468 & y_1 &= 1.639575414656912 \\
x_2 &= -1.960902462113829 & y_2 &= -1.605062781926992 \\
x_3 &= -0.884036092149343 & y_3 &= -1.024007671749708 \\
v_{0x} &= -0.588406317185329 & v_{0y} &= -0.507208447977203 \\
v_{1x} &= 0.377133148141246 & v_{1y} &= 0.459577663854508 \\
v_{2x} &= -0.377133148141246 & v_{2y} &= -0.459577663854508 \\
v_{3x} &= 0.588406317185329 & v_{3y} &= 0.507208447976862
\end{aligned} \tag{48}$$

Use the function `reb_output_orbit` (in C) or the information given in the object `sim.particles` (Python) to get a typical period time scale t_p of the problem. Evolve the system for 100 t_p and plot the orbits. You should get a picture like Fig. 2. There are an infinite number of planar N-Body choreographies. See Robert Vanderbei's homepage for more.

n	linear stability analysis γ	numerical simulation analysis γ
8	2.412	2.4121
10	2.375	2.3753
36	2.306	2.3066
100	2.300	2.2999

Table 1: Values for the parameter γ found by Vanderbei & Kolemen.

4.3 Stability of Saturn's rings

The idea for this exercise is based on the paper by Vanderbei & Kolemen (2007)[4]. They give a self-contained modern linear stability analysis of a system of n equal mass bodies in circular orbit around a single more massive body. We will investigate their analytical findings with simulations.
Consider following scenario: one high mass body with mass M is orbited by $n-1$ lower mass bodies with equal masses m at distance $r=1$. The initial angular velocity ω of the lower mass bodies is

$$\omega = \sqrt{\frac{GM}{r^3} + \frac{GmI_n}{r^3}}, \tag{49}$$

where I_n is given by

$$I_n = \sum_{k=1}^{n-1} \frac{1}{4} \frac{1}{\sin(\pi k/n)}.$$

The result of Vanderbei & Kolemen is that the system is linear stable as long as

$$m \leq \frac{\gamma M}{n^3}, \tag{50}$$

with the values for γ given in table 1. The goal of this exercise is to reproduce the analysis by Vanderbei & Kolemen. Can you reproduce their result with your simulations? Use Saturn's mass for the central object ($M = 2.857\,166\,56 \times 10^{-4} \mathrm{M}_\odot$), set the initial distance of $n-1$ lower mass bodies to $r=1$ and the velocity according to eq. (49), use the leap frog integrator. Vary both n and the mass m of the $n-1$ smaller objects, check if you expect a stable system according to eq. (50) and integrate the system for 100 t_p. If the masses are still orbiting Saturn, we consider the systems as stable. Try different systems using values from table 1, i.e. start with $n=8$ and vary the mass from a value which gives a stable configuration to a value that yields instability in 10-20 steps, proceed in the same manner for $n = 10, 36, 100$.

4.4 Jupiter and Kirkwood gaps

In this problem we want to address the influence of Jupiter on the asteroid belt. A Kirkwood gap is a gap in the distribution of the semi-major axes of the asteroids. They correspond to the locations of orbital resonances with Jupiter. We will use inactive particles to model the asteroids and consider only the Sun, Jupiter and Mars as gravitating objects (`r->N_active=3`). Hence, the asteroids have no feedback on Sun, Jupiter and Mars and are just tracer particles. Set up the Sun-Mars-Jupiter system with the help of the NASA Horizons web-interface to

[4] `https://arxiv.org/abs/astro-ph/0606510`

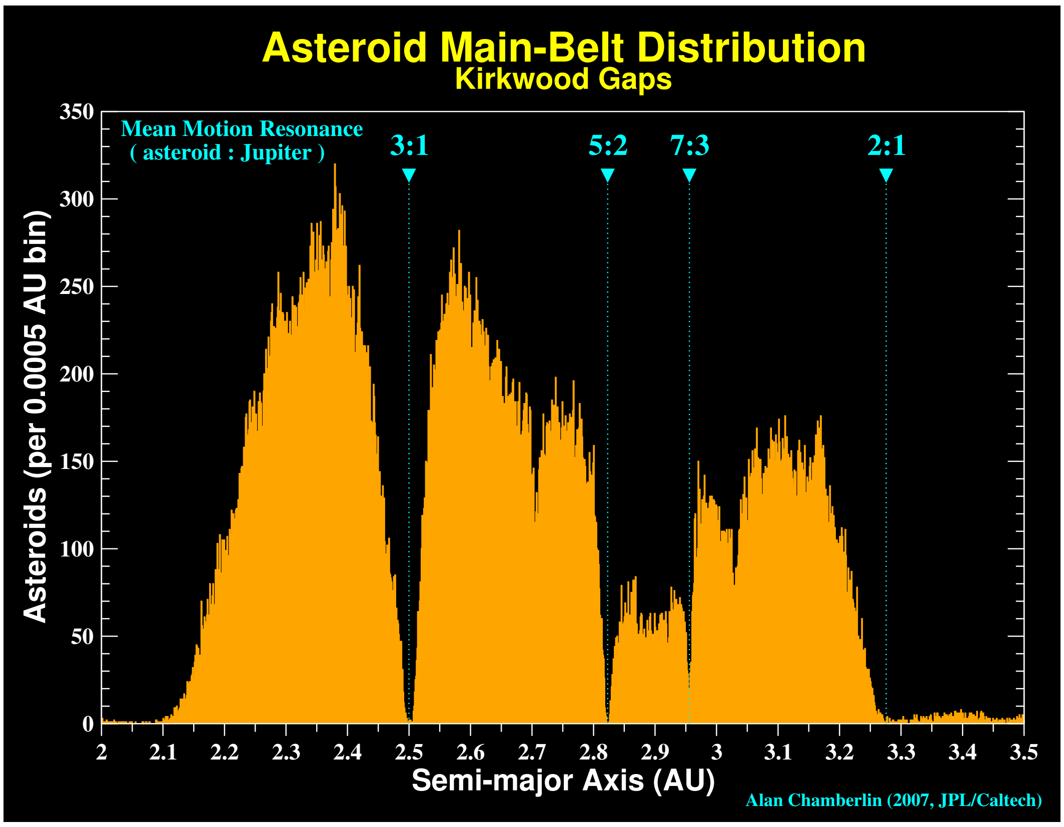


Figure 3: Kirkwood gaps in the asteroid belt (Chamberlin 2007).

get initial conditions for Mars and Jupiter (Ephemeris Type VECTORS) and add randomly 10 000 inactive particles between 2 and 4 au in the midplane.
Let the system evolve for several one hundred thousands of years and look for the Kirkwood gaps (cf. Figure 3).
Note: This will probably require to run the code over night (depending on your hardware).

4.4.1 Solution with C

The following code snippet can be used to generate testparticles moving on a Keplerian orbit around the sun

```
#define TWOPI 6.283185307179586

    double au = 1.496e11;
    while (r->N < 10003) {
        // here we need to randomly generate points between 2 and 4 au
        // random radius
        double radius = 4*((double)rand()/RAND_MAX);
        double phi = TWOPI*(double)rand()/RAND_MAX;
        double x = radius * cos(phi);
        double y = radius * sin(phi);
        double a    = sqrt(x*x+y*y);
        if (a<2) continue;
        if (a>4) continue;
        a *= au;
        // star is our sun
        double vkep = sqrt(r->G*star.m/a);
        struct reb_particle testparticle = {0};
        testparticle.x  = x*au;
        testparticle.y  = y*au;
        testparticle.z  = 0.0;
        testparticle.vx = -vkep*sin(phi);
        testparticle.vy = vkep*cos(phi);
        reb_add(r, testparticle);
    }
```

4.4.2 Solution with Python

The following code snippet can be used to generate testparticles moving on a Keplerian orbit around the sun

```
solarmass = 1.989e30
au = 1.496e11
gravitationalconstant = 6.67408e-11

# create a sim object
sim = rebound.Simulation()
# use SI units
sim.G = gravitationalconstant

# add 10000 inactive tracerparticles between 2 and 4 au
N_testparticle = 10000
a_ini = np.linspace(2*au, 4*au, N_testparticle)
for a in a_ini:
    sim.add(a=a, f=np.random.rand()*2.*np.pi, e=np.random.rand()) # mass is
    set to 0 by default, random true anomaly, random eccentricity

# set a reasonable time step
# we use 1e-2 of an orbit at 2 au
orbit = 2*np.pi*np.sqrt(8*au*au*au/(gravitationalconstant*solarmass));
sim.dt = orbit*1e-2
```

Generate meaningful plots (eccentricity over semi-major axis, histogram plot of number of asteroids over semi-major axis) at different times (e.g., every one hundred thousand years) until at least one million years to show the formation of the dips. Add the known Kirkwood gaps to your plots. Do you find agreement with your results?

Happy Coding! ☺ ☕